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Characterization of the Matrix Class $(\ell_\alpha, \ell_\beta)$, $0 < \alpha \leq \beta \leq 1$

P. N. Natarajan[a]

[a] *Old No. 2/3; New No. 3/3; Second Main Road; R.A. Puram; Chennai 600 028; INDIA*

Abstract. Throughout the present paper, entries of sequences, infinite series and infinite matrices are real or complex numbers. In this paper, we characterize the matrix class $(\ell_\alpha, \ell_\beta)$, $0 < \alpha \leq \beta \leq 1$.

1. Introduction and Preliminaries

Throughout the present paper, entries of sequences, infinite series and infinite matrices are real or complex numbers; α, β are real numbers satisfying $0 < \alpha \leq \beta \leq 1$.

We need the following sequence space in the sequel.

$$\ell_\alpha = \left\{ x = \{x_k\} / \sum_{k=0}^{\infty} |x_k|^\alpha < \infty \right\}, \alpha > 0.$$

If $A = (a_{nk})$, $n, k = 0, 1, 2, \ldots$ is an infinite matrix, we write

$$A \in (\ell_\alpha, \ell_\beta), \alpha, \beta > 0,$$

if

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,$$

is defined, $n = 0, 1, 2, \ldots$ and the sequence $A(x) = \{(Ax)_n\} \in \ell_\beta$, whenever $x = \{x_k\} \in \ell_\alpha$. $A(x)$ is called the A-transform of x.

We now present a short summary of the research done so far regarding the characterization of the matrix class $(\ell_\alpha, \ell_\beta)$. A complete characterization of the matrix class $(\ell_\alpha, \ell_\beta)$, $\alpha, \beta \geq 2$, does not seem to be available in the literature. The latest result in this direction [3] characterizes only non-negative matrices in $(\ell_\alpha, \ell_\beta)$, $\alpha \geq \beta > 1$. A known simple sufficient condition ([4], p. 174, Theorem 9) for $A = (a_{nk}) \in (\ell_\alpha, \ell_\alpha)$ is

$$A \in (\ell_\infty, \ell_\infty) \cap (\ell_1, \ell_1).$$

Sufficient conditions or necessary conditions for $A \in (\ell_\alpha, \ell_\beta)$ are available in the literature (for instance, see [7]). Necessary and sufficient conditions for $A \in (\ell_1, \ell_1)$ are due to Mears [5] (for alternative proofs, see Knopp and Lorentz [2], Fridy [1]). In [6], Natarajan characterized the matrix class $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$.

In the context of the above survey, the main result of the present paper is interesting.

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Email address: pinnangudinatarajan@gmail.com (P. N. Natarajan)

2. Main Result

In this section, we need the following lemma.

Lemma 2.1. *[([4], p. 22)]*

(i)

$$||a|^{\alpha} - |b|^{\alpha}| \leq |a+b|^{\alpha} \leq |a|^{\alpha} + |b|^{\alpha}, 0 < \alpha \leq 1; \tag{1}$$

(ii)

$$\sum_{k=0}^{\infty} |a_k + b_k|^{\alpha} \leq \sum_{k=0}^{\infty} |a_k|^{\alpha} + \sum_{k=0}^{\infty} |b_k|^{\alpha}, 0 < \alpha \leq 1. \tag{2}$$

We now take up the main result of the paper.

