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On Numerical Pricing of Put-Call Parities for Asian Options Driven by New Time-Fractional Black-Scholes Evolution Equation

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Abstract. The objective of this paper is twofold. Firstly, to derive time-fractional evolution equation modeling the No-Arbitrage premium of Asian option (with arithmetic and geometric averages) contingent upon an underlying asset that satisfies the fractional stochastic differential equation, in a setting when the strike price is fixed and floating. Secondly, we have computed the four versions of the put-call parities for Asian options, by solving the time-fractional Black-Scholes evolution modeling the difference of the premiums of put and call Asian options, through Fractional Reduced Differential Transform (FRDT) algorithm. We have also established the convergence and the error estimates for the FRDT Algorithm for the two independent variables.

1. Introduction

This paper focuses on developing a version of the fractional Black-Scholes evolution equation, that enables the price of the put-call parities for Asian options, when the underlying stock price process $(S(t))_{t\geq 0}$, satisfies the following fractional stochastic differential equation (FSDE),

$$d^{\alpha}S(t) = \mu S(t)(dt)^{\alpha} + \sigma S(t)dW(t), \tag{1.1}$$

where $0 < \alpha \leq 1$, constants μ and σ are expected rate of return and volatility respectively, and $W(t)$ is the classical Wiener process. The motivation of SDE (1.1) comes from the fact that large volatility of class of the underlying assets can be modeled being proportional to the time-fractional differential $(dt)^{\alpha}$, see [22] and hence SDE (1.1) becomes a natural model for the dynamics of the stock price process. Guy Jumarie, has extensively worked on the theory and applications of such SDEs. In [25] Jumarie studied the fractional stochastic calculus and properties of fractional SDEs involving the terms $(dt)^{\alpha}$, moreover, in [25], [23] and [22] Jumarie focused on applying fractional SDEs of type (1.1) into variety of contexts in finance and physics. In general, the fractional stochastic differential equations are the natural generalizations of the standard stochastic differential equations. Another, significant advantage of considering fractional SDE of type (1.1) is to incorporate the Memory effect in financial pricing. For instance, in a financial system studying long-term memory is important because a rational participant of the market can make quick decisions based on

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the memory and experience of the rational decisions made by other market participants. Fractional SDEs of type (1.1) can model the memory effects in the financial market effectively [49].

The theory of pricing the financial options contracts has been proven to be of prime importance in risk-management from its inception in celebrated [15]. Financial derivatives are the financial contracts that allow its holders to hedge risks to which they are exposed due to investments in random financial entities, such as, stocks; exchange rates; commodities (gold, oil, gas, etc), volatility indices, and many more. The payoff of such financial derivatives are the functions of the random dynamics of the underlying entity, so the central question in quantitative finance is to determine the premium of the risk hedging derivatives at an instant t before the expiry date T of the contract. Out of the huge class of financial derivatives options are the fundamental financial derivatives, even most of the financial contacts which are not options can be described in the terms of options, therefore this makes the options as the contract of prime importance.

Asian style derivatives are the exotic path-dependent financial contracts whose payoff depends upon the average (geometric or arithmetic) of the underlying asset price over the lifetime of the contract. The dependence of a derivative on average underlying assets, make them a lot cheaper and important for several reasons. For example, the manipulation of underlying assets for the sake of the financial incentive by executives can be sharply restricted if their incentives are linked with Asian options, because then the incentive that they are going on earn is going depend upon the average performance of the company. Therefore, it becomes important to price the Asian options as efficiently and accurately as possible. The pricing of the Asian style options turned out to be very challenging, especially in a continuous-time setting, even under the standard Black-Scholes assumption. Moreover, there is no closed-form formula for the premium of Asian option because the computation of the distribution of continuous-time arithmetic average price of the underlying asset i.e. the integral of the underlying asset is challenging. The Asian option has been priced in several interesting settings, analytically and numerically both. The geometric version of the Asian option was analytically was first priced by Kemna in [5] explicitly, based on the central idea that the product of lognormal random variable is lognormal. While, since the product of lognormal is not lognormal so there is explicit pricing formula for geometric Asian option cf. Turnbull [54]. Rogers in [38], Curran in [39] and Chen in [37] used versions of conditioning methods to price the Asian options. Monte-Carlo is the central method employed vastly in literature to price the Asian options numerically, for example see [48], [31], [5], [11]. Another, interesting approach to price the Asian option as the solution of versions of deterministic Black-Scholes evolution equations can be found [30], [32], [16] and [35]. Moreover in [12], [64], [34], [63] the authors used the integral transform method to solve evolution equation modeling pricing Asian option, and computed the price in the form of special functions such as hyper-geometric functions. The issue with special functions is that it is challenging to evaluate and invert them numerically. The more recent attempts are pricing through Wavelet methods by Leitao in [7] and [6]. The Fourier transform methods were employed price Asian style option by Zhang in [10] and Krikby in [36]. The attempt through finite difference scheme was made by Zvan in [52], but their stability of solution was a problem due to convection terms in PDE. Asymptotic pricing was done by Geman in [27]. For some of the recent interesting applications of fractional and deterministic and stochastic differential equations in computational finance, we refer to [4] and [3] by Nikan, Golbabai, Nikazad; moreover, in the context of computational physics [47], [44], [46] and [45] by Nikan, Avazzadeh, Tenreiro, Golbabai, and Rashidinia.

Fractional evolution equations have turned out to be a reliable tool to model the physical phenomenon, in particular, several processes arising in physics, mechanics, electricity, economics can be modeled through fractional evolution equations. Recently, a good amount of progress has been made to model the variety of financial contracts through fractional evolution equations. For instance, Wyss in [61], Jumarie in [22], and A. Farhadi in [2] studied fractional versions of the Black-Scholes equation modeling a variety of financial derivatives for risk hedging. Wyss in [61] introduced time-fractional Black-Scholes equation to price European style option. Jumarie in [22] introduced two versions of fractional Black-Scholes equation and numerically computer their solutions. A. Farhadi in [2] incorporated tend memory in the pricing of financial derivatives and derived new time-fractional order Black-Scholes equation, and RVIM algorithm was used to compute the solution. Some of the recent work on treatments of fractional Black-Scholes equation includes, [13] by Prathumwan, which focuses on 2D time fractional-order Black–Scholes equation for a European put option; [41] by Mianfu, Lili, Renxuan, Dongfang developed a novel scheme for time-fractional based on the Chebyshev-Galerkin spectral method. Golbabai, Nikan, and Nikazad in [3] deduced fractional Black-Scholes Model based on the price variation of the correlated fractal transmission system and solved through the method of radial basis functions. In [55] Edeki and Jena considered the time-space fractional Black-Scholes equation and computed its analytical solution through coupled transform method.

