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A Matrix Transform on Function Space with Related Topics

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Abstract. In this paper, we define a matrix transform of functionals via the bounded linear operators on function space. We then establish the existence of the matrix transform for exponential functionals. Finally, we obtain some fundamental formulas for the matrix transform involving the generalized first variations.

1. Introduction

For a positive real number T, let the triple $(C_0[0,T],\mathcal{M},m)$ be the one-parameter Wiener space and let $K\equiv K_0[0,T]$ be the space of complex-valued continuous functions defined on $[0,T]$ which vanish at $t=0$. We denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)dm(x).$$

For certain values of the parameters γ and β and for certain classes of functionals, the Fourier-Wiener transform, the modified Fourier-Wiener transform, the Fourier-Feynman transform and the Gauss transform are special cases of Lee's integral transform $\mathcal{F}_{\gamma,\beta}$ defined by the formula

$$\mathcal{F}_{\gamma,\beta}(F)(y)=\int_{C_0[0,T]} F(\gamma x+\beta y)dm(x), \qquad y\in K. \tag{1}$$

Lee showed that the solution of a differential equation (Cauchy problem) can be obtained by the integral transform [14]. Many mathematicians have attempted to define a more generalized form of the integral transform. One of them is the integral transform defined by the formula

$$\mathcal{F}^{h_1,h_2}_{\gamma,\beta}(F)(y)=\int_{C_0[0,T]} F(\gamma Z_{h_1}(x,\cdot)+\beta Z_{h_2}(y,\cdot))dm(x), \qquad y\in K,$$

for some appropriate functions h_1 and h_2 on $[0,T]$, where $Z_h(x,t)=\int_0^t h(s)dx$ is the Paley-Wiener-Zygmund (PWZ) stochastic integral. One can show that $\mathcal{F}^{1,1}_{\gamma,\beta}(F)(y)=\mathcal{F}_{\gamma,\beta}(F)(y)$, see [4]. The other example of generalized versions is the integral transform defined by the formula

$$\mathcal{G}_{S,R}(F)(y)=\int_{C_0[0,T]} F(Sx+Ry)dm(x), \qquad y\in K, \tag{2}$$

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where S and R are bounded linear operators on K, see [2, 6, 7, 15].

Recently, a number of studies have been published on the generalized version of Lee's integral transform. It is natural that arise the needness to us a more generalized version of them. We have felt the need for a new integral transform that encompasses all of the concepts of integral transforms to date, and we thought that a new type of integral transform was needed. We are forcing to define a more generalized version of integral transform. In order to do this, we define a matrix transform of functionals on function space, and then we establish some fundamental formulas for the matrix transform involving the generalized first variations. All results and formulas in previous papers [1–3, 8, 9, 12, 17] are corollaries of our results and formulas in this paper.

2. Definitions and preliminaries

We first list some definitions and preliminaries to understand this paper.

A subset B of $C_0[0,T]$ is said to be scale-invariant measurable provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.) [16].

Let

$$C_0' \equiv C_0'[0,T] = \Big\{w \in C_0[0,T] : w(t) = \int_0^t z_w(s)ds,\ z_w \in L_2[0,T]\Big\}.$$

Then it is a separable infinite dimensional Hilbert space with inner product

$$(w_1, w_2)_{C_0'} = \int_0^T w_1'(t)w_2'(t)dt.$$

Also, let K' be the complexification of C_0'. As is known, $(C_0'[0,T], C_0[0,T], m)$ is an example of abstract Wiener spaces [10, 11, 13]. Thus we have $\widetilde{C_0[0,T]} \subset \widetilde{C_0'[0,T]} \approx C_0'[0,T] \subset C_0[0,T]$, where $\tilde{S}$ is the topological dual space of a normed space S. Let X and Y be normed spaces and let $\mathcal{L}(X : Y)$ be the space of all bounded linear operators from X to Y. Hence the space $\mathcal{L} \equiv \mathcal{L}(K : K)$ is the set of all bounded linear operators on K into K.

For $v \in L_2[0,T]$ and $x \in C_0[0,T]$, let $\langle v, x\rangle$ denote the PWZ stochastic integral. One can show that for each $v \in L_2[0,T]$, $\langle v, x\rangle$ exists for a.e. $x \in C_0[0,T]$ and if $v \in L_2[0,T]$ is a function of bounded variation on $[0,T]$, $\langle v, x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_0[0,T]$. Also, $\langle v, x\rangle$ has the expected linearity property. Furthermore, $\langle v, x\rangle$ is a Gaussian random variable with mean 0 and variance $\|v\|_2^2$. For a more detailed study of the PWZ stochastic integral, see [1, 5, 7, 9, 14, 15, 18].

