Identifying Latent Components of the $TINAR$(1) Model

Miodrag S. Djordjević[a], Miroslav M. Ristić[a], Bogdan Pirković[b]

[a]University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia
[b]University of Kragujevac, Faculty of Science, Radoja Domanovića 12, 34000 Kragujevac, Serbia

Abstract. In this paper we give a solution for the problem of identifying and predicting latent components of the integer-valued time series with skewed Skellam marginal distribution. At the beginning, expressions for latent components identification and prediction are derived. These expressions give us a possibility of revealing the values of two hidden, immeasurable components which affect the integer-valued time series with skewed Skellam marginals. Yule-Walker estimators of the unknown parameters are obtained. Also, quality of identification and lag-one prediction is tested on simulated data. At the very end, the model validation is performed in application on the real-life data.

1. Introduction

The area of integer-valued autoregressive ($INAR$) time series has been the subject of research during several decades. The major breakthrough in this area was made, independently of each other, by [13] and [2]. In the period that followed, development of $INAR$ models proceeded in two directions. Some authors introduced new thinning operators, as it was done by [3], [12], [17, 18] or [16]. On the other hand, some of them discussed marginal distributions, as it was done by [14], [1], [4], [5] etc. The most of introduced models were non-negative, although in many real-life situations there are processes which may consist of integer values including both, positive and negative numbers.

A step forward in this direction was made by [11], who introduced true integer-valued time series with Skellam marginal distribution, defined in distribution as a difference between two nonnegative, independent $INAR$ time series with Poisson marginal distributions. Similar idea was used by [7], who introduced a integer-valued autoregressive time series with geometric-Poisson marginals. Lately, [8] proposed a new $INAR$ time series with generalized Poisson difference marginal distributions based on difference of two quasi-binomial thinning operators. On the other hand, [15] and [6] introduced $INAR$ models with symmetric discrete Laplace and skewed discrete Laplace marginal distributions, defined in distribution as a difference between two non-negative and independent $NGINAR$(1) time series with geometric marginal distributions. However, none of the mentioned researches doesn't discuss the problem of latent components identification and their prediction. This topic was first analyzed by [10], who introduced a new time series

of order 1 with skewed discrete Laplace marginals and managed to derive expressions for identification and prediction of its latent components. Also, similar was done in [9], except that the time series of order p came in focus of the author.

Here, we give a method for quantifying two opposite, latent dimensions having only their difference - the resultant of these two hidden forces. Knowing the values of latent components can help us a lot in understanding the primary time series we want to analyze. Predicting their values give us a clue how the primary time series will behave in the future. Inspired by ideas given in [9, 10], we can conclude it would be useful to determine expressions for identification and prediction of latent components of $TINAR(1)$ time series with Skellam marginal distribution. In particular, we will pay attention on $TINAR(1)$ time series $\{Z_n\}$ with skewed Skellam marginal distribution, which can be presented in distribution as the difference between two nonnegative, independent $INAR(1)$ components $\{X_n\}$ and $\{Y_n\}$ with different Poisson marginal distributions, i.e. $Z_n \overset{d}{=} X_n - Y_n$. Expression for calculating the mathematical expectation of a random variable X_{n+k}, $k = 0, 1$, given Z_n, represents the foundation of the obtained results.

Section 2 brings us expressions for identification and prediction of latent components, as a functions of random sequence $\{Z_n\}$ and latent components marginal distributions parameters. In Section 3, Yule-Walker (YW) estimates of unknown parameters of the model are provided. In Section 4, the quality of extraction and lag-one prediction of latent components is examined on simulated data. Application of this kind of modeling on selected real-life data is presented in Section 5..

2. Extracting and predicting latent components of the skewed $TINAR(1)$ time series

For any time series $\{Z_n\}$ that is defined in distribution as a difference of two independent time series $\{X_n\}$ and $\{Y_n\}$,

$$Z_n \overset{d}{=} X_n - Y_n,$$

it could be easily shown that conditional moment of order $r, r \in \mathbb{N}$, of the variable X_{n+k}, given $Z_n = z$, is of the form

$$E(X_{n+k}^r|Z_n = z) = \begin{cases} \dfrac{\sum\limits_{j=0}^{\infty} E(X_{n+k}^r|X_n=j+z,Y_n=j)P(X_n=j+z)P(Y_n=j)}{P(Z_n=z)}, & z > 0, \\[4mm] \dfrac{\sum\limits_{j=0}^{\infty} E(X_{n+k}^r|X_n=j,Y_n=j-z)P(X_n=j)P(Y_n=j-z)}{P(Z_n=z)}, & z \leq 0. \end{cases} \tag{1}$$

Namely, let us suppose that z is greater than 0. We have that

$$E(X_{n+k}^r|Z_n = z) = \sum_{i=0}^{\infty} i^r P(X_{n+k} = i|Z_n = z) = \sum_{i=0}^{\infty} i^r \frac{P(X_{n+k} = i, Z_n = z)}{P(Z_n = z)}.$$

With $z > 0$, we have that $P(Z_n = z) = \sum\limits_{j=0}^{\infty} P(X_n = j + z, Y_n = j)$, so it holds

$$\begin{aligned} E(X_{n+k}^r|Z_n = z) &= \frac{1}{P(Z_n = z)} \sum_{i=0}^{\infty} i^r \sum_{j=0}^{\infty} P(X_{n+k} = i, X_n = j + z, Y_n = j) \\ &= \frac{1}{P(Z_n = z)} \sum_{i=0}^{\infty} i^r \sum_{j=0}^{\infty} P(X_{n+k} = i|X_n = j + z, Y_n = j) \\ &\quad \times P(X_n = y + z)P(Y_n = y). \end{aligned}$$

Exchanging the order of summation, we get

$$
\begin{aligned}
E(X_{n+k}^r|Z_n = z) &= \frac{1}{P(Z_n = z)} \sum_{j=0}^{\infty} P(X_n = j + z)P(Y_n = j) \sum_{i=0}^{\infty} i^r P(X_{n+k} = i|X_n = j + z, Y_n = j) \\
&= \frac{1}{P(Z_n = z)} \sum_{j=0}^{\infty} E(X_{n+k}^r|X_n = j + z, Y_n = j)P(X_n = j + z)P(Y_n = j).
\end{aligned}
$$

Corresponding equality can be obtained in case of $z \leq 0$ as well. And, using the same procedure suggested above, similar result can be derived for the other latent component, i.e.

$$
E(Y_{n+k}^r|Z_n = z) = \begin{cases} \dfrac{\sum\limits_{i=0}^{\infty} E(Y_{n+k}^r|X_n=i+z,Y_n=i)P(X_n=i+z)P(Y_n=i)}{P(Z_n=z)}, & z > 0, \\ \dfrac{\sum\limits_{i=0}^{\infty} E(Y_{n+k}^r|X_n=i,Y_n=i-z)P(X_n=i)P(Y_n=i-z)}{P(Z_n=z)}, & z \leq 0. \end{cases} \tag{2}
$$

As it was previously mentioned, *TINAR*(1) model with skewed Skellam marginal distribution could be presented as the difference of two nonnegative, independent *INAR*(1) time series with different Poisson marginal distributions, hence (1) and (2) holds for *TINAR*(1). Using these equations, we shall try to show in this paper our approach in identifying and predicting its latent components.