Theorem 2.2. *$A = (a_{nk}) \in (\ell_\alpha, \ell_\beta)$, $0 < \alpha \leq \beta \leq 1$, if and only if*

$$\sup_{k \geq 0} \sum_{n=0}^{\infty} |a_{nk}|^{\beta} < \infty. \tag{3}$$

Proof. Sufficiency. Let (3) hold. We first claim that $0 < \alpha \leq \beta \leq 1$ implies that $\ell_\alpha \subseteq \ell_\beta \subseteq \ell_1$. Let $x = \{x_k\} \in \ell_\alpha$, i.e., $\sum_{k=0}^{\infty} |x_k|^{\alpha} < \infty$. So $x_k \to 0$, $k \to \infty$. We can find a positive integer N such that

$$|x_k| < 1, k \geq N.$$

Since $\frac{\beta}{\alpha} \geq 1$,

$$|x_k|^{\frac{\beta}{\alpha}} \leq |x_k|,$$
$$i.e., |x_k|^{\beta} \leq |x_k|^{\alpha}, k \geq N.$$

Thus,

$$\sum_{k=N}^{\infty} |x_k|^{\beta} \leq \sum_{k=N}^{\infty} |x_k|^{\alpha} < \infty$$

and so $\sum_{k=0}^{\infty} |x_k|^{\beta} < \infty$, i.e., $x = \{x_k\} \in \ell_\beta$. Hence $\ell_\alpha \subseteq \ell_\beta$. Similarly, $\beta \leq 1$ implies that $\ell_\beta \subseteq \ell_1$. Consequently, $\ell_\alpha \subseteq \ell_\beta \subseteq \ell_1$. Now, let $x = \{x_k\} \in \ell_\alpha$. So $\{x_k\} \in \ell_1$, i.e., $\sum_{k=0}^{\infty} |x_k| < \infty$. Using (3), $\sup_{n,k} |a_{nk}| < \infty$. Hence

$$\sum_{k=0}^{\infty} |a_{nk} x_k| \leq \left(\sup_{n,k} |a_{nk}|\right)\left(\sum_{k=0}^{\infty} |x_k|\right) < \infty,$$

from which it follows that $\sum_{k=0}^{\infty} a_{nk} x_k$ converges, $n = 0, 1, 2, \ldots$.

So,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$$

is defined, $n = 0, 1, 2, \ldots$. Since $\ell_\alpha \subseteq \ell_\beta$, $\sum_{k=0}^{\infty} |x_k|^\beta < \infty$.
Now, using Lemma 2.1 and condition (3), we get

$$\begin{aligned}\sum_{n=0}^{\infty} |(Ax)_n|^\beta &= \sum_{n=0}^{\infty} \left|\sum_{k=0}^{\infty} a_{nk} x_k\right|^\beta \\ &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_{nk}|^\beta |x_k|^\beta \\ &= \sum_{k=0}^{\infty} |x_k|^\beta \sum_{n=0}^{\infty} |a_{nk}|^\beta \\ &\leq \left(\sup_{k\geq 0} \sum_{n=0}^{\infty} |a_{nk}|^\beta\right)\left(\sum_{k=0}^{\infty} |x_k|^\beta\right) \\ &< \infty.\end{aligned}$$

Hence $\{(Ax)_n\} \in \ell_\beta$, i.e., $A \in (\ell_\alpha, \ell_\beta)$.

Necessity. Let $A \in (\ell_\alpha, \ell_\beta)$. First, we note that

$$B_n = \sup_{k\geq 0} |a_{nk}|^\alpha < \infty, n = 0, 1, 2, \ldots. \tag{4}$$

Suppose not. Then, for some positive integer m,

$$B_m = \sup_{k\geq 0} |a_{mk}|^\alpha = \infty.$$

We can now choose a strictly increasing sequence $\{k(i)\}$ of positive integers such that

$$|a_{m,k(i)}|^\alpha > i^2, i = 1, 2, \ldots.$$

Define the sequence $x = \{x_k\}$ by

$$x_k = \begin{cases} \frac{1}{a_{m,k(i)}}, & \text{if } k = k(i); \\ 0, & \text{if } k \neq k(i), i = 1, 2, \ldots. \end{cases}$$

$x = \{x_k\} \in \ell_\alpha$, for,

$$\begin{aligned}\sum_{k=0}^{\infty} |x_k|^\alpha = \sum_{i=1}^{\infty} |x_{k(i)}|^\alpha &= \sum_{i=1}^{\infty} \frac{1}{|a_{m,k(i)}|^\alpha} \\ &< \sum_{i=1}^{\infty} \frac{1}{i^2} \\ &< \infty.\end{aligned}$$