For $x \in C_0[0,T]$ and $w \in C_0'[0,T]$ with $w(t) = \int_0^t z(s)ds$ for $t \in [0,T]$, $(w, x)^\sim \equiv \langle D_t w, z\rangle = \langle z, x\rangle$ is a well-defined Gaussian random variable with mean 0 and variance $\|w\|_{C_0'}^2$. Then we have the following observations :

(i) For $x \in K$ and $w \in C_0'[0,T]$, let $(w, x)^\sim = (w, Re(x))^\sim + i(w, Im(x))^\sim$.
(ii) For $x \in C_0[0,T]$ and $w \in K'$, let $(w, x)^\sim = (Re(w), x)^\sim + i(Im(w), x)^\sim$.
(iii) In view of (ii) and (iii), for $x \in K$ and $w \in K'$, it follows that

$$\begin{aligned}(w, x)^\sim = &(Re(w), Re(x))^\sim + i(Im(w), Re(x))^\sim \\ &+ i(Re(w), Im(x))^\sim - (Im(w), Im(x))^\sim.\end{aligned}$$

In this case $(\cdot,\cdot)^\sim$ is a complex bilinear form on $\widetilde{K} \times K$.

Let $\mathcal{F}$ be an arbitrary set. For natural numbers k and n, let $\mathcal{M}_{k\times n}^{\mathcal{F}}$ be the set of all matrices whose components are in $\mathcal{F}$, namely,

$$\mathcal{M}_{k\times n}^{\mathcal{F}} = \{A = (a_{ij})_{k\times n} | a_{ij} \in \mathcal{F}, 1 \le i \le k, 1 \le j \le n\}.$$

Example 2.1. *When $k = 2, n = 1$ and $\mathcal{F} = K$,*

$$\mathcal{M}^K_{2\times1} = \left\{ X = \begin{pmatrix} x \\ y \end{pmatrix} : \ x, y \in K \right\}.$$

If $k = n = 2$ and $\mathcal{F} = \mathcal{L}$, then

$$\mathcal{M}^{\mathcal{L}}_{2\times2} = \left\{ A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} : \ T_1, T_2, T_3, T_4 \in \mathcal{L} \right\}.$$

3. Algebraic calculations on $\mathcal{M}^{\mathcal{L}}_{k\times n}$

In this section, we explain algebraic calculations on $\mathcal{M}^{\mathcal{F}}_{k\times n}$ to develop our theories and results. According to the various algebraic calculations for the matrices with constant coefficients, we shall introduce some algebraic calculations on $\mathcal{M}^{\mathcal{F}}_{k\times n}$. Before do this, we investigate an algebraic structure on $\mathcal{M}^{\mathcal{L}}_{k\times n}$.

Define a function $\|\cdot\|_0$ on $\mathcal{M}^{\mathcal{L}}_{k\times n}$ by the formula

$$\|A\|_0 = \max\{\|T_{ij}\|_{op} \ : \ 1 \le i \le k, 1 \le j \le n\},$$

where $A = (T_{ij}) \in \mathcal{M}^{\mathcal{L}}_{k\times n}$ and $\|T\|_{op}$ is the operator norm of $T \in \mathcal{L}$. One can see that $\|\cdot\|_0$ is a norm on $\mathcal{M}^{\mathcal{L}}_{k\times n}$and hence $(\mathcal{M}^{\mathcal{L}}_{k\times n}, \|\cdot\|_0)$ is a normed space.

Theorem 3.1. *The space $(\mathcal{M}^{\mathcal{L}}_{k\times n}, \|\cdot\|_0)$ is a Banach space.*

Proof. Let $(A_l) \equiv (A_l)_{l=1}^{\infty}$ be a Cauchy sequence in $\mathcal{M}^{\mathcal{L}}_{k\times n}$. It suffices to show that the sequence (A_l) converges. Let $A_l = (T^{(l)}_{ij})$. Since (A_l) is Cauchy, for every $\epsilon > 0$ there is a positive integer N so that for any $s, r > N$,

$$\|A_s - A_r\|_0 = \max\{\|T^{(s)}_{ij} - T^{(r)}_{ij}\|_{op} \ : \ 1 \le i \le k, 1 \le j \le n\} < \epsilon.$$