Let the $\{Z_n\}$ be a *TINAR*(1) time series of the first order with skewed *Skellam*$(\mu/(1-\alpha), \nu/(1-\beta))$ marginal distribution. And let $\{X_n\}$ and $\{Y_n\}$ be two independent *INAR*(1) time series with Poisson, $\mathcal{P}(\mu/(1-\alpha))$ and $\mathcal{P}(\nu/(1-\beta))$ distribution, defined using the binomial thinning operator in the following way:

$$
X_{n+1} = \alpha \circ X_n + \varepsilon_{n+1} \text{ i } Y_{n+1} = \beta \circ Y_n + \eta_{n+1}, \tag{3}
$$

where $"\alpha \circ"$ is the binomial thinning operator, elements of counting series are independent and identically distributed random variables with Bernoulli distributions with parameter α and parameter β. Let those two time series be latent components of the time series $\{Z_n\}$ and let

$$
Z_n \stackrel{d}{=} X_n - Y_n.
$$

In order to determine expressions for identification and prediction of latent components upon realization of the time series $\{Z_n\}$, we should find, for $k = 0, 1$, conditional expectations $E(X_{n+k}|Z_n = z)$ and $E(Y_{n+k}|Z_n = z)$, whereby the conditional expectation of the variable X_n, given $Z_n = z$, is defined as

$$
E(X_n|Z_n = z) = \sum_{j=0}^{\infty} j \cdot p_{X_n|Z_n}(j|z)
$$

and $p_{X_n|Z_n}(j|z)$ represents the conditional probability mass function of the variable X_n, given $Z_n = z$.

With that goal in mind, we use (1) and get the following. Let's suppose that $k = 0$, $r = 1$ and $z > 0$. In that case, we get that

$$
E(X_n|Z_n = z) = \frac{1}{P(Z_n = z)} \sum_{j=0}^{\infty} E(X_n|X_n = j + z, Y_n = j)P(X_n = j + z)P(Y_n = j)
$$

Bearing in mind that distributions of random variables X_n and Y_n are Poisson with parameters $\frac{\mu}{1-\alpha}$ and $\frac{\nu}{1-\beta}$, respectively, with the probability mass functions given as

$$
P(X_n = x) = \frac{\left(\frac{\mu}{1-\alpha}\right)^x}{x!} e^{-\frac{\mu}{1-\alpha}}, \quad P(Y_n = y) = \frac{\left(\frac{\nu}{1-\beta}\right)^y}{y!} e^{-\frac{\nu}{1-\beta}}, \tag{4}
$$

it can be shown that

$$
\begin{aligned}
E(X_n|Z_n = z) &= \frac{e^{-\left(\frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\right)}}{P(Z_n = z)} \sum_{j=0}^{\infty} \frac{\left(\frac{\mu}{1-\alpha}\right)^{j+z} \left(\frac{\nu}{1-\beta}\right)^j}{j!(j+z)!} E(X_n|X_n = j+z, Y_n = j) \\
&= \frac{e^{-\left(\frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\right)}}{P(Z_n = z)} \sum_{j=0}^{\infty} \frac{\left(\frac{\mu}{1-\alpha}\right)^{j+z} \left(\frac{\nu}{1-\beta}\right)^j}{j!(j+z)!} (j+z) \\
&= \frac{e^{-\left(\frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\right)}}{P(Z_n = z)} \sum_{j=0}^{\infty} \frac{\left(\frac{\mu}{1-\alpha}\right)^{j+z} \left(\frac{\nu}{1-\beta}\right)^j}{j!(j+z-1)!}.
\end{aligned}
$$

Considering the fact that the marginal distribution of the time series $\{Z_n\}$ is *Skellam*$(\mu/(1-\alpha), \nu/(1-\beta))$, i.e.

$$
P(Z_n = z) = e^{-\left(\frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\right)} \left(\frac{\mu(1-\beta)}{\nu(1-\alpha)}\right)^{\frac{z}{2}} I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right), \tag{5}
$$

where $I_z(x)$ is modified Bessel function of the first order, defined as

$$
I_z(x) = \sum_{j=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2j+z}}{j!\Gamma(j+z+1)}, \tag{6}
$$

it becomes easy to obtain

$$
E(X_n|Z_n = z) = \sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}} \frac{I_{z-1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}. \tag{7}
$$

Using the same procedure given above, for $z \leq 0$, it could be obtain in the similar way that

$$
E(X_n|Z_n = z) = \sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}} \frac{I_{-z+1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_{-z}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}. \tag{8}
$$

In order to present the results in a clearer form, (7) and (8) may be pooled in

$$
E(X_n|Z_n = z) = \sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}} \frac{I_{|z|-1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_{|z|}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}.
$$

Since $E(Z_n|Z_n = z) = E(X_n - Y_n|Z_n = z) = E(X_n|Z_n = z) - E(Y_n|Z_n = z) = z$, it is obvious that $E(Y_n|Z_n = z) = E(X_n|Z_n = z) - z$. This will be an excellent starting point for revealing formula for calculating the conditional expectation $E(Y_n|Z_n = z)$. Namely, using (7), for $z > 0$, it holds

$$
E(Y_n|Z_n = z) = E(X_n|Z_n = z) - z = \sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}} \frac{I_{z-1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)} - z.
$$