Along with traditional numerical methods to deal with the fractional partial differential equations there is another class of method available, which provide the analytical series solution for the Fractional PDEs. The method includes fractional homotopy analysis, a fractional variation of iteration method, and fractional reduced differential transform method, for a more detailed description and applications of these methods we refer the reader to the [56]. In particular, we are interested in the fractional reduced differential transform method (FRDTM), which is gaining much attention of the researchers due to the fact that it can be applied to almost all kinds of nonlinear PDEs and is simple to execute. The novel aspect of FRDTM is that almost no assumptions are needed, it is computationally less expensive and evades the round-off errors. Some of the recent applications of FRDTM to Fluid mechanics can be found in [28], for solving Multiterm Time-Fractional Diffusion Equations in [53], [55] Edeki and Jena considered time-space fractional Black-Scholes equation and computed its analytical solution through coupled transform method.

As mentioned earlier that there ample literature available on fractional Black-Scholes pricing of European option but for pricing exotic option in the fractional framework is relatively new and very limited literature available in this direction. Some of the interesting recent work includes [40] which focuses on studying time-fractional exotic options via efficient local meshless methods; [29] focuses on demonstrating that the space-fractional diffusion equations can sever as the model for the class of exotic options.

Finally, before providing the layout of the paper we would like to briefly motivate the reader that why it is important to study the put-call parity for the financial derivatives in general and exotic options in particular? We know that in the absence of arbitrage, the premiums P and C for putting and call options, respectively, contingent upon the same underlying asset, same maturity T with strike K must satisfy a relation of the following form:

$$P + S = C + K + e^{-rT},$$

where S is the price of the underlying asset at T and r is the risk-free interest rate. This put-call parities for financial options despite being a very simple relation is a powerful no-arbitrage relation because its violation implies the existence of arbitrage and hence the trading strategies can be created to have a *free lunch*. We emphasize to call put-call parity *simple* because without doing a complicated analysis of data or without even knowing the future distribution of underlying asset prices, the absence or presence of arbitrage opportunities can be determined. Do the violations of the put-call parity exists the answer is positive. Klemkosky in [51] and Resnick in [50] determined the violations of put-call parities and the existence of arbitrage opportunities. Nissim and Tchahi in [8] found evidence of the violations of put-call parities in the Israeli derivative markets. Taylor in [57] studied the relative prices of exchange-traded puts and calls and found evidence of a violation of the put-call parity, especially at times really near to expiry prior to expiry. The applications of put-call parities are not only limited to determine the presence or absence of arbitrage. For instance, in Finucane in [58], using put-call parities for indexed options, derived the measure of relative prices of put and call that contains the information about the future returns of the underlying assets while a similar study was done in [60] by the Hsieh, Lee, and Yuan in Taiwan between index futures and index options. Goh and Allen in [17] studied the efficiency of the London Traded Options Market, while Vipul in [59] did the same for the Indian option market. Recently, Lecuyer and Lefort in [14] generalized the fundamental theorem of asset pricing in the financial markets with frictions where the put-call parity holds.

In this paper, a new time-fractional-order evolution equation for pricing the Put-Call parities Asian option has been derived by running the no-arbitrage argument, under the assumption that the underlying asset price satisfies a time-fractional stochastic differential equation. Finally, an approximate analytical solution has been constructed for our time-fractional-order evolution equation for pricing the Asian option. Next, we aim to solve this evolution equation, by Fractionally reduced differential transform algorithm, modeling following the Put-Call parities for Arithmetic and geometric Asian option with fixed and floating strikes.

We believe that the results presented in this paper are interesting, new, and can be used as a comparison in future studies. In short, our paper makes the following major contributions:
1. This is the first paper in the direction of the theory of exotic (not vanilla) option pricing that makes use of the Fractional reduced differential transform method to price the option by solving effectively solving the governing fractional evolution equations for put-call parties of Asian options.

2. Theoretically, we have proved a new result (Theorem 3.3) for the convergence of the fractional reduced differential transform algorithm for the two independent variables.

3. Based on a more realistic fractional stochastic model for underlying stock price (i.e. FSDE (1.1)) and using no-arbitrage argument and Ito Lemma we have derived a new time-fractional Black-Scholes evolution equation that is satisfied by the price of the Asian option.

4. Finally, the put-call parities for Asian option prices, for the following four cases, have been obtained through solving the governing time-fractional evolution equation by fractionally reduced differential transform algorithm,
i. Put-call Parity for arithmetic average Asian Option with Fixed Strike Price,
ii. Put-call Parity for arithmetic average Asian Option with Floating Strike Price,
iii. Put-call Parity for geometric average Asian Option with Fixed Strike Price,
iv. Put-call Parity for geometric average Asian Option with Floating Strike Price.

5. We believe that our put-call parities for the Asian option will in a fractional environment encourage more researchers in the future to develop put-call parities for more exotic options.

2. Fractional Calculus preliminaries and some important results

In this section, we will provide basic definitions and results from the fractional calculus and the time-fractional stochastic differential equation that we assume to model our underlying asset. For details we refer to Jumarie [26], [25], [24], [22] and Farhadi [2].

Definition 2.1. *[22] For a continuous function $f : \mathbb{R} \to \mathbb{R}$ (not necessarily differentiable), the fractional difference of order $\alpha \in (0, 1]$ of $f(x)$ can be defined as,*

$$\Delta^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k) h], \tag{2.1}$$

where $h > 0$ is constant discretization period, $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}$ and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *[22] (Modified Riemann-Liouville Derivative). For a continuous function $f : \mathbb{R} \to \mathbb{R}$ (not necessarily differentiable), the fractional derivative of order α of f can be defined as,*

$$f^{(\alpha)}(x) := \frac{d^{\alpha} f(x)}{dx^{\alpha}} = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int\limits_0^x (x-\xi)^{-\alpha-1} (f(\xi) - f(0))\, d\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0))\, d\xi, & 0 < \alpha \le 1, \\ \frac{d^n\left(f^{(\alpha-n)}(x)\right)}{dx^n}, & n < \alpha \le n+1, \end{cases} \tag{2.2}$$

where n denotes a positive integer.

Proposition 2.3. *[26] With the notation in Definition 2.1, the following equality holds:*

$$f^{(\alpha)}(x) = \lim_{h\to 0} \frac{\Delta^{\alpha} f(x)}{h^{\alpha}}. \tag{2.3}$$

Proposition 2.4. *[26] If f be a real-valued continuous function that is differentiable upto order $k\alpha$ in fractional sense, for all integers $k > 0$ and for all $\alpha \in (0, 1]$. Then for any $h > 0$ we have,*

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(1+\alpha k)} f^{(\alpha k)}(x). \tag{2.4}$$

Here $f^{(\alpha k)}(x)$ denotes the αk-th order derivative of f.