This shows that for each fixed i and j with $1 \le i \le k$ and $1 \le j \le n$, the sequence $(T^{(p)}_{ij})$ is a Cauchy sequence in $(\mathcal{L}, \|\cdot\|_{op})$ and so $(T^{(p)}_{ij})$ converges, say, $T^{(p)}_{ij} \to T^{(0)}_{ij}$ as $p \to \infty$. Using these $k \times n$ limits, we define $A = (T^{(0)}_{ij})$. Clearly, $A \in \mathcal{M}^{\mathcal{L}}_{k\times n}$ and hence we can conclude that for any $p > N$

$$\|A_p - A\|_0 < \epsilon.$$

Hence we have the desired results. □

From now on, we are going to explain algebraic calculations on $\mathcal{M}^{\mathcal{L}}_{k\times n}$ for $k = 2$ and $n = 1, 2$ via the properties of matrices for real or complex numbers as below :

(a) For $A \in \mathcal{M}^{\mathcal{L}}_{2\times2}$ and $X \in \mathcal{M}^K_{2\times1}$, let

$$AX = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} T_1 x + T_2 y \\ T_3 x + T_4 y \end{pmatrix}. \tag{3}$$

Then $AX \in \mathcal{M}^K_{2\times1}$.

(b) For $A_1, A_2 \in \mathcal{M}^{\mathcal{L}}_{2\times2}$, let

$$A_1A_2 = \begin{pmatrix} T_{11} & T_{12} \\ T_{13} & T_{14} \end{pmatrix} \begin{pmatrix} T_{21} & T_{22} \\ T_{23} & T_{24} \end{pmatrix} = \begin{pmatrix} T_{11}T_{21} + T_{12}T_{23} & T_{11}T_{22} + T_{12}T_{24} \\ T_{13}T_{21} + T_{14}T_{23} & T_{13}T_{22} + T_{14}T_{24} \end{pmatrix} \tag{4}$$

where T_1T_2 is the composition of T_1 and T_2. Then A_1A_2 is also in $\mathcal{M}^{\mathcal{L}}_{2\times2}$ and $A_1A_2 \ne A_2A_1$ generally.

(c) We next shall introduce a transform $\mathcal{H}$ on $\mathcal{M}^K_{2\times 1}$ into $\mathcal{M}^K_{1\times 1}$ defined by the formula

$$\mathcal{H}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x + y. \tag{5}$$

In fact, $\mathcal{M}^K_{1\times 1} = K$. Then $\mathcal{H}$ is a linear operator on $\mathcal{M}^K_{2\times 1}$ into $\mathcal{M}^K_{1\times 1}$.

(d) Let F be a complex-valued functional on K. According to three algebraic calculations (3), (4) and (5), one can observe that $(F \circ \mathcal{H})$ is defined on $\mathcal{M}^K_{2\times 1}$ into $\mathbb{C}$ and

$$(F \circ \mathcal{H})(X) = F(\mathcal{H}(X)) = F(x + y)$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$. Furthermore, we can see that for $A \in \mathcal{M}^{\mathcal{L}}_{2\times 2}$ and $X \in \mathcal{M}^K_{2\times 1}$,

$$(F \circ \mathcal{H})(AX) = F((T_1 + T_3)x + (T_2 + T_4)y)$$

where $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$.

(e) Using some technic and methods on the theory of matrices, we have the following observations.

(i) For $A \in \mathcal{M}^{\mathcal{L}}_{2\times 2}$ with $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$, let $A_{(1)}$ and $A_{(2)}$ be sub-matrices of A, namely, A can be written by

$$A = \left(A_{(1)} | A_{(2)}\right)$$

where $A_{(1)} = \begin{pmatrix} T_1 \\ T_3 \end{pmatrix}$ and $A_{(2)} = \begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$.

(ii) For $A \in \mathcal{M}^{\mathcal{L}}_{k\times n}$, let A^t be the transposed matrix of A. Then A^t is an element of $\mathcal{M}^{\mathcal{L}}_{n\times k}$.

(iii) For $A = (T_{ij}) \in \mathcal{M}^{\mathcal{L}}_{k\times n}$, let $A^* = (T^*_{ij})$ be the matrix of adjoint operators, where T^* is the adjoint operator of T. For example, if $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$, then $A^* = \begin{pmatrix} T^*_1 & T^*_2 \\ T^*_3 & T^*_4 \end{pmatrix}$.