Using (6) and extracting the factor $\dfrac{\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}$, we have

$$
\begin{aligned}
E(Y_n|Z_n = z) &= \frac{\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}\left(\sum_{j=0}^{\infty}\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z-1}}{j!(j+z-1)!} - \sum_{j=0}^{\infty}z\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z-1}}{j!(j+z)!}\right) \\[2mm]
&= \frac{\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}\left(\sum_{j=0}^{\infty}\frac{(j+z)\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z-1}}{j!(j+z)!} - \sum_{j=0}^{\infty}z\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z-1}}{j!(j+z)!}\right) \\[2mm]
&= \frac{\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}\sum_{j=1}^{\infty}\frac{j\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z-1}}{j!(j+z)!}.
\end{aligned}
$$

Finally, by introducing the replacement $k = j - 1$ and using again (6), we obtain that

$$
E(Y_n|Z_n = z) = \frac{\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}\sum_{k=0}^{\infty}\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2k+z+1}}{k!(k+z+1)!} = \sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\frac{I_{z+1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}. \tag{9}
$$

The aforementioned procedure of obtaining the conditional expectation $E(Y_n|Z_n = z)$ can be carried out in the case of $z \le 0$ as well, which results in

$$
E(Y_n|Z_n = z) = \sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\frac{I_{-z-1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_{-z}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}. \tag{10}
$$

Also, (9) and (10) can be presented in shortened unified form,

$$
E(Y_n|Z_n = z) = \sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\frac{I_{|z+1|}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_{|z|}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}.
$$

Regarding the prediction of latent components, derivation takes place in a similar way as it was done with their extraction. Namely, starting from (1), for $k = 1$ and $r = 1$, we can get the expression for the mean of future values of latent component X_{n+1}, conditioned by Z_n.

Let it be $z > 0$. Using (1) and (4), we have that

$$
\begin{aligned}
E(X_{n+1}|Z_n = z) &= \frac{1}{P(Z_n = z)}\sum_{j=0}^{\infty}E(X_{n+1}|X_n = j+z, Y_n = j)P(X_n = j+z)P(Y_n = j) \\[2mm]
&= \frac{e^{-\left(\frac{\mu}{1-\alpha}+\frac{\nu}{1-\beta}\right)}}{P(Z_n = z)}\sum_{j=0}^{\infty}\frac{\left(\frac{\mu}{1-\alpha}\right)^{j+z}\left(\frac{\nu}{1-\beta}\right)^{j}}{j!(j+z)!}E(X_{n+1}|X_n = j+z, Y_n = j).
\end{aligned}
$$

Due to the properties of the binomial thinning operator and the fact that $\{X_n\}$ is the Poisson $INAR(1)$ time series given by (3), whose innovation process $\{\varepsilon_n\}$ has $\mathcal{P}(\mu)$ distribution, it becomes easy to obtain that

$E(X_{n+1}|X_n = j + z, Y_n = j) = E(X_{n+1}|X_n = j + z) = \alpha(j + z) + \mu$. Then, extracting the factor $\frac{e^{-\left(\frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\right)}}{P(Z_n=z)}\left(\sqrt{\frac{\frac{\mu}{1-\alpha}}{\frac{\nu}{1-\beta}}}\right)^z$, we have

$$E(X_{n+1}|Z_n = z) = \frac{e^{-\left(\frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\right)}}{P(Z_n = z)} \sum_{j=0}^{\infty} \frac{\left(\frac{\mu}{1-\alpha}\right)^{j+z}\left(\frac{\nu}{1-\beta}\right)^j}{j!(j+z)!}(\alpha(j+z) + \mu)$$

$$= \frac{e^{-\left(\frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\right)}}{P(Z_n = z)}\left(\sqrt{\frac{\frac{\mu}{1-\alpha}}{\frac{\nu}{1-\beta}}}\right)^z\left[\alpha\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\sum_{j=0}^{\infty}\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z-1}}{j!(j+z-1)!} + \mu\sum_{j=0}^{\infty}\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z}}{j!(j+z)!}\right].$$

Now, (6) gives us that

$$\alpha\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\sum_{j=0}^{\infty}\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z-1}}{j!(j+z-1)!} + \mu\sum_{j=0}^{\infty}\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z}}{j!(j+z)!} =$$

$$\alpha\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}I_{z-1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right) + \mu I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right),$$

which, with the help of (5), enables us to obtain

$$E(X_{n+1}|Z_n = z) = \alpha\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\frac{I_{z-1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)} + \mu. \tag{11}$$

In the similar way, for $z \le 0$, we have that

$$E(X_{n+1}|Z_n = z) = \alpha\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\frac{I_{-z+1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_{-z}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)} + \mu. \tag{12}$$

Finally, we will pool (11) and (12) in

$$E(X_{n+1}|Z_n = z) = \alpha\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\frac{I_{|z-1|}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_{|z|}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)} + \mu.$$

For the other latent component Y, we have the following. First of all, let z be greater then 0. Starting from (2), it holds

$$E(Y_{n+1}|Z_n = z) = \frac{1}{P(Z_n = z)}\sum_{j=0}^{\infty}E(Y_{n+1}|X_n = j + z, Y_n = j)P(X_n = j + z)P(Y_n = j).$$

Due to (4) and the fact that $\{Y_n\}$ is the Poisson $INAR(1)$ time series given with (3), whose innovation process $\{\eta_n\}$ has $\mathcal{P}(\nu)$ distribution, it becomes easy to obtain that

$$E(Y_{n+1}|Z_n = z) = \frac{e^{-\left(\frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\right)}}{P(Z_n = z)}\sum_{j=0}^{\infty}\frac{\left(\frac{\mu}{1-\alpha}\right)^{j+z}\left(\frac{\nu}{1-\beta}\right)^j}{j!(j+z)!}E(Y_{n+1}|X_n = j + z, Y_n = j)$$

$$= \frac{e^{-\left(\frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\right)}}{P(Z_n = z)}\sum_{j=0}^{\infty}\frac{\left(\frac{\mu}{1-\alpha}\right)^{j+z}\left(\frac{\nu}{1-\beta}\right)^j}{j!(j+z)!}(\beta j + \nu).$$