On the other hand,

$$a_{m,k(i)} x_{k(i)} = 1 \not\to 0, i \to \infty,$$

which implies that

$$(Ax)_m = \sum_{k=0}^{\infty} a_{mk} x_k$$

is not defined, a contradiction, proving (4). For $k = 0, 1, 2, \ldots$, the sequence $x = \{x_k\} = \{0, 0, \ldots, 0, 1, 0, \ldots\}$, 1 occurring in the kth place, is in ℓ_α for which $(Ax)_n = a_{nk}$. $\{(Ax)_n\} = \{a_{nk}\}_{n=0}^{\infty} \in \ell_\beta$ implies that

$$\mu_k = \sum_{n=0}^{\infty} |a_{nk}|^\beta < \infty, k = 0, 1, 2, \ldots.$$

We, now, claim that $\{\mu_k\}$ is bounded. Suppose not, i.e., $\{\mu_k\}$ is unbounded. Choose a positive integer $k(1)$ such that

$$\mu_{k(1)} > 3.$$

We now choose a positive integer $n(1)$ such that

$$\sum_{n=n(1)+1}^{\infty} |a_{n,k(1)}|^\beta < 1,$$

so that

$$\mu_{k(1)} = \sum_{n=0}^{n(1)} |a_{n,k(1)}|^\beta + \sum_{n=n(1)+1}^{\infty} |a_{n,k(1)}|^\beta,$$

$$\begin{aligned} \text{i.e., } \sum_{n=0}^{n(1)} |a_{n,k(1)}|^\beta &= \mu_{k(1)} - \sum_{n=n(1)+1}^{\infty} |a_{n,k(1)}|^\beta \\ &> 3 - 1 \\ &= 2. \end{aligned}$$

More generally, having chosen the positive integers $k(j), n(j), j \leq m-1$, choose the positive integers $k(m), n(m)$ such that $k(m) > k(m-1)$, $n(m) > n(m-1)$,

$$\sum_{n=n(m-2)+1}^{n(m-1)} \sum_{k=k(m)}^{\infty} B_n^{\beta/\alpha} k^{-2} < 1, \tag{5}$$

$$\mu_{k(m)} > 2 \sum_{n=0}^{n(m-1)} B_n + \rho^{-\alpha} m^2 \left\{ 2 + \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} \right\} \tag{6}$$

and

$$\sum_{n=n(m)+1}^{\infty} |a_{n,k(m)}|^\beta < \sum_{n=0}^{n(m-1)} B_n, \tag{7}$$

where, $0 < \rho < 1$. Now, using (6) and (7), we get

$$\begin{aligned} \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta &= \mu_{k(m)} - \sum_{n=0}^{n(m-1)} |a_{n,k(m)}|^\beta - \sum_{n=n(m)+1}^{\infty} |a_{n,k(m)}|^\beta \\ &> 2 \sum_{n=0}^{n(m-1)} B_n + \rho^{-\alpha} m^2 \left\{ 2 + \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} \right\} - \sum_{n=0}^{n(m-1)} B_n - \sum_{n=0}^{n(m-1)} B_n \\ &= \rho^{-\alpha} m^2 \left\{ 2 + \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} \right\}. \end{aligned} \tag{8}$$

Now, for every $i = 1, 2, 3, \ldots$, we can choose a non-negative integer $\lambda(i)$ such that

$$\rho^{\lambda(i)+1} \le i^{-\frac{2}{\alpha}} < \rho^{\lambda(i)}. \tag{9}$$

Define the sequence $x = \{x_k\}$, where

$$x_k = \begin{cases} \rho^{\lambda(i)+1}, & \text{if } k = k(i); \\ 0, & \text{if } k \neq k(i), i = 1, 2, \ldots. \end{cases}$$