(iv) In view of (i), (ii) and (iii), one can see that for all $A \in \mathcal{M}^{\mathcal{L}}_{k\times n}$, $(A^*)^t = (A^t)^*$. Furthermore, for $A \in \mathcal{M}^{\mathcal{L}}_{2\times 2}$ with $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$, we have

$$A_{(1)}(A^*_{(1)})^t = \begin{pmatrix} T_1 \\ T_3 \end{pmatrix} \begin{pmatrix} T^*_1 & T^*_3 \end{pmatrix} = \begin{pmatrix} T_1 T^*_1 & T_1 T^*_3 \\ T_3 T^*_1 & T_3 T^*_3 \end{pmatrix}$$

and it is an element of $\mathcal{M}^{\mathcal{L}}_{2\times 2}$.

4. Matrix transform with some fundamental formulas

In this section, we define a matrix transform of functionals on K, and then establish some fundamental formulas for the matrix transform.

We start this section by giving the class of functionals on function space used in this paper. For each $w \in \widetilde{K}$, let Φ_w be a functional on K of the form

$$\Phi_w(x) = \exp\{(w, x)^\sim\}. \tag{6}$$

The functionals of the form (6) are called the exponential type functionals. We note that $\tilde{K} \subset \tilde{K}' \approx K' \subset K$ and so for each $T \in \mathcal{L}$ and $w \in \widetilde{K}$, we have

$$\exp\{(w, Tx)^\sim\} = \exp\{(T^* w, x)^\sim\}.$$

We next state the following useful formula for Wiener integrals; namely that for $w \in \widetilde{K}$ and $x \in C_0[0,T]$,

$$\int_{C_0[0,T]} \exp\{(w,x)^{\sim}\}dm(x) = \exp\Big\{\frac{1}{2}(w,w)^{\sim}\Big\}. \tag{7}$$

If $w \in C_0'$, then it follows that

$$\int_{C_0[0,T]} \exp\{(w,x)^{\sim}\}dm(x) = \exp\Big\{\frac{1}{2}\|w\|^2_{C_0'}\Big\}.$$

We are ready to define a matrix transform via the bounded linear operators on function space.

Definition 4.1. *Let F be a functional on K and let $\mathcal{H}$ be as in equation* (5). *For a given $A \in \mathcal{M}^{\mathcal{L}}_{2\times 2}$, the $\mathcal{H}$-matrix transform $\mathbb{T}^{\mathcal{H}}_A(F)$ of F is defined by the formula*

$$\mathbb{T}^{\mathcal{H}}_A(F)(y) = \int_{C_0[0,T]} (F\circ\mathcal{H})(AX)dm(x), \qquad y \in K \tag{8}$$

if it exists, where $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Remark 4.2. *When $A = \begin{pmatrix} T_1 & 0 \\ 0 & T_4 \end{pmatrix}$, we have*

$$\mathbb{T}^{\mathcal{H}}_A(F)(y) = \mathcal{G}_{T_1,T_4}(F)(y)$$

for all $y \in K$ if they exist, where $\mathcal{G}_{T_1,T_4}$ is the generalized integral transform defined by equation (2). *In particular, for nonzero complex numbers γ and β, if $A = \begin{pmatrix} \gamma & 0 \\ 0 & \beta \end{pmatrix}$, then*

$$\mathbb{T}^{\mathcal{H}}_A(F)(y) = \mathcal{F}_{\gamma,\beta}(F)(y)$$

for all $y \in K$ if they exist, where $\mathcal{F}_{\gamma,\beta}$ is the generalized integral transform defined by equation (1).

In order to establish various fundamental formulas and results, we need the following Lemma 4.3.

Lemma 4.3. *Let $A \in \mathcal{M}^{\mathcal{L}}_{2\times 2}$ with its sub-matrices $A_{(1)}$ and $A_{(2)}$ and let $\mathcal{H}$ be ginve by equation* (5). *Then we have*

$$\mathcal{H}\Big(A_{(1)}(A^*_{(1)})^t\begin{pmatrix} w \\ w \end{pmatrix}\Big) = (T_1T_1^* + T_1T_3^* + T_3T_1^* + T_3T_3^*)w \tag{9}$$

and

$$(A^*_{(2)})^t\begin{pmatrix} w \\ w \end{pmatrix} = (T_2+T_4)^*w \tag{10}$$

Proof. Let $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ and hence $A_{(1)} = \begin{pmatrix} T_1 \\ T_3 \end{pmatrix}$, $A_{(2)} = \begin{pmatrix} T_2 \\ T_4 \end{pmatrix}$ and $A^* = \begin{pmatrix} T_1^* & T_2^* \\ T_3^* & T_4^* \end{pmatrix}$. One can see that