Extracting the factor $\frac{e^{-\left(\frac{\mu}{1-\alpha}+\frac{\nu}{1-\beta}\right)}}{P(Z_n=z)}\left(\sqrt{\frac{\frac{\mu}{1-\alpha}}{\frac{\nu}{1-\beta}}}\right)^z$ and using (6), will give

$$
\begin{aligned}
E(Y_{n+1}|Z_n = z) &= \frac{e^{-\left(\frac{\mu}{1-\alpha}+\frac{\nu}{1-\beta}\right)}}{P(Z_n = z)}\left(\sqrt{\frac{\frac{\mu}{1-\alpha}}{\frac{\nu}{1-\beta}}}\right)^z\left[\beta\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\sum_{j=0}^{\infty}\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z+1}}{j!(j+z+1)!}\right.\\
&\quad + \left.\nu\sum_{j=0}^{\infty}\frac{\left(\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)^{2j+z}}{j!(j+z)!}\right]\\
&= \frac{e^{-\left(\frac{\mu}{1-\alpha}+\frac{\nu}{1-\beta}\right)}}{P(Z_n = z)}\left(\sqrt{\frac{\frac{\mu}{1-\alpha}}{\frac{\nu}{1-\beta}}}\right)^z\beta\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}I_{z+1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)\\
&\quad + \frac{e^{-\left(\frac{\mu}{1-\alpha}+\frac{\nu}{1-\beta}\right)}}{P(Z_n = z)}\left(\sqrt{\frac{\frac{\mu}{1-\alpha}}{\frac{\nu}{1-\beta}}}\right)^z\nu I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right),
\end{aligned}
$$

which finally, with the help of (5), provides that

$$
E(Y_{n+1}|Z_n = z) = \beta\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\frac{I_{z+1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_z\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)} + \nu. \tag{13}
$$

Using the previously given procedure, for $z \le 0$, it could be shown in the similar way that

$$
E(Y_{n+1}|Z_n = z) = \beta\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\frac{I_{-z-1}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_{-z}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)} + \nu. \tag{14}
$$

In order to present the results in a clearer form, (13) and (14) can be pooled in

$$
E(Y_{n+1}|Z_n = z) = \beta\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\frac{I_{|z+1|}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)}{I_{|z|}\left(2\sqrt{\frac{\mu}{1-\alpha}\frac{\nu}{1-\beta}}\right)} + \nu.
$$

3. Parameter estimation

In order to estimate unknown parameters of the model, we used the Yule-Walker method. As an initial step, we took the mean, the variance and autocovariances of orders one and two,

$$
\begin{aligned}
E(Z_t) &= \frac{\mu}{1-\alpha} - \frac{\nu}{1-\beta}\\
Var(Z_t) &= \frac{\mu}{1-\alpha} + \frac{\nu}{1-\beta}\\
\gamma_Z(1) &= \alpha\frac{\mu}{1-\alpha} + \beta\frac{\nu}{1-\beta}\\
\gamma_Z(2) &= \alpha^2\frac{\mu}{1-\alpha} + \beta^2\frac{\nu}{1-\beta}.
\end{aligned}
$$

By equalizing the true moments with the corresponding sample moments and by solving the system thus obtained, we get the following results:

$$\hat{\mu} = \frac{\left((\bar{S}_Z)^2 - \hat{\gamma}_Z(1)\right)\left((\bar{S}_Z)^2 + \bar{Z}\right) - \sqrt{\left(\hat{\gamma}_Z(2)(\bar{S}_Z)^2 - \hat{\gamma}_Z^2(1)\right)\left(\left((\bar{S}_Z)^2\right)^2 - (\bar{Z})^2\right)}}{2(\bar{S}_Z)^2}$$

$$\hat{\nu} = \frac{\left((\bar{S}_Z)^2 - \hat{\gamma}_Z(1)\right)\left((\bar{S}_Z)^2 - \bar{Z}\right) + \sqrt{\left(\hat{\gamma}_Z(2)(\bar{S}_Z)^2 - \hat{\gamma}_Z^2(1)\right)\left(\left((\bar{S}_Z)^2\right)^2 - (\bar{Z})^2\right)}}{2(\bar{S}_Z)^2}$$

$$\hat{\alpha} = \frac{\hat{\gamma}_Z(1)\left((\bar{S}_Z)^2 + \bar{Z}\right) + \sqrt{\left(\hat{\gamma}_Z(2)(\bar{S}_Z)^2 - \hat{\gamma}_Z^2(1)\right)\left(\left((\bar{S}_Z)^2\right)^2 - (\bar{Z})^2\right)}}{\left((\bar{S}_Z)^2\right)^2 + (\bar{S}_Z)^2\bar{Z}}$$

$$\hat{\beta} = \frac{\hat{\gamma}_Z(1)\left((\bar{S}_Z)^2 - \bar{Z}\right) - \sqrt{\left(\hat{\gamma}_Z(2)(\bar{S}_Z)^2 - \hat{\gamma}_Z^2(1)\right)\left(\left((\bar{S}_Z)^2\right)^2 - (\bar{Z})^2\right)}}{\left((\bar{S}_Z)^2\right)^2 - (\bar{S}_Z)^2\bar{Z}}.$$

After dividing numerators and denominators of the aforementioned results with the sample variance $(\bar{S}_Z)^2$, we obtain following estimators:

$$\hat{\mu} = \frac{(1 - \hat{\rho}_Z(1))\left(1 + \frac{\bar{Z}}{(\bar{S}_Z)^2}\right) - \sqrt{\left(\hat{\rho}_Z(2) - \hat{\rho}_Z^2(1)\right)\left(1 - \left(\frac{\bar{Z}}{(\bar{S}_Z)^2}\right)^2\right)}}{2}$$

$$\hat{\nu} = \frac{(1 - \hat{\rho}_Z(1))\left(1 - \frac{\bar{Z}}{(\bar{S}_Z)^2}\right) - \sqrt{\left(\hat{\rho}_Z(2) - \hat{\rho}_Z^2(1)\right)\left(1 - \left(\frac{\bar{Z}}{(\bar{S}_Z)^2}\right)^2\right)}}{2}$$

$$\hat{\alpha} = \frac{\hat{\rho}_Z(1)\left(1 + \frac{\bar{Z}}{(\bar{S}_Z)^2}\right) + \sqrt{\left(\hat{\rho}_Z(2) - \hat{\rho}_Z^2(1)\right)\left(1 - \left(\frac{\bar{Z}}{(\bar{S}_Z)^2}\right)^2\right)}}{1 + \frac{\bar{Z}}{(\bar{S}_Z)^2}}$$

$$\hat{\beta} = \frac{\hat{\rho}_Z(1)\left(1 - \frac{\bar{Z}}{(\bar{S}_Z)^2}\right) + \sqrt{\left(\hat{\rho}_Z(2) - \hat{\rho}_Z^2(1)\right)\left(1 - \left(\frac{\bar{Z}}{(\bar{S}_Z)^2}\right)^2\right)}}{1 - \frac{\bar{Z}}{(\bar{S}_Z)^2}}.$$

In this paper we didn't analyze properties of the estimators. The asymptotical behavior of the obtained estimators is going to be in focus of authors' next research.