Corollary 2.5. *[26] Keeping the assumptions of Proposition 2.4, for $\alpha \in (0, 1]$, we have following,*
i)

$$d^{\alpha} f(x) \cong \Gamma(1+\alpha)\, df(x). \tag{2.5}$$

ii) Following relation between dx^{α} and dx holds,

$$dx^{\alpha} = \Gamma(1+\alpha)\Gamma(2-\alpha)\, x^{\alpha-1} dx, \quad 0 < \alpha \le 1. \tag{2.6}$$

iii) The following equality holds,

$$f_x^{(\alpha)}[u(x)] = \Gamma(2-\alpha)\, u^{\alpha-1}(x)\, f_x^{(\alpha)}[u(x)]\, u_x^{(\alpha)}(x). \tag{2.7}$$

3. Fractional Reduced Differential Transform Algorithm

This section has been dedicated to describe definitions and properties of related to Fractional reduced differential transform algorithm (FRDTA). For a detailed account of FRDTA we refer to [9].

3.1. *Definition and key properties of Fractional Reduced Differential Transform Algorithm*

Let us start by defining the reduced differential transform of a smooth (i.e. $C^{\infty}(\mathbb{R})$ function of three variables.

Definition 3.1. *[9] Suppose function $u(t, x, y)$ be $C^{\infty}(\mathbb{R})$-function and is analytic in a neighbourhood $(t_0, t_0 + r)$ of t_0, then the fractional differential transform differential transform (FRDTM) of $u(t, x, y)$ is following,*

$$u_k(x,y) = \frac{1}{\Gamma(1+k\alpha)} \left[\frac{\partial^{\alpha k} u}{\partial t^{\alpha k}}(x,y,t) \right]_{t=t_0}, \tag{3.1}$$

where α is order of Modified Riemann-Liouville Derivative, $u_k(x, y)$ can be treated as the the t-dimensional spectrum transformed function.
The inverse fractional reduced differential transform (FRDT) of $u_k(x, y)$ is defined as follows:

$$u(x,y,t) \;:=\; \sum_{k=0}^{\infty} u_k(x,y)(t-t_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+k\alpha)} \left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} u(x,y,t) \right]_{t=t_0} (t-t_0)^{k\alpha}. \tag{3.2}$$

Based on above we have the following theorem listing the basic properties of reduce differential transform,

Theorem 3.2. *[9] For any smooth functions u, v the reduce differential transform of u and v satisfies following properties,*
i) Linearity: For any linear combination of u and v, i.e $v(t, x, y) = au(t, x, y) + bv(t, x, y)$, where $a, b \in \mathbb{R}$, the reduced differential transform is $v_k(x, y) = au_k(x, y) + v_k(x, y)$, $\quad k \in \mathbb{N}$. where u_k, v_k and $\mathcal{W}_k$ are differential transforms of u, v and w respectively.
ii) If $v(x,y,t) = \frac{\partial^{N\alpha} u}{\partial t^{N\alpha}}(x,y,t)$, then $v_k(x,y) = \frac{\Gamma(1+(k+N)\alpha)}{\Gamma(1+k\alpha)} u_{k+N}(x,y)$.
iii) If $v(x,y,t) = t^{\alpha} \frac{\partial^{\alpha} u}{\partial t^{\alpha}}(x,y,t)$, then $v_k(x,y) = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} u_k(x,y)$.

3.2. Application of Fractional Reduced Differential Transform Algorithm to nonlinear evolution equations: A generic algorithm

Consider the following general nonlinear partial differential equation:

$$\begin{aligned} D_t^{\alpha} u(t,x,y) &= \mathcal{A}u(t,x,y) + \mathcal{B}u(t,x,y) + f(t,u(t,x,y)), \\ u(0,x,y) &= h(x,y). \end{aligned} \tag{3.3}$$

Here D_t^{α} is an easily invertible linear operator, Here $\mathcal{A}$ is linear operator, $\mathcal{B}$ is nonlinear linear operator and f is some smooth function of x, y. We can look for the solution $u(t,x,y)$ based on the properties of two dimensional differential transform, the function $u(t,x,y)$ can be represented as follows:

$$u(x,y,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i,j)\, x^i y^j \sum_{k=0}^{\infty} g(k)\, t^k = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k}^{\infty} W(i,j)\, x^i y^j t^k = \sum_{k}^{\infty} \mathcal{U}_k(x,y)\, t^k. \tag{3.4}$$

where $\mathcal{U}_k(x,y)$ is called t-dimensional spectrum function of $u(t,x,y)$. Now, let us write the nonlinear term,

$$\mathcal{B}(u,t) = \sum_{n=0}^{\infty} \mathcal{B}_n\left(\mathcal{U}_0(x,y), \mathcal{U}_1(x,y), \cdots, \mathcal{U}_n(x,y)\right) t^{n\alpha}, \tag{3.5}$$

where $\mathcal{B}_n$ is the appropriate Adomian's polynomials [55]. In this specific nonlinearity, we use the general form of the formula for $\mathcal{B}_n$ Adomian's polynomials as,

$$\mathcal{B}_n\left(\mathcal{U}_0(x,y), \mathcal{U}_1(x,y), \cdots, \mathcal{U}_n(x,y)\right) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\mathcal{B}\left(\sum_{i=0}^{\infty} \lambda^i \mathcal{U}_i(x,y)\right)\right]_{\lambda=0}.$$

Now, applying Riemann-Liouville integral J^{α} both sides of equation (3.4), we have,

$$u(t,x,y) = \Phi + J^{\alpha}\mathcal{A}u(t,x,y) + J^{\alpha}\mathcal{B}u(t,x,y) + J^{\alpha} f(t,u(t,x,y)),$$

where from the initial condition $\Phi = u(0,x,y) = h(x,y)$. Substituting equations (3.4) and (3.5), for $u(t,x,y)$ and $\mathcal{B}(u,t)$, respectively, in equation (3.3) yields:

$$\sum_{k=0}^{\infty} \mathcal{U}_k(x,y) t^{\alpha k} = h(x,y) + J^{\alpha}\left(\mathcal{A}\left(\sum_{k=0}^{\infty} \mathcal{U}_k(x,y) t^{\alpha k}\right)\right) + J^{\alpha}\left(\mathcal{B}\left(\sum_{k=0}^{\infty} \mathcal{U}_k(x,y) t^{\alpha k}\right)\right) + J^{\alpha}\left(\sum_{k=0}^{\infty} f_k(x,y,\mathcal{U}_k(x,y)) t^{\alpha k}\right),$$

where $f(t,x,y,u(t,x,y)) = \sum_{k=0}^{\infty} f_k\left(x,y,\mathcal{U}_k(x,y)\right) t^{\alpha k}$, and $f_k\left(x,y,\mathcal{U}_k(x,y)\right)$ is the transformed function of $f(t,x,y,u(t,x,y))$. After carry out Riemann-Liouville integral J^{α}, we obtain,

$$\begin{aligned} \sum_{k=0}^{\infty} \mathcal{U}_k(x,y) t^{\alpha k} = h(x,y) &+ \left(\mathcal{A}\left(\sum_{k=0}^{\infty} \mathcal{U}_k(x,y) \frac{t^{\alpha(k+1)}\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right)\right) + \left(\sum_{k=0}^{\infty} \mathcal{B}_k(x,y) \frac{t^{\alpha(k+1)}\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right) \\ &+ \left(\sum_{k=0}^{\infty} f_k\left(x,y,\mathcal{U}_k(x,y)\right) \frac{t^{\alpha(k+1)}\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right). \end{aligned}$$