$$A_{(1)}(A^*_{(1)})^t = \begin{pmatrix} T_1T_1^* & T_1T_3^* \\ T_3T_1^* & T_3T_3^* \end{pmatrix}$$

and so

$$A_{(1)}(A^*_{(1)})^t\begin{pmatrix} w \\ w \end{pmatrix} = \begin{pmatrix} T_1T_1^* & T_1T_3^* \\ T_3T_1^* & T_3T_3^* \end{pmatrix}\begin{pmatrix} w \\ w \end{pmatrix} = \begin{pmatrix} T_1T_1^*w + T_1T_3^*w \\ T_3T_1^*w + T_3T_3^*w \end{pmatrix}.$$

This yields equation (9). Also, we have

$$(A^*_{(2)})^t \begin{pmatrix} w \\ w \end{pmatrix} = \begin{pmatrix} T^*_2 & T^*_4 \end{pmatrix} \begin{pmatrix} w \\ w \end{pmatrix} = T^*_2 w + T^*_4 w$$

and so equation (10) is obtained. $\square$

For notational convenience we adopt the following notation: for $A_1, A_2, \dots, A_n \in \mathcal{M}^{\mathcal{L}}_{2\times 2}$ and $w \in \widetilde{K}$, let

$$\mathbf{B}(A_1, \dots, A_n : w) \equiv \exp\left\{ \frac{1}{2} \sum_{j=1}^{n} \Big(\mathcal{H}(A_{(j1)}(A^*_{(j1)})^t W), w \Big)^{\sim} \right\}, \tag{11}$$

where $W = \begin{pmatrix} w \\ w \end{pmatrix}$ and $A_{(j1)}$ is the first part of sub-matrix of A_j for each $j = 1, 2, \dots, n$. Note that the symmetric property for $\mathbf{B}(\cdot; w)$ holds. That is to say,

$$\mathbf{B}(A_1, A_2, \dots, A_n : w) = \mathbf{B}(A_{\pi(1)}, A_{\pi(2)}, \dots, A_{\pi(n)} : w)$$

for any permutation π of $\{1, \dots, n\}$.

In Theorem 4.4, we establish the existence of $\mathcal{H}$-matrix transform $\mathbb{T}^{\mathcal{H}}_A(\Phi_w)$ of an exponential functional Φ_w.

Theorem 4.4. *Let Φ_w be an exponential type functional and let $A, A_{(1)}$ and $A_{(2)}$ be as in Lemma 4.3 above. Then the $\mathcal{H}$-matrix transform $\mathbb{T}^{\mathcal{H}}_A(\Phi_w)$ of Φ_w exists and is given by the formula*

$$\mathbb{T}^{\mathcal{H}}_A(\Phi_w)(y) = \mathbf{B}(A : w)\Phi_{(T_2+T_4)^* w}(y) \tag{12}$$

for $y \in K$. Furthermore, using the algebraic calculations, we have

$$\mathbb{T}^{\mathcal{H}}_A(\Phi_w)(y) = \mathbf{B}(A : w)\Phi_{(A^*_{(2)})^t W}(y) \tag{13}$$

for $y \in K$.

Proof. We start to establish (12). By using equations (7), (8) and (6). Then we have

$$\begin{aligned}
\mathbb{T}^{\mathcal{H}}_A(\Phi_w)(y) &= \int_{C_0[0,T]} (\Phi_w \circ \mathcal{H})(AX) dm(x) \\
&= \int_{C_0[0,T]} \Phi_w((T_1 + T_3)x + (T_2 + T_4)y) dm(x) \\
&= \int_{C_0[0,T]} \exp\{((T^*_1 + T^*_3)w, x)^{\sim} + ((T^*_2 + T^*_4)w, y)^{\sim}\} dm(x) \\
&= \Phi_{(T_2+T_4)^* w}(y) \\
&\times \exp\left\{ \frac{1}{2} \Big[(T_1 T^*_1 w, w)^{\sim} + (T_3 T^*_3 w, w)^{\sim} + (T_1 T^*_3 w, w)^{\sim} + (T_3 T^*_1 w, w)^{\sim} \Big] \right\}
\end{aligned}$$

for $y \in K$. Now using equation (9) in Lemma 4.3 and equation (11) above, we can establish equation (12) as desired. Also, using equation (10), equation (13) is also obtained. Hence we have the desired results. $\square$

In the second theorem, we give a formula for the commutativity of $\mathcal{H}$-matrix transform of exponential functionals.