Finally, we can formulate statistics that can be used in order to extract or predict the latent components of a time series with skewed *Skellam* marginal distribution. More precisely, statistics for extracting latent components are given as

$$\hat{X}_n = \sqrt{\frac{\hat{\mu}}{1 - \hat{\alpha}}\frac{\hat{\nu}}{1 - \hat{\beta}}}\frac{I_{|Z_n - 1|}\left(2\sqrt{\frac{\hat{\mu}}{1 - \hat{\alpha}}\frac{\hat{\nu}}{1 - \hat{\beta}}}\right)}{I_{|Z_n|}\left(2\sqrt{\frac{\hat{\mu}}{1 - \hat{\alpha}}\frac{\hat{\nu}}{1 - \hat{\beta}}}\right)}, \quad \hat{Y}_n = \sqrt{\frac{\hat{\mu}}{1 - \hat{\alpha}}\frac{\hat{\nu}}{1 - \hat{\beta}}}\frac{I_{|Z_n + 1|}\left(2\sqrt{\frac{\hat{\mu}}{1 - \hat{\alpha}}\frac{\hat{\nu}}{1 - \hat{\beta}}}\right)}{I_{|Z_n|}\left(2\sqrt{\frac{\hat{\mu}}{1 - \hat{\alpha}}\frac{\hat{\nu}}{1 - \hat{\beta}}}\right)},$$

whereas statistics for predicting latent componenets are of the form

$$\hat{X}_{n+1} = \hat{\alpha} \sqrt{\frac{\hat{\mu}}{1-\hat{\alpha}} \frac{\hat{v}}{1-\hat{\beta}}} \frac{I_{|Z_n-1|}\left(2\sqrt{\frac{\hat{\mu}}{1-\hat{\alpha}} \frac{\hat{v}}{1-\hat{\beta}}}\right)}{I_{|Z_n|}\left(2\sqrt{\frac{\hat{\mu}}{1-\hat{\alpha}} \frac{\hat{v}}{1-\hat{\beta}}}\right)} + \hat{\mu}, \quad \hat{Y}_{n+1} = \hat{\beta} \sqrt{\frac{\hat{\mu}}{1-\hat{\alpha}} \frac{\hat{v}}{1-\hat{\beta}}} \frac{I_{|Z_n+1|}\left(2\sqrt{\frac{\hat{\mu}}{1-\hat{\alpha}} \frac{\hat{v}}{1-\hat{\beta}}}\right)}{I_{|Z_n|}\left(2\sqrt{\frac{\hat{\mu}}{1-\hat{\alpha}} \frac{\hat{v}}{1-\hat{\beta}}}\right)} + \hat{v}.$$

4. Simulations

In order to check performances of suggested statistics we simulated four skewed *TINAR*(1) time series by creating appropriate pairs of independent Poisson time series and calculating their differences. We simulated samples of size 5000. We decided to simulate skewed *TINAR*(1) time series with different combinations of parameters, small numbers, large numbers, close and distant numbers. For the first case we used $\mu = 0.6, v = 0.8, \alpha = 0.2, \beta = 0.7$. These are relatively small and close values for parameters μ and v. For the second case we used relatively different values $\mu = 2, v = 0.5, \alpha = 0.6, \beta = 0.3$. Parameters for the third case were $\mu = 4, v = 8, \alpha = 0.2, \beta = 0.4$, which are relatively large and different values for parameters μ and v. We decided for an almost symmetric model for the fourth case, i.e. the same values for parameters μ and v, and very close values for parameters α and β, $\mu = v = 3, \alpha = 0.6, \beta = 0.5$.

First of all, we estimated parameters for each of the four cases. In Table 1 parameter estimates are derived for subsamples of sizes 200, 500, 1000 and 5000 and corresponding standard errors are calculated. With the increase of sample size, all estimates are convergent with the standard errors decreasing towards 0. Upon obtained parameter values in case of $n = 5000$ we extracted latent components. According to *RMSE* values given in Table 2, regarding true mean values of the original Poisson time series, we can conclude that this estimation is quite good. In Figure 1, we show parts of original Poisson time series trajectories and the proposed estimation. We can see pretty good fitting capacity of extracted latent components.

Table 1: True and estimated values (with standard errors in parentheses) of the parameters of simulated skewed *TINAR*(1) time series

n	$\hat{\mu}$	$\hat{v}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\mu}$	$\hat{v}$	$\hat{\alpha}$	$\hat{\beta}$
True values	$\mu = 0.6$	$v = 0.8$	$\alpha = 0.2$	$\beta = 0.7$	$\mu = 4$	$v = 8$	$\alpha = 0.2$	$\beta = 0.4$
200	0.543	2.865	0.357	0.812	4.867	6.687	0.057	0.499
	(0.221)	(0.498)	(0.293)	(0.236)	(1.519)	(1.990)	(0.284)	(0.218)
500	0.567	2.943	0.313	0.796	4.689	6.992	0.069	0.478
	(0.146)	(0.350)	(0.205)	(0.0164)	(1.052)	(1.472)	(0.252)	(0.161)
1000	0.570	3.003	0.300	0.756	4.686	7.007	0.071	0.469
	(0.122)	(0.238)	(0.166)	(0.011)	(0.920)	(1.132)	(0.209)	(0.123)
5000	0.595	0.757	0.271	0.742	4.499	7.325	0.076	0.448
	(0.048)	(0.117)	(0.104)	(0.063)	(0.619)	(0.837)	(0.160)	(0.075)
True values	$\mu = 2$	$v = 0.5$	$\alpha = 0.6$	$\beta = 0.3$	$\mu = 3$	$v = 3$	$\alpha = 0.6$	$\beta = 0.5$
200	2.252	0.690	0.701	0.209	2.867	3.901	0.752	0.382
	(0.472)	(0.386)	(0.272)	(0.180)	(1.806)	(1.260)	(0.293)	(0.227)
500	1.799	0.652	0.669	0.251	2.339	3.688	0.707	0.401
	(0.303)	(0.288)	(0.180)	(0.109)	(1.025)	(0.906)	(0.211)	(0.182)
1000	1.806	0.642	0.661	0.257	2.386	3.679	0.704	0.419
	(0.290)	(0.259)	(0.153)	(0.077)	(0.874)	(0.836)	(0.182)	(0.135)
5000	1.866	0.597	0.639	0.277	2.531	3.405	0.681	0.432
	(0.143)	(0.107)	(0.067)	(0.044)	(0.591)	(0.483)	(0.115)	(0.095)