We note that $x = \{x_k\} \in \ell_\alpha$, since

$$\begin{aligned} \sum_{k=0}^{\infty} |x_k|^\alpha &= \sum_{i=1}^{\infty} |x_{k(i)}|^\alpha \\ &= \sum_{i=1}^{\infty} \rho^{(\lambda(i)+1)\alpha} \\ &\le \sum_{i=1}^{\infty} \frac{1}{i^2}, \text{ using (9)} \\ &< \infty. \end{aligned}$$

In view of Lemma 2.1, we have

$$\sum_{n=n(m-1)+1}^{n(m)} |(Ax)_n|^\beta \ge \Sigma_1 - \Sigma_2 - \Sigma_3,$$

where

$$\Sigma_1 = \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta |x_{k(m)}|^\beta,$$

$$\Sigma_2 = \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=1}^{m-1} |a_{n,k(i)}|^\beta |x_{k(i)}|^\beta$$

and

$$\Sigma_3 = \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=m+1}^{\infty} |a_{n,k(i)}|^\beta |x_{k(i)}|^\beta.$$

Now,

$$\begin{aligned} \Sigma_1 &= \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta \rho^{(\lambda(m)+1)\alpha} \\ &> \rho^\alpha \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta m^{-2}, \text{ using (9)} \\ &= \rho^\alpha m^{-2} \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta \\ &> 2 + \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)}, \text{ using (8);} \end{aligned} \tag{10}$$

$$\begin{aligned}
\Sigma_2 &= \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=1}^{m-1} |a_{n,k(i)}|^{\beta} \rho^{(\lambda(i)+1)\beta} \\
&< \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=1}^{m-1} |a_{n,k(i)}|^{\beta} \rho^{(\lambda(i)+1)\alpha}, \\
&\qquad\qquad \text{since } 0 < \rho < 1 \text{ and } \alpha \leq \beta \\
&\leq \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=1}^{m-1} |a_{n,k(i)}|^{\beta} i^{-2}, \text{ using (9)} \\
&= \sum_{i=1}^{m-1} i^{-2} \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(i)}|^{\beta} \\
&\leq \sum_{i=1}^{m-1} i^{-2} \sum_{n=0}^{\infty} |a_{n,k(i)}|^{\beta} \\
&= \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_3 &= \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=m+1}^{\infty} |a_{n,k(i)}|^{\beta} \rho^{(\lambda(i)+1)\beta} \\
&< \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=m+1}^{\infty} |a_{n,k(i)}|^{\beta} \rho^{(\lambda(i)+1)\alpha} \\
&\qquad\qquad \text{since } 0 < \rho < 1 \text{ and } \alpha \leq \beta \\
&\leq \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=m+1}^{\infty} B_n^{\beta/\alpha} i^{-2}, \text{ using (4)} \\
&< 1, \text{ using (5).} \qquad (12)
\end{aligned}$$

Now, using (10), (11) and (12), we get

$$\begin{aligned}
\sum_{n=n(m-1)+1}^{n(m)} |(Ax)_n|^{\beta} &> 2 + \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} - \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} - 1 \\
&= 1, m = 2, 3, \ldots,
\end{aligned}$$

from which it follows that $\{(Ax)_n\} \notin \ell_\beta$, while, $x = \{x_k\} \in \ell_\alpha$, which is a contradiction. Thus (3) is necessary, completing the proof of the theorem. □

Corollary 2.3. *If we put $\beta = \alpha$, we get a characterization of the matrix class $(\ell_\alpha, \ell_\alpha)$, $0 < \alpha \leq 1$, which was obtained by the author in [6].*

Corollary 2.4. $A = (a_{nk}) \in (\ell_\alpha, \ell_\beta)$, $0 < \alpha < \beta \leq 1$ *if and only if*

$A \in (\ell_\beta, \ell_\beta)$.

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