Theorem 4.5. *Let Φ_w be an exponential type functional and let*

$$A_1 = \begin{pmatrix} T_{11} & T_{12} \\ T_{13} & T_{14} \end{pmatrix} \quad \textit{and} \quad A_2 = \begin{pmatrix} T_{21} & T_{22} \\ T_{23} & T_{24} \end{pmatrix}$$

be elements of $\mathcal{M}^{\mathcal{L}}_{2\times 2}$.Then the Fubini theorem for $\mathcal{H}$-matrix transform is given by the formula

$$\mathbb{T}^{\mathcal{H}}_{A_2}(\mathbb{T}^{\mathcal{H}}_{A_1}(\Phi_w))(y) = \mathbb{T}^{\mathcal{H}}_{A_1}(\mathbb{T}^{\mathcal{H}}_{A_2}(\Phi_w))(y) \tag{14}$$

for $y \in K$ if and only if the following conditions : $T_{12} + T_{14}$ and $T_{22} + T_{24}$ are commutative, and

$$\mathbf{B}(A_1 : w)\mathbf{B}(A_2 : (T_{12} + T_{14})^* w) = \mathbf{B}(A_2 : w)\mathbf{B}(A_1 : (T_{22} + T_{24})^* w) \tag{15}$$

hold. Furthermore, the composition formula is given by the formula

$$\mathbb{T}^{\mathcal{H}}_{A_2}(\mathbb{T}^{\mathcal{H}}_{A_1}(\Phi_w))(y) = \mathbb{T}^{\mathcal{H}}_{A_3}(\Phi_w)(y) \tag{16}$$

for $y \in K$, where $A_3 = \begin{pmatrix} T_{31} & T_{32} \\ T_{33} & T_{34} \end{pmatrix}$, if and only if the following conditions :

$$\mathbf{B}(A_1 : w)\mathbf{B}(A_2 : (T_{12} + T_{14})^* w) = \mathbf{B}(A_3 : w) \tag{17}$$

and

$$(T_{22} + T_{24})^*(T_{12} + T_{14})^* = (T_{32} + T_{34})^* \tag{18}$$

hold.

Proof. We first use equations (7), (8), (3) and (12) to establish the left-hand side of equation (14). Then we have

$$\begin{aligned}
&\mathbb{T}^{\mathcal{H}}_{A_2}(\mathbb{T}^{\mathcal{H}}_{A_1}(\Phi_w))(y) \\
&= \int_{C_0[0,T]} \mathbb{T}^{\mathcal{H}}_{A_1}(\Phi_w)((T_{21} + T_{23})x + (T_{22} + T_{24})y)dm(x) \\
&= \mathbf{B}(A_1 : w) \int_{C_0[0,T]} \Phi_{(T_{12}+T_{14})^* w}((T_{21} + T_{23})x + (T_{22} + T_{24})y)dm(x) \\
&= \mathbf{B}(A_1 : w)\mathbf{B}(A_2 : (T_{12} + T_{14})^* w) \exp\Big\{\Big((T_{22} + T_{24})^*(T_{12} + T_{14})^* w, y\Big)^{\sim}\Big\} \\
&= \mathbf{B}(A_1 : w)\mathbf{B}(A_2 : (T_{12} + T_{14})^* w)\Phi_{(T_{22}+T_{24})^*(T_{12}+T_{14})^* w}(y).
\end{aligned}$$

Also using equations (7), (8), (4) and (12) again, we obtain that

$$\mathbb{T}^{\mathcal{H}}_{A_1}(\mathbb{T}^{\mathcal{H}}_{A_2}(\Phi_w))(y) = \mathbf{B}(A_2 : w)\mathbf{B}(A_1 : (T_{22} + T_{24})^* w)\Phi_{(T_{12}+T_{14})^*(T_{22}+T_{24})^* w}(y).$$

Hence we can complete the proof of Theorem 4.5 as desired. □

In our next theorem, we establish the inverse transform for $\mathcal{H}$-matrix transform.