Table 2: Values of the root mean square errors of extraction and prediction for given true parameter values and estimated parameter values (with standard errors in parentheses) in case of $n = 5000$

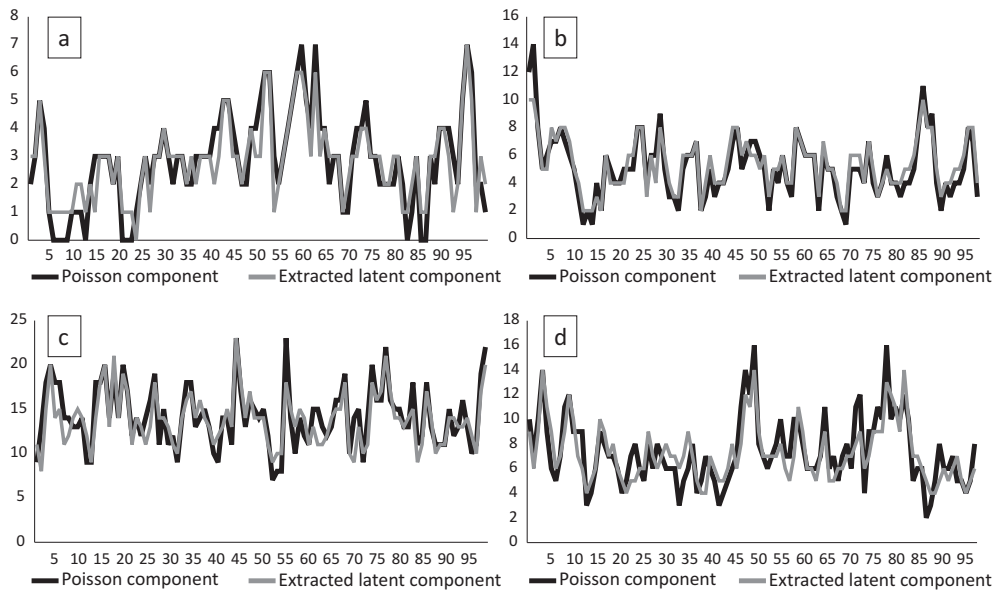| True values				Estimated values					
μ	ν	α	β	$\hat{\mu}$	$\hat{\nu}$	$\hat{\alpha}$	$\hat{\beta}$	*RMSE*	*RMSE*(1)
0.6	0.8	0.2	0.7	0.595	0.757	0.271	0.742	0.775	1.030
				(0.048)	(0.117)	(0.104)	(0.063)		
2	0.5	0.6	0.3	1.866	0.597	0.639	0.277	0.860	1.875
				(0.143)	(0.107)	(0.068)	(0.044)		
4	8	0.2	0.4	4.499	7.325	0.076	0.448	1.871	3.404
				(0.619)	(0.837)	(0.160)	(0.075)		
3	3	0.6	0.5	2.531	3.405	0.681	0.432	1.800	3.381
				(0.591)	(0.483)	(0.115)	(0.095)		

Figure 1: Parts of the trajectories of simulated Poisson components and their estimations extracted from the skewed *TINAR*(1) time series; (a) $\mu = 0.6, \nu = 0.8, \alpha = 0.2, \beta = 0.7$; (b) $\mu = 2, \nu = 0.5, \alpha = 0.6, \beta = 0.3$; (c) $\mu = 4, \nu = 8, \alpha = 0.2, \beta = 0.4$; (d) $\mu = 3, \nu = 3, \alpha = 0.6, \beta = 0.5$
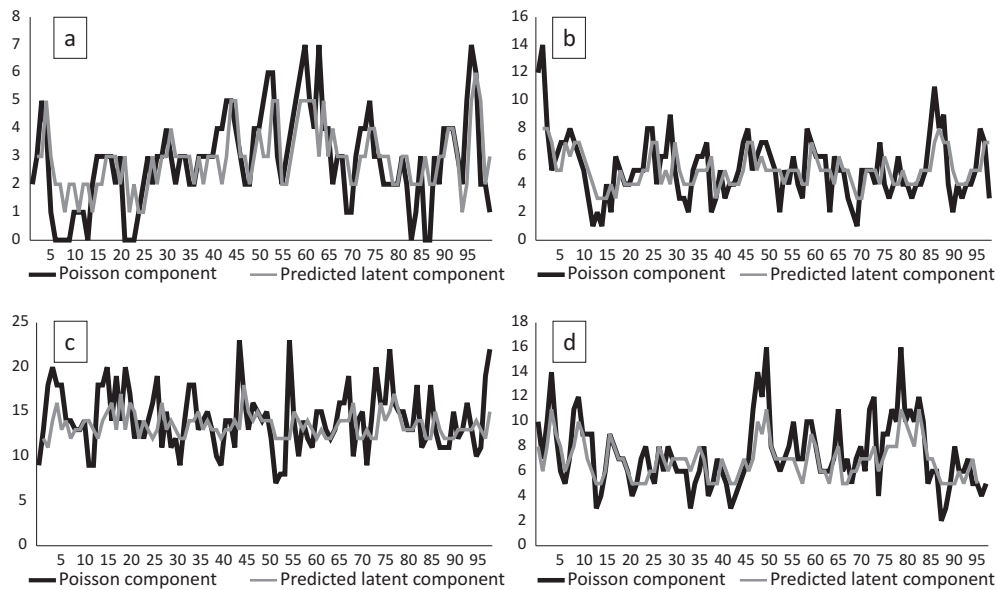
Figure 2: Parts of the trajectories of simulated Poisson components and their predictions of lag-one obtained from the skewed *TINAR*(1) time series; (a) $\mu = 0.6, \nu = 0.8, \alpha = 0.2, \beta = 0.7$; (b) $\mu = 2, \nu = 0.5, \alpha = 0.6, \beta = 0.3$; (c) $\mu = 4, \nu = 8, \alpha = 0.2, \beta = 0.4$; (d) $\mu = 3, \nu = 3, \alpha = 0.6, \beta = 0.5$

As expected, results of lag-one prediction are not as good as results obtained for extraction of latent components. The prediction *RMSE* (denoted as *RMSE*(1)), given in Table 2, is quite larger that the extraction one, but again, regarding the mean value of the Poisson components, is still acceptable. In Figure 2 there are parts of original Poisson time series trajectories and the proposed prediction of lag-one. In comparison with extracted components, we can notice a bit reduced flexibility and fitting power.