Finally, equating coefficients of like powers of t, we derive the following recursive formula,

$$\mathcal{U}_0(x,y) = h(x,y),$$

and for $k \geq 0$,

$$\mathcal{U}_{k+1}(x,y) = \mathcal{A}\left(\mathcal{U}_k(x,y)\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right) + \mathcal{B}_k(x,y)\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} + f_k\left(x,y,\mathcal{U}_k(x,y)\right)\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}. \tag{3.6}$$

Using the known $\mathcal{U}_0(x,y)$, all components $\mathcal{U}_1(x,y), \mathcal{U}_2(x,y), \cdots, \mathcal{U}_n(x,y), \cdots$, etc., are determinable by using equation (3.6). Substituting these $\mathcal{U}_n(x,y), \mathcal{U}_1(x,y), \mathcal{U}_2(x,y), \cdots, \mathcal{U}_n(x,y), \cdots$, etc. in equations (3.4), the approximate solution can be obtained as,

$$\tilde{u}_n(t,x,y) = \sum_{k=0}^{n} \mathcal{U}_k(x,y) t^{k\alpha}, \tag{3.7}$$

where n is the order of approximate solution. Therefore, the corresponding exact solution is given by

$$u(t,x,y) = \lim_{n\to\infty} \tilde{u}_n(t,x,y) = \sum_{k=0}^{\infty} \mathcal{U}_k(x,y) t^{k\alpha}. \tag{3.8}$$

3.3. Convergence of Fractional Reduced Differential Transform Algorithm

In the following result we assume that X denotes the space of real-valued continuous function on $[t_0, t_0+\delta]$ i.e. $X := C[t_0, t_0+\delta]$ with usual sup norm $\|\cdot\|_\infty$.

Theorem 3.3. *The solution series $\sum_{k=0}^{\infty} u_k(x,y)t^{k\alpha}$ (as described in (3.8)) converges, for all t in any neighbourhood $(t, t_0+\delta)$ of t_0, for any $\delta > 0$; if there exists $\gamma \in (0,1)$ such that $\left\|u_{k+1}(x,y)t^{k+1}\right\| \le \gamma \left\|u_k(x,y)t^k\right\|$, for all $k \in \mathbb{N}\cup\{0\}$. Indeed, it is well-known that $(X, \|\cdot\|_\infty)$ is a Banach space. Moreover, if $\sum_{k=0}^{\infty} u_k(x,y)t^k$ converges to $u(t,x,y)$ then error between the truncated sum $\sum_{k=0}^{n} u_k(x,y)t^k$ and $u(t,x,y)$ can be controlled by following inequality, $\|\cdot\|_\infty$*

$$\left\| u(t,x,y) - \sum_{k=0}^{n} u_k(x,y)t^k \right\|_\infty \le \frac{\gamma^{n+1}}{\gamma - 1} \left\| u_0(x,y) \right\|_\infty.$$

Proof. In order to show that convergence of formal series $\sum_{k=0}^{\infty} u_k(x,y)t^k$ in X, we aim to show that sequence of $\left(\sum_{k=0}^{n} u_k(x,y)t^k\right)_{n\in\mathbb{N}}$ is convergent, and since $(X, \|\cdot\|_\infty)$ is complete normed space so it is enough to show that $\left(\sum_{k=0}^{n} u_k(x,y)t^k\right)_{n\in\mathbb{N}}$ is Cauchy sequence. From hypothesis mentioned in the statement of theorem, it follows that,

$$\begin{aligned}\left\| \sum_{k=0}^{n+1} u_k(x,y)t^k - \sum_{k=0}^{n} u_k(x,y)t^k \right\|_\infty &= \left\| u_{n+1}(x,y)t^{n+1} \right\|_\infty \le \gamma \left\| u_n(x,y)t^n \right\|_\infty \le \gamma^2 \left\| u_{n-1}(x,y)t^{n-1} \right\|_\infty \\ &\dots \le \gamma^{n+1} \left\| u_0(x,y) \right\|_\infty.\end{aligned} \tag{3.9}$$

For every $n, m \in \mathbf{N}, n \ge m$, using telescopic sum, we get,

$$\left\| \sum_{k=0}^{n} u_k(x,y)t^k - \sum_{k=0}^{m} u_k(x,y)t^k \right\|_\infty = \left\| \begin{array}{c} \left(\sum_{k=0}^{n} u_k(x,y)t^k - \sum_{k=0}^{n-1} u_k(x,y)t^k\right) \\ + \left(\sum_{k=0}^{n-1} u_k(x,y)t^k - \sum_{k=0}^{n-2} u_k(x,y)t^k\right) \\ +\dots + \left(\sum_{k=0}^{m+1} u_k(x,y)t^k - \sum_{k=0}^{m} u_k(x,y)t^k\right) \end{array} \right\|_\infty.$$

Using triangle inequality, we may infer,

$$\begin{aligned}\left\| \sum_{k=0}^{n} u_k(x,y)t^k - \sum_{k=0}^{m} u_k(x,y)t^k \right\|_\infty &\le \left\| \sum_{k=0}^{n} u_k(x,y)t^k - \sum_{k=0}^{n-1} u_k(x,y)t^k \right\|_\infty + \left\| \sum_{k=0}^{n-1} u_k(x,y)t^k - \sum_{k=0}^{n-2} u_k(x,y)t^k \right\|_\infty \\ &+\dots + \left\| \sum_{k=0}^{m+1} u_k(x,y)t^k - \sum_{k=0}^{m} u_k(x,y)t^k \right\|_\infty.\end{aligned}$$

Now using inequality (3.9) and fact that $0 < \gamma < 1$, we it follows that,

$$\begin{aligned}\left\|\sum_{k=0}^{n} u_k(x,y)t^k - \sum_{k=0}^{m} u_k(x,y)t^k\right\|_\infty &\leq \gamma^n \left\|u_0(x,y)\right\|_\infty + \gamma^{n-1}\left\|u_0(x,y)\right\|_\infty + ... + \gamma^m \left\|u_0(x,y)\right\|_\infty \\ &\leq \frac{1-\gamma^{n-m}}{1-\gamma}\gamma^{m+1}\left\|u_0(x,y)\right\|_\infty \to 0 \text{ as } n,m\to\infty. \end{aligned} \tag{3.10}$$