Theorem 4.6. *Let Φ_w be an exponential type functional and let*

$$A = \begin{pmatrix} T_{11} & T_{12} \\ T_{13} & T_{14} \end{pmatrix}$$

be an element of $\mathcal{M}^{\mathcal{L}}_{2\times 2}$. Then the equation

$$\mathbb{T}^{\mathcal{H}}_{A_0}(\mathbb{T}^{\mathcal{H}}_{A}(\Phi_w))(y) = \Phi_w(y) = \mathbb{T}^{\mathcal{H}}_{A}(\mathbb{T}^{\mathcal{H}}_{A_0}(\Phi_w))(y)$$

holds if and only if $T_{12} + T_{14}$ *and* $T_{02} + T_{04}$ *are commutative,*

$$\mathbf{B}(A_1 : w)\mathbf{B}(A_0 : (T_{12} + T_{14})^* w) = \mathbf{B}(A_0 : w)\mathbf{B}(A_1 : (T_{02} + T_{04})^* w),$$

$$\mathbf{B}(A_1 : w)\mathbf{B}(A_0 : (T_{12} + T_{14})^* w) = 1$$

and

$$(T_{02} + T_{04})^*(T_{12} + T_{14})^* = I$$

for $y \in K$, *where* $A_0 = \begin{pmatrix} T_{01} & T_{02} \\ T_{03} & T_{04} \end{pmatrix}$. *These tell that the inverse matrix transform exists and is given by the formula*

$$(\mathbb{T}_A^{\mathcal{H}})^{-1} = \mathbb{T}_{A_0}^{\mathcal{H}}.$$

Proof. The proof of Theorem 4.6 is established from equations (14) thru (18) in Theorem 4.5 above. □

5. More formulas involving the generalized first variations

In this section, we give some fundamental formulas for the $\mathcal{H}$-matrix transform involving the generalized first variation.

We state the definition of the generalized first variation of functionals on K, see [8, 9].

Definition 5.1. *Let* Ψ *be a functional on* K *and let* $S \in \mathcal{L}$. *Then the generalized first variation* $\delta_S(\Psi|u)$ *of* Ψ *is defined by the formula (if it exists)*

$$\delta_S\Psi(x|u) = \frac{\partial}{\partial k}\Psi(x + kSu)\Big|_{k=0}, \qquad x, u \in K. \tag{19}$$

Remark 5.2. *The generalized first variation* $\delta_S\Psi(x|u)$ *acts like a directional derivative in the direction of* Su. *If* $S = I$, *where* I *is the identity operator, then* $\delta_S\Psi(x|u) = \delta\Psi(x|u)$ *which was used in [4, 6]. We will explain the meaning of the generalized first variation* $\delta_S\Psi(x|u)$ *as follows: For an orthonormal set* $\{\phi_1, \cdots, \phi_n\}$ *in* C_0', *let* $S_1 : C_0' \to C_0'$ *be the linear operator defined by*

$$S_1 x(t) = \sum_{j=1}^{n} (\phi_j, x)_{C_0'} \phi_j(t).$$

Also, let $S_2 : C_0' \to C_0'$ *be the linear operator defined by*

$$S_2 w(t) = \int_0^t w(s) db(s).$$

Then

$$S_2^* w(t) = w(T)t - \int_0^t w(s)ds = \int_0^t [w(T) - w(s)]ds$$

and so the linear operator $A = S_2^* S_2$ *is given by*

$$Aw(t) = \int_0^T \min\{s, t\} w(s)ds.$$

And hence A *is a self-adjoint operator on* C_0' *and*

$$(w_1, Aw_2)_{C_0'} = (S_2 w_1, S_2 w_2)_{C_0'} = \int_0^T w_1(s) w_2(s) ds$$

for all $w_1, w_2 \in C_0'$. *Thus* A *is a positive definite operator. From two examples, we can tell that our generalized first variation is better to explain various circumvents.*

In our next theorem, we establish a relationship between the $\mathcal{H}$-matrix transform $\mathbb{T}_A^{\mathcal{H}}(\Phi_w)$ and the generalized first variation $\delta_S(\Phi_w)$ of exponential functionals.

Theorem 5.3. *Let Φ_w be an exponential type functional and let $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be an element of $\mathcal{M}_{2\times 2}^{\mathcal{L}}$. Let $S \in \mathcal{L}$ and let $u \in C_0'$. Then*

$$\mathbb{T}_A^{\mathcal{H}}(\delta_{(T_2+T_4)S}\Phi_w(\cdot|u))(y) = \delta_S(\mathbb{T}_A^{\mathcal{H}}(\Phi_w))(y|u) \tag{20}$$

for $y \in K$.