5. Real life application

Regarding the application to real-life data, the goal is to check real power of our statistics in extracting or predicting a latent component from an integer valued series, i.e. extracting or predicting the minuend and the diminutive knowing only the difference. The difficulty might be the fact that we don't know original values of latent components since they are latent. In order to bypass this difficulty, we should find a difference made as subtraction of two measurable components and try to reconstruct or predict those components.

Thus, we applied statistics on "artificial" differences which components are not "so latent" and as such eligible for comparison. We used goal differences of the Southampton FC and BV Borussia Dortmund in order to extract and predict number of goals scored by each of these teams or their opponents. Data were collected from website *www.worldfootball.net*. For Southampton FC, we took the data from the season 2010/11 to the season 2014/15, and for BV Borussia Dortmund the data from the season 2000/01 to the season 2018/19. Their goal difference distributions were tested if they significantly differ from *Skellam* distribution. χ^2−test revealed that in both cases we should accept the null hypothesis, which claims that the distributions of goal differences are *Skellam*. The results of the χ^2−test are shown in Table 3.

We also observed differences in the number of criminal acts reported to two police stations and tried to reconstruct and to predict number of criminal acts reported to each police station. The data were collected from the website *www.forecastingprinciples.com*. We focused on the difference in the number of robberies reported to two police stations, number 8800 and 9602, both in Rochester, New York, USA, during the period from the January 1991 to the December 2001 and the difference in the number of aggravated assaults reported to two police stations, number 700 and 1700, also, both in Rochester, New York, USA, during the

same period. According those data, we extracted and predicted particular number of reported criminal acts for each police station. Basic statistics of all four real-life data sequences are given in Table 4.

Table 3: Results of χ^2−test for testing differences between empirical and *Skellam* distributions

Variable	χ^2	*p*-value
Southampton FC goal difference	10.709	0.218
BV Borussia Dortmund goal difference	10.463	0.314
Difference in the number of robberies	8.182	0.146
Difference in the number of aggravated assaults	4.319	0.889
Test distribution: *Skellam*		

Table 4: Basic statistics of the real-life data

Basic statistics	Southampton FC goal difference	BV Borusia Dortmund goal difference	Difference in the number of robberies	Difference in the number of aggravated assaults
Sample mean	−0.362	1.002	1.098	1.073
Sample variance	1.742	1.953	1.497	1.406
Median	0	1	1	1
Maximum	4	7	9	8
Minimum	−9	−6	−2	−3
First quartile	−1	0	0	0
Third quartile	1	2	2	2

During the estimation of model parameters, the obtained value for parameter β was negative, so we replaced it by a small positive number. This could be associated to relatively small sample sizes.

Again, we used *RMSE* as a measure of fitting quality. As we can see from Table 5, regarding observed values, level of errors in extracting the components are really good. The quality of extraction is presented in Figure 3.

As it was the case with the simulations, the prediction has a bit higher values of *RMSE* (denoted as *RMSE*(1)), which can be also seen from Table 5. A lack of ability to reach extremes could be noticed from Figure 4, but predictions follow the form of original data.

Table 5: Parameter estimations and *RMSE*s of extraction and prediction of real-life data

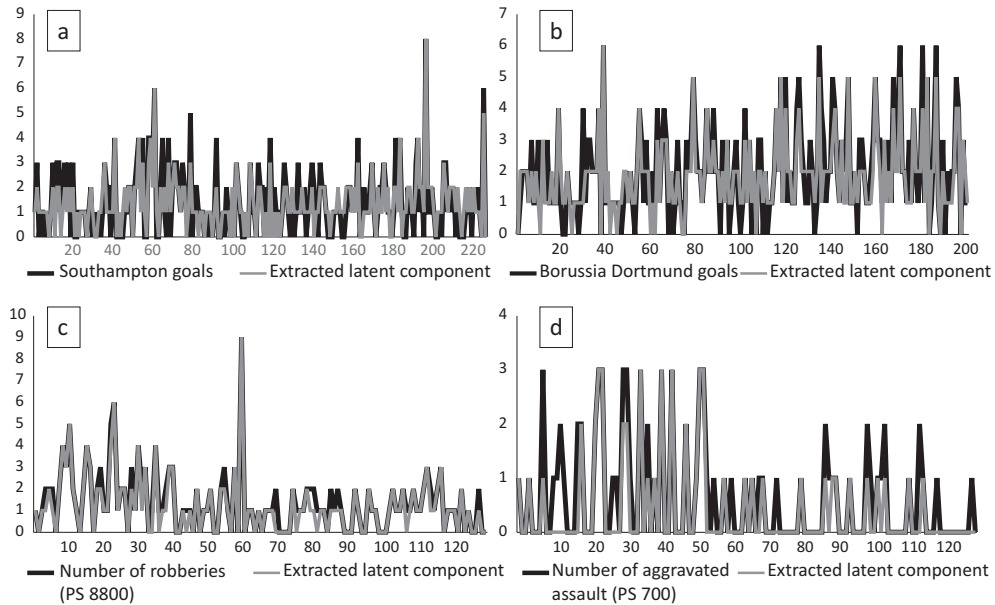| Variable	$\hat{\mu}$	$\hat{\nu}$	$\hat{\alpha}$	$\hat{\beta}$	*RMSE*	*RMSE*(1)
Southampton FC goal difference	1.359	2.200	0.275	0.001	0.799	1.422
BV Borussia Dortmund goas difference	1.545	1.876	0.253	0.001	0.823	1.558
Difference in the number of robberies	1.195	0.851	0.281	0.001	0.426	1.153
Difference in the number of aggravated assaults	0.631	1.222	0.295	0.001	0.537	1.069