Hence sequence of $\left(\sum_{k=0}^{n} u_k(x,y)t^k\right)_{n\in\mathbb{N}}$ is convergent in X. Thus $\sum_{k=0}^{\infty} u_k(x,y)t^k$ converges in X and the series makes sense.
Next, lets move towards the error estimate. Using last inequality (3.10), and as $\gamma \in (0,1)$ so $\gamma^{n-m} < 1$,

$$\left\|\sum_{k=0}^{n} u_k t^k - \sum_{k=0}^{m} u_k t^k\right\|_\infty \leq \frac{1-\gamma^{n-m}}{1-\gamma}\gamma^{m+1}\left\|u_0\right\|_\infty \leq \frac{\gamma^{m+1}}{1-\gamma}\left\|u_0\right\|_\infty. \tag{3.11}$$

Now take limit $n \to \infty$ and keeping in view the fact that $u(t,x,y) = \sum_{k=0}^{\infty} u_k(x,y)t^k$., we get the following required inequality,

$$\left\|u - \sum_{k=0}^{m} u_k t^k\right\|_\infty \leq \frac{\gamma^{m+1}}{1-\gamma}\left\|u_0\right\|_\infty. \tag{3.12}$$

□

4. Derivation of Time-fractional model for pricing Asian Options

This section is devoted to the derivation of the premium of Asian Style options at any instant t before expiry it's T, in standard Black-Scholes assumptions except the assumption about stock price process $(S(t))_{t\geq 0}$, which we assume that it satisfies fractional stochastic differential equation (1.1). Our strategy would be to use Ito Lemma to create a self-financing portfolio that neutralizes the risk and arrive at a time-fractional PDE that models the Asian option. We refer the readers to [33] where the similar approach has been employed to derive the models for large class of exotic options.

4.1. Derivation based on No-arbitrage Principle

Let us start with assuming that stock price process $(S(t))_{t\geq 0}$, satisfies following time-fractional stochastic differential equation,

$$d^\alpha S(t) = \mu S(t)dt^\alpha + \sigma S(t)dW(t), \tag{4.1}$$

where $0 < \alpha \leq 1$, (α may be also interpreted as Hurst index), constants μ and σ are expected rate of return and volatility respectively, and $W(t)$ is standard 1D Wiener process. For $\alpha = 1$, SDE (4.1) reduces to a geometric Brownian motion.

Let $V(t,S(t),J(t))$ denotes the premium of Asian style at $t \in [0,T]$, with expiry T and Strike price K, contingent upon stock price process described $(S(t))_{t\geq 0}$ in (4.1). Here $(J(t))_{t\geq 0}$ is average (arithmetic or geometric) of the process $(S(t))_{t\geq 0}$, up to time t. We assume that that $V(t,S(t),J(t))$ is continuously differentiable in t (in fractional sense) up to order $\alpha \in [0,1)$, and it is twice differentiable w.r.t S. Consider the following fractional Taylor's expansion of $V(t,S(t),J(t))$ introduced by Jumarie in [22],

$$dV = \frac{1}{\Gamma(1+\alpha)}\frac{\partial^\alpha V}{\partial t^\alpha}dt^\alpha + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial J}dJ + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2. \tag{4.2}$$

Taking into consideration the Maruyama's notation for Brownian motion and equation $d^\alpha f(x) \cong \Gamma(1+\alpha)df(x)$ (see corollary 2.1), the fractional SDE (4.1) yields,

$$dS = \frac{\mu S}{\Gamma(1+\alpha)}dt^\alpha + \frac{\sigma S}{\Gamma(1+\alpha)}dW(t), \tag{4.3}$$

$$(dS)^2 = \left(\frac{\mu S}{\Gamma(1+\alpha)}\right)^2 dt^{2\alpha} + \left(\frac{\sigma S}{\Gamma(1+\alpha)}\right)^2 (dW(t))^2 + 2\left(\frac{\mu\sigma S^2}{\Gamma^2(1+\alpha)}\right)dt^\alpha dW(t). \tag{4.4}$$

Using Ito rules for Wiener process i.e. $(dW(t))^2 \approx dt$, so $dt^\alpha dW(t) = dt^{\alpha+\frac{1}{2}}$ and ignoring the powers of dt higher than 1, we infer that,

$$(dS)^2 = \frac{\sigma^2 S^2}{\Gamma^2(1+\alpha)}dt. \tag{4.5}$$

Substituting the expressions for dS and $(dS)^2$ in equation (4.2), it follows that,

$$dV = \frac{1}{\Gamma(1+\alpha)}\frac{\partial^\alpha V}{\partial t^\alpha}dt^\alpha + \frac{\mu S}{\Gamma(1+\alpha)}\frac{\partial V}{\partial S}dt^\alpha + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial J}dJ + \frac{\sigma S}{\Gamma(1+\alpha)}\frac{\partial V}{\partial S}dW(t). \tag{4.6}$$

Using property (ii) i.e. equation (2.6) from Proposition 2.4, for dt, it follws that,

$$\begin{aligned} dV &= \frac{1}{\Gamma(1+\alpha)}\frac{\partial^\alpha V}{\partial t^\alpha}\Gamma(1+\alpha)\Gamma(2-\alpha)t^{\alpha-1}dt + \frac{\mu S}{\Gamma(1+\alpha)}\frac{\partial V}{\partial S}\Gamma(1+\alpha)\Gamma(2-\alpha)t^{\alpha-1}dt \\ &\quad + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial J}dJ + \frac{\sigma S}{\Gamma(1+\alpha)}\frac{\partial V}{\partial S}dW(t). \end{aligned}$$

On simplification of last equation we get the following dynamics of $V(t, S_t, J_t)$ satisfies the Ito SDE,

$$dV = \left(\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha V}{\partial t^\alpha} + \mu\Gamma(2-\alpha)t^{\alpha-1}S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2}\right)dt + \frac{\partial V}{\partial J}dJ + \frac{\sigma S}{\Gamma(1+\alpha)}\frac{\partial V}{\partial S}dW(t). \tag{4.7}$$

Let us use no-arbitrage argument (See chapter 2 of [33]) to derive time-fractional model for premium of Asian option. Suppose stocks pay dividend q on each unit of stock. Assume that there is no transaction cost or any form of tax in trading of stocks, and there is no arbitrage in market.
Consider a self-financing portfolio in which we have a long position in Asian option and short position the Δ stocks i.e.

$$\Pi(t) = V(t, S, J) - \Delta S - qS. \tag{4.8}$$

We will Δ in such way that portfolio $(\Pi(t))_{t\geq 0}$ becomes risk-free in $(t, t+dt)$ i.e. $\Pi(t)$ satisfied following deterministic differential equation,

$$d\Pi = r\Pi dt = r(V - \Delta S - qS)dt. \tag{4.9}$$

Taking into account the dividends, the value of the portfolio $(\Pi(t))_{t\geq 0}$ at infinitesimally closed instant $t+dt$ to dt, can be given as,

$$\Pi(t+dt) = V(t+dt) - \Delta(t)S(t)q(t)dt - \Delta(t)S(t+dt). \tag{4.10}$$

Taking limit $dt \to 0$ on difference of equations (4.10) and (4.8), it follows that,

$$d\Pi = dV - \Delta dS - q\Delta S dt. \tag{4.11}$$

Using property (ii) of Proposition 2.4 in (4.3), we may rewrite the dynamics of stock price process $(S(t))_{t\geq 0}$ in following form,

$$dS = \mu S\Gamma(2-\alpha)t^{\alpha-1}dt + \frac{\sigma S}{\Gamma(1+\alpha)}dW(t). \tag{4.12}$$

Substituting (4.7) and (4.12) into (4.11), we infer that value of portfolio process $(\Pi(t))_{t\geq 0}$ satisfies following Ito's stochastic differential equation,

$$\begin{aligned}
d\Pi &= \left(\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha V}{\partial t^\alpha} + \mu\Gamma(2-\alpha)t^{\alpha-1}S\frac{\partial V}{\partial S} + \frac{\sigma^2S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2}\right)dt + \frac{\partial V}{\partial J}dJ \\
&\quad + \frac{\sigma S}{\Gamma(1+\alpha)}\frac{\partial V}{\partial S}dW(t) - \Delta\left(\mu S\Gamma(2-\alpha)t^{\alpha-1}dt + \frac{\sigma S}{\Gamma(1+\alpha)}dW(t)\right) - q\Delta S dt, \\
d\Pi &= \left(\begin{array}{c} \Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha V}{\partial t^\alpha} + \mu\Gamma(2-\alpha)t^{\alpha-1}S\frac{\partial V}{\partial S} + \frac{\sigma^2S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2} - \Delta\mu S\Gamma(2-\alpha)t^{\alpha-1} \\ -q\Delta S + \frac{\partial V}{\partial J}\frac{dJ}{dt} \end{array}\right)dt \\
&\quad + \left(\frac{\sigma S}{\Gamma(1+\alpha)}\frac{\partial V}{\partial S} - \Delta\frac{\sigma S}{\Gamma(1+\alpha)}\right)dW(t).
\end{aligned} \tag{4.13}$$

Compare (4.9) and (4.14) for risk free situation on both sides of equation, coefficient of random term $dW(t)$ must be zero, we can choose $\Delta = \frac{\partial V}{\partial S}$,

$$\begin{aligned}
rV - rS\frac{\partial V}{\partial S} &= \Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha V}{\partial t^\alpha} + \mu\Gamma(2-\alpha)t^{\alpha-1}S\frac{\partial V}{\partial S} + \frac{\sigma^2S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2} \\
&\quad -\mu\Gamma(2-\alpha)t^{\alpha-1}S\frac{\partial V}{\partial S} - q\Delta S + \frac{\partial V}{\partial J}\frac{dJ}{dt}, \\
&= \Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\sigma^2S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2} - q\Delta S + \frac{\partial V}{\partial J}\frac{dJ}{dt}.
\end{aligned}$$

Or

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\partial V}{\partial J}\frac{dJ}{dt} + \frac{\sigma^2S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \tag{4.14}$$

where

$$J(t) = \begin{cases} \frac{1}{t}\int\limits_0^t S(\tau)d\tau, & \text{(arithmetic average)} \\ e^{\frac{1}{t}\int\limits_0^t \ln S(\tau)d\tau}. & \text{(geometric average).} \end{cases} \tag{4.15}$$

Therefore

$$\frac{dJ(t)}{dt} = \begin{cases} \frac{1}{t}\left[S(t) - J(t)\right], & \text{(arithmetic average)} \\ J(t)\left[\frac{\ln S(t) - \ln J(t)}{t}\right]. & \text{(geometric average).} \end{cases} \tag{4.16}$$

Hence, in order price the Asian options with arithmetic averages, we need to consider the following terminal value problem in the domain $\{0 \leq S < \infty, 0 \leq J < \infty, 0 \leq t \leq T\}$ and for $0 < \alpha \leq 1$,

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{S-J}{t}\frac{\partial V}{\partial J} + \frac{\sigma^2S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0. \tag{4.17}$$

$$V(t,S,J) = \begin{cases} (J-K)^+, & \text{(call option with fixed strike price)} \\ (K-J)^+, & \text{(put option with fixed strike price)} \\ (S-J)^+, & \text{(call option with floating strike price)} \\ (J-S)^+, & \text{(put option with floating strike price).} \end{cases} \tag{4.18}$$

Similarly time-fractional geometric average Asian option can be formulate as the following terminal value problem in the domain $\{0 \leq S < \infty, 0 \leq J < \infty, 0 \leq t \leq T\}$ and for $0 < \alpha \leq 1$:

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha V}{\partial t^\alpha} + J\frac{\ln S - \ln J}{t}\frac{\partial V}{\partial J} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \tag{4.19}$$

$$V(t,S,J) = \begin{cases} (J-K)^+, & \text{(call option with fixed strike price)} \\ (K-J)^+, & \text{(put option with fixed strike price)} \\ (S-J)^+, & \text{(call option with floating strike price)} \\ (J-S)^+, & \text{(put option with floating strike price).} \end{cases} \tag{4.20}$$

Note for $\alpha = 1$ in equations (4.17) and (4.19), we recover the standard arithmetic average Asian option pricing model and geometric average Asian option pricing model [33] respectively.

5. Computation of Put-call Parities for Asian Option

The computation of the put-call parities are a simple extremely important tool to spot the presence or absence or the arbitrage in the financial markets see [51], [50] and [8]. Finucane in [58] even concluded that, if frictions are present in the market then relative put and call prices contain information about future gains of the underlying asset, and demonstrated that measure of relative index option prices leads the stock market by at least 15 minutes. We have given more details that why it is interesting to develop and study put-call parities? in the introduction of the paper.

In this section by employing the terminal value problems derived in the last section and using the FRDT algorithm, we will compute closed-form expressions for the following:
i. Put-call Parity for arithmetic average Asian option with Fixed strike price,
ii. Put-call Parity for arithmetic average Asian option with Floating Strike Price,
iii. Put-call Parity for geometric average Asian option with Fixed strike price,
iv. Put-call Parity for geometric average Asian option with Floating strike price.
Throughout the section, we will assume the notations and the framework of section 4. Further, we also assume that we are in the domain

$$\{0 \leq t \leq T, 0 \leq S(t) < \infty, 0 \leq J(t) < \infty\}.$$

For sake of simplicity, we will instead of writing $S(t)$ and $J(t)$ we will write S and J The above domain can financially be interpreted as, the issuer of the Asian option is exposed to infinite risk (at least theoretically) as the price of the underlying asset price S can be extremely expensive or cheap at expiry, due to variety of reason such as political instability or pandemic, etc. Hence, the average stock price can go very high or drop. Moreover, instant t signifies that we are interested in determining the put-call parities for the at any instant t prior to expiry T.

5.1. *Computation of Put-call Parity for arithmetic average Asian Option with Fixed Strike Price.*

Suppose $C(t,S,J)$ be premium of the Asian call option with expiry T and fixed strike price K, at instant t prior to expiry, satisfies the following terminal value problem,

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha C}{\partial t^\alpha} + \frac{S-J}{t}\frac{\partial C}{\partial J} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC = 0, \tag{5.1}$$

subject to terminal condition (i.e. payoff of call option),

$$C(T,S(T),J(T)) = (J(T)-K)^+ = \max\{J(T)-K, 0\}.$$

The terminal condition can financially be interpreted as the payoff of the option at maturity whether it is exercises or not. Now similarly, let $P(t, S, J)$ be premium of the Asian put option, contingent upon the same underlying asset and with the same expiry T and same fixed strike price K that of call option described above, at the instant t prior to expiry, satisfied the following terminal value problem,

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^{\alpha}P}{\partial t^{\alpha}} + \frac{S-J}{t}\frac{\partial P}{\partial J} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 P}{\partial S^2} + (r-q)S\frac{\partial P}{\partial S} - rP = 0, \tag{5.2}$$

subject to terminal condition (i.e. payoff of put option),

$$P(T, S(T), J(T)) = (K - J(T))^+ = \max\{K - J(T), 0\}.$$

Recall the following put-call parity satisfied by premiums Call and put options,

$$C(t, S(t), J(t)) - P(t, S(t), J(t)) = Ke^{-r(T-t)} - S(t). \tag{5.3}$$

In the view of last equation, the computation of put-call parity is equivalent (See chapter 9 of [33]) to the explicitly computing the difference $W(t, S, J) := C(t, S, J) - P(t, S, J)$. Indeed, the difference $W(t, S, J)$ satisfies, following terminal-value problem,

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^{\alpha}W}{\partial t^{\alpha}} + \frac{S-J}{t}\frac{\partial W}{\partial J} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 W}{\partial S^2} + (r-q)S\frac{\partial W}{\partial S} - rW = 0, \tag{5.4}$$

$$W(T, S(T), J(T)) = (J(T) - K)^+ - (K - J(T))^+ = J(T) - K. \tag{5.5}$$

In order to solve the above terminal value problem, we first turn into the initial value problem by introducing the transformation $\tau = T - t$. Using property (iii) from Corollary 2.5, it can be easily seen that,

$$\frac{\partial^{\alpha}W}{\partial t^{\alpha}} = -\tau^{\alpha-1}(T-\tau)^{1-\alpha}\frac{\partial^{\alpha}W}{\partial \tau^{\alpha}}. \tag{5.6}$$

On using (5.6) in (5.4), the transformation $\tau = T - t$, and the approximation $\left(1-\frac{\tau}{T}\right)^{-1} \approx 1 + \frac{\tau}{T}$, the terminal value problem (5.4)-(5.4) takes following form,

$$\begin{cases} \Gamma(2-\alpha)\tau^{\alpha}\frac{\partial^{\alpha}W}{\partial\tau^{\alpha}} = \frac{\sigma^2S^2}{2\Gamma^2(1+\alpha)}\tau\frac{\partial^2W}{\partial S^2} + \frac{S-J}{T}\left(\tau + \frac{\tau^2}{T}\right)\frac{\partial W}{\partial J} + (r-q)S\tau\frac{\partial W}{\partial S} - r\tau W. \\ W|_{\tau=0} = J - K. \end{cases} \tag{5.7}$$

Now, let us apply the FRDT Algorithm using its properties mentioned in Theorem 3.2 onto the initial value problem 5.7, set of initial value recursive system,

$$\begin{aligned} \frac{\Gamma(2-\alpha)\Gamma(1+m\alpha)}{\Gamma(1+(m-1)\alpha)}\mathcal{W}_m(S,J) &= \frac{\sigma^2S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2\mathcal{W}_{m-1}}{\partial S^2}(S,J) + \frac{S-J}{T}\frac{\partial\mathcal{W}_{m-1}}{\partial J}(S,J) + \frac{S-J}{T^2}\frac{\partial\mathcal{W}_{m-2}}{\partial J}(S,J) \\ &\quad + (r-q)S\frac{\partial\mathcal{W}_{m-1}}{\partial S}(S,J) - r\mathcal{W}_{m-1}(S,J) \\ W|_{\tau=0} &= \mathcal{W}_0(S,J) = J - K. \end{aligned} \tag{5.8}$$

Here $\mathcal{W}_k(S, J)$, for all $k \in \mathbb{N}$, are t-dimensional spectrum functions of $W(t, S, J)$. Using the known $\mathcal{W}_0(S, J)$, all components $\mathcal{W}_1(S, J), \mathcal{W}_2(S, J), \cdots, \mathcal{W}_n(S, J), \cdots$, etc., can also be interpreted as Adomian polynomials, see [56], and all can be determined through the recursive system (5.8).
For $m = 1$ in the recursive relation (5.8) takes following form,

$$\begin{aligned} \Gamma(2-\alpha)\Gamma(1+\alpha)\mathcal{W}_1(S,J) &= \frac{\sigma^2S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2\mathcal{W}_0}{\partial S^2}(S,J) + \frac{S-J}{T}\frac{\partial\mathcal{W}_0}{\partial J}(S,J) \\ &\quad + (r-q)S\frac{\partial\mathcal{W}_0}{\partial S}(S,J) - r\mathcal{W}_0(S,J). \end{aligned} \tag{5.9}$$

Using initial condition $\mathcal{W}_0(S,J) = (J-K)$, and so $\frac{\partial \mathcal{W}_0}{\partial S} = \frac{\partial^2 \mathcal{W}_0}{\partial S^2} = 0$, and $\frac{\partial \mathcal{W}_0}{\partial J} = 1$. Therefore, from the last equation (5.9) we get,

$$\mathcal{W}_1(S,J) = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{\frac{S-J}{T} - r(J-K)\right\}. \tag{5.10}$$

For $m = 2$, the recursive relation (5.8) turns out to be following,

$$\begin{aligned}\Gamma(2-\alpha)\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\mathcal{W}_2(S,J) &= \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_1}{\partial S^2}(S,J) + \frac{S-J}{T}\frac{\partial \mathcal{W}_1}{\partial J}(S,J)\\ &+ \frac{S-J}{T^2}\frac{\partial \mathcal{W}_0}{\partial J}(S,J) + (r-q)S\frac{\partial \mathcal{W}_1}{\partial S}(S,J) - r\mathcal{W}_1(S,J).\end{aligned} \tag{5.11}$$

Using computed W_1 from equation (5.10), we have computed the following required partial derivatives,

$$\frac{\partial \mathcal{W}_1}{\partial S} = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left(\frac{1}{T}\right), \quad \frac{\partial^2 \mathcal{W}_1}{\partial S^2} = 0, \quad \frac{\partial \mathcal{W}_1}{\partial J} = -\frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left(\frac{1}{T} + r\right), \quad \frac{\partial \mathcal{W}_0}{\partial J} = 1.$$

Using the above partial derivatives in the equation (5.11), we infer that,

$$\mathcal{W}_2(S,J) = \frac{\frac{2rJ}{T} - \frac{S(r^2-q^2)}{T(r-q)} + r^2(J-K) - \frac{S-J}{T^2}}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}\left\{\frac{S-J}{T^2}\right\}. \tag{5.12}$$

Since we have computed Adomian polynomials $\mathcal{W}_0, \mathcal{W}_1$ and $\mathcal{W}_2$, so we can employ inverse FRDT algorithm described in Definition 3.1 and section 3.2 the solution of terminal value problem (5.4) can be recovered by substituting the computed functions $\mathcal{W}_0, \mathcal{W}_1$ and $\mathcal{W}_2$ and so on. Indeed,

$$\begin{aligned}W(\tau,S,J) &= \sum_{m=0}^{\infty}\mathcal{W}_m(S,J)\tau^{\alpha m} = \mathcal{W}_0(S,J) + \mathcal{W}_1(S,J)\tau^{\alpha} + \mathcal{W}_2(S,J)\tau^{2\alpha} + \cdots,\\ &= (J-K) + \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{\frac{S-J}{T} - r(J-K)\right\}\tau^{\alpha}\\ &+ \left[\frac{\frac{2rJ}{T} - \frac{S(r^2-q^2)}{T(r-q)} + r^2(J-K) - \frac{S-J}{T^2}}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}\left\{\frac{S-J}{T^2}\right\}\right]\tau^{2\alpha} + \cdots.\end{aligned}$$

Returning $\tau = T - t$, and restricting ourselves to second-order approximation in time, we get the following approximate solution to terminal value problem (5.4)-(5.4),

$$\begin{aligned}W(t,S,J) &\approx (J-K) + \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{\frac{S-J}{T} - r(J-K)\right\}(T-t)^{\alpha}\\ &+ \left[\frac{\frac{2rJ}{T} - \frac{S(r^2-q^2)}{T(r-q)} + r^2(J-K) - \frac{S-J}{T^2}}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}\left\{\frac{S-J}{T^2}\right\}\right](T-t)^{2\alpha}.\end{aligned} \tag{5.13}$$

Consider a six-month Asian call and put options option on stock. For $S_0 = K = \$145, r = 6, q = 3$ and $\sigma = 29.5$ following graph of solution of (5.14) has been computed using Mathematica over the interval $(t,S) \in [0,200] \times [0,150]$, which describes the evolution of prices of Asian call option. Finally the required following form of put-call parity for arithmetic average Asian options with Fixed strike price has been explicitly obtained by substituting the $W(t,S,J)$ into (5.3),

$$\begin{aligned}Ke^{-r(T-t)} - S &= (J-K) + \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{\frac{S-J}{T} - r(J-K)\right\}(T-t)^{\alpha}\\ &+ \left[\frac{\frac{2rJ}{T} - \frac{S(r^2-q^2)}{T(r-q)} + r^2(J-K) - \frac{S-J}{T^2}}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}\left\{\frac{S-J}{T^2}\right\}\right](T-t)^{2\alpha}.\end{aligned} \tag{5.14}$$

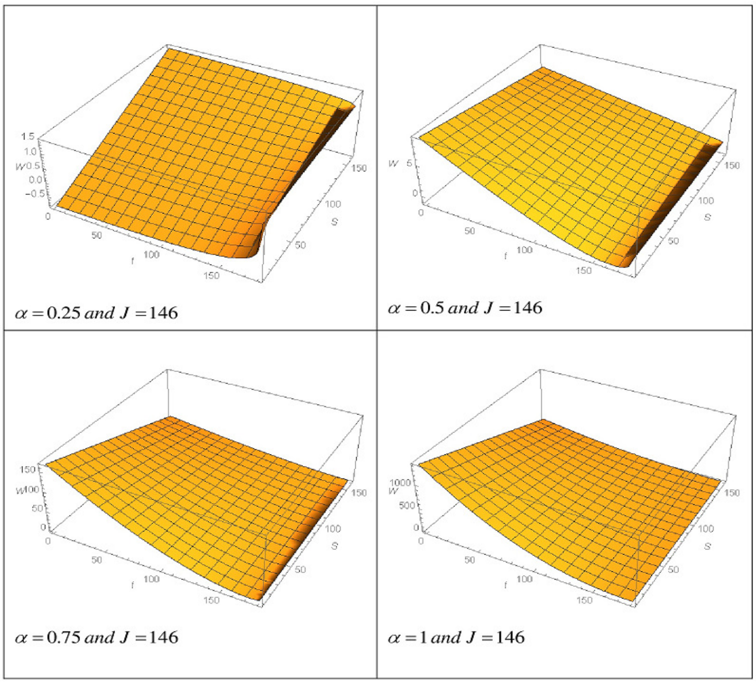

Figure 1: Plot of $W(t, S, J)$ when $S_0 = K = 145, r = 6, q = 3$ and $\sigma = 29.5$

The set of surfaces in Fig. 1, computed through Mathematica, show the evolution of $W(t, S, J)$ when time t and stock price S are changing. The results are consistent with difference of premiums computed in [33] (See chapter 9).

Keep in view that from an economic aspect this put-call is extremely important because its violation will reflect there is an arbitrage in the derivative market, hence the investor can create a self-financing strategy of investing into bonds and stocks that will allow the investor to have a risk-free profit with an actual investment.

5.2. Computation of Put-call Parity for arithmetic average Asian Option with Floating Strike Price.

In this subsection we will work closely on the lines of the subsection 5.1. Let $C(t, S, J)$ and $P(t, S, J)$ be premium of the Asian call and put options respectively, with same expiry T and same floating strike price $S(T)$ i.e. value of the stock at the maturity (which is obviously unknown at any instant strictly prior to T), and J is the arithmetic average of underling asset over the interval $[0, T]$. Indeed, $C(t, S, J)$ and $P(t, S, J)$ satisfies the fractional evolution equations respectively equations