Proof. Using equations (8), (12) and (19), it follows that for all $y \in K$,

$$\begin{aligned} \mathbb{T}_A^{\mathcal{H}}(\delta_{(T_2+T_4)S}\Phi_w(\cdot|u))(y) &= \int_{C_0[0,T]} ((\delta_{(T_2+T_4)S}\Phi_w) \circ \mathcal{H})(AX|u)dm(x) \\ &= \int_{C_0[0,T]} \frac{\partial}{\partial k}(\Phi_w \circ \mathcal{H})(AX')\bigg|_{k=0} dm(x) \\ &= \frac{\partial}{\partial k}\int_{C_0[0,T]} \Phi_w(\mathcal{H}(AX) + k(T_2+T_4)Su)dm(x)\bigg|_{k=0} \\ &= (S^*(T_2+T_4)^*w, u)^\sim \mathbf{B}(A:w)\Phi_{(T_2+T_4)^*w}(y) \end{aligned} \tag{21}$$

where $X' = X + \begin{pmatrix} k(T_2+T_4)Su \\ 0 \end{pmatrix}$. Also, using equations (8), (12) and (19), it follows that for all $y \in K$,

$$\begin{aligned} &\delta_S(\mathbb{T}_A^{\mathcal{H}}(\Phi_w))(y|u) \\ &= \frac{\partial}{\partial k}\mathbb{T}_A^{\mathcal{H}}(\Phi_w)(y+kSu)\bigg|_{k=0} \\ &= \frac{\partial}{\partial k}\bigg(\mathbf{B}(A:w)\Phi_{(T_2+T_4)^*w}(y+kSu)\bigg)\bigg|_{k=0} \\ &= \frac{\partial}{\partial k}\bigg(\mathbf{B}(A:w)\exp\{((T_2+T_4)^*w, y)^\sim + k(S^*(T_2+T_4)^*w, u)^\sim\}\bigg)\bigg|_{k=0} \\ &= (S^*(T_2+T_4)^*w, u)^\sim \mathbf{B}(A:w)\Phi_{(T_2+T_4)^*w}(y). \end{aligned} \tag{22}$$

Comparing two equations (21) and (22), we complete the proof of Theorem 5.3 as desired. □

In the last theorem, we give a fundamental formula involving the generalized first variation.

Theorem 5.4. *Let Φ_w, A_1, A_2, A_3 be as in Theorem 4.5, and let S and u be as in Theorem 5.3. Then we have*

$$\mathbb{T}_{A_2}^{\mathcal{H}}(\mathbb{T}_{A_1}^{\mathcal{H}}(\delta_{(T_{12}+T_{14})(T_{22}+T_{24})S}\Phi_w(\cdot|u)))(y) = \mathbb{T}_{A_3}^{\mathcal{H}}(\delta_{(T_{32}+T_{34})S}\Phi_w(\cdot|u)))(y) \tag{23}$$

for $y \in K$.

Proof. First using equation (20) twice, it follows that for all $y \in K$,

$$\begin{aligned} &\mathbb{T}_{A_2}^{\mathcal{H}}(\mathbb{T}_{A_1}^{\mathcal{H}}(\delta_{(T_{12}+T_{14})(T_{22}+T_{24})S}\Phi_w(\cdot|u)))(y) \\ &= \mathbb{T}_{A_2}^{\mathcal{H}}(\delta_{(T_{22}+T_{24})S}\mathbb{T}_{A_1}^{\mathcal{H}}(\Phi_w)(\cdot|u))(y) \\ &= \delta_S\mathbb{T}_{A_2}^{\mathcal{H}}(\mathbb{T}_{A_1}^{\mathcal{H}}(\Phi_w))(y|u). \end{aligned}$$

We now use equation (13) to yield the equation

$$\mathbb{T}_{A_2}^{\mathcal{H}}(\mathbb{T}_{A_1}^{\mathcal{H}}(\delta_{(T_{12}+T_{14})(T_{22}+T_{24})S}\Phi_w(\cdot|u)))(y) = \delta_S\mathbb{T}_{A_3}^{\mathcal{H}}(\Phi_w)(y|u).$$

Using equation (20) again, we obtain

$$\mathbb{T}^{\mathcal{H}}_{A_3}(\delta_{(T_{32}+T_{34})S}(\Phi_w)(\cdot|u)))(y) = \delta_S \mathbb{T}^{\mathcal{H}}_{A_3}(\Phi_w)(y|u),$$

which completes the proof of Theorem 5.4 as desired. □

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