Figure 3: Trajectories of Poisson components obtained from real data and their estimations obtained from the appropriate skewed *TINAR*(1) time series; Number of goals scored by Southampton (a) and Borussia Dortmund (b) and their estimation extracted from goal differences; Number of robberies reported to the Rochester police station No. 8800 (c) and number of aggravated assaults reported to the Rochester police station No. 700 (d) and their estimations extracted from differences in the number of crimes reported to two police stations
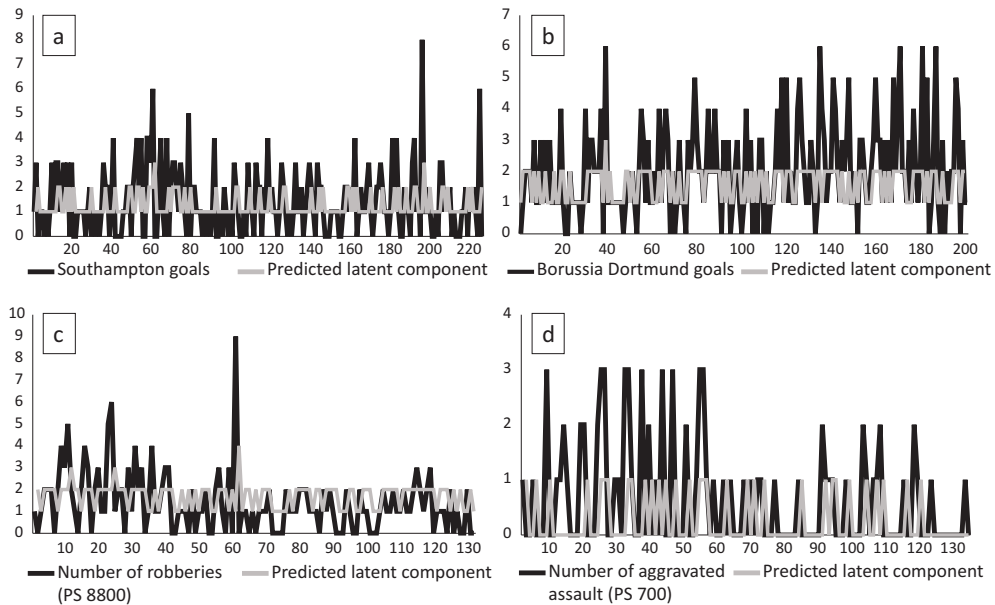

Figure 4: Trajectories of Poisson components obtained from real data and their lag-one predictions obtained from the appropriate skewed *TINAR*(1) time series; Number of goals scored by Southampton (a) and Borussia Dortmund (b) and their predictions extracted from goal differences; Number of robberies reported to the Rochester police station No. 8800 (c) and number of aggravated assaults reported to the Rochester police station No. 700 (d) and their predictions extracted from differences in the number of crimes reported to two police stations

6. Conclusion

The possibility of identifying and predicting the latent components of the integer-valued time series with skewed Skellam marginals has been introduced in this paper. The main feature of these components is that, by affecting in opposite directions, they give as a result the observed skewed *TINAR*(1) time series. Besides expressions for identification and prediction of latent components, estimation of the model parameters was carried out using the method of moments. Quality of extraction and lag-one prediction of latent components is examined on the simulated data series, each of length 5000. At the very end, application of this kind of modeling on appropriate real-life data is presented.

Of course, this does not exhaust all issues related to this subject matter. Further research might be focused on models in which latent components are not independent. Also, a step forward might be made on the issue of parameter estimation, for instance, by using the conditional least square method.

References

[1] Al-Osh, M.A., Aly, E.E.A.A. (1992) First order autoregressive time series with negative binomial and geometric marginals. Communication in Statistics - Theory and Methods, 21, 2483-2492.
[2] Al-Osh, M.A., Alzaid, A.A. (1987) First-order integer-valued autoregressive (INAR(1)) process. Journal of Time Series Analysis, 8, 261-275.
[3] Aly, E.E.A.A., Bouzar, N. (1994) On Some Integer-Valued Autoregressive Moving Average Models. Journal of Multivariate Analysis, 50, 132-151.
[4] Alzaid, A. A., Al-Osh, M. A. (1993) Some autoregressive moving average processes with generalized Poisson marginal distributions. Annals of the Institute of Statistical Mathematics, 45, 223-232.
[5] Bakouch, H.S., Ristić, M.M. (2010) Zero Truncated Poisson Integer Valued AR(1) Model. Metrika 72(2), 265-280.
[6] Barreto-Souza, W., Bourguignon, M. (2015) A skew INAR(1) process on $\mathbb{Z}$. Advances in Statistical Analysis, 99, 189-208.
[7] Bourguignon, M., Vasconcellos, K.L.P. (2016) A new skew integer valued time series process. Statistical Methodology, 31, 8-19.
[8] da Cunha, E.T., Vasconcellos, K.L.P., Bourguignon, M. (2018) A skew integer-valued time-series process with generalized Poisson difference marginal distribution. Journal of Statistical Theory and Practice, 12, 718-743.
[9] Djordjević, M.S. (2016) A combined SDLINAR(p) model and identification and prediction of its latent components. Facta universitatis, 31, 919-946.
[10] Djordjević, M.S. (2017) An extension on INAR models with discrete Laplace marginal distributions. Communication in Statistics - Theory and Methods, 46, 5896-5913.
[11] Freeland, R.K. (2010), True integer value time series. Advances in Statistical Analysis, 94, 217-229.
[12] Latour, A. (1998) Existence and stochastic structure of a non-negative integer-valued autoregressive process. Journal of Time Series Analysis, 19, 439-455.
[13] McKenzie, E. (1985) Some simple models for discrete variate time series. Water Resources Bulletin, 21, 645-650.
[14] McKenzie, E. (1986) Autoregressive moving-average processes with negative binomial and geometric distributions. Advances in Applied Probability, 18, 679-705.
[15] Nastić, A.S., Ristić, M.M., Djordjević, M.S. (2016) An INAR model with discrete Laplace marginal distributions. Brazilian Journal of Probability and Statistics, 30(1), 107-126.
[16] Ristić, M.M., Bakouch, H.S., Nastić, A.S. (2009) A new geometric first-order integer-valued autoregressive (NGINAR(1)) process. Journal of Statistical Planning and Inference, 139, 2218-2226.
[17] Zheng, H., Basawa, I.V., Datta, S. (2006) Inference for pth-order random coefficient integer-valued autoregressive processes. Journal of Time Series Analysis, 27, 411-440.
[18] Zheng, H., Basawa, I.V., Datta, S. (2007) First-order random coefficient integer-valued autoregressive processes. Journal of Statistical Planning and Inference, 137, 212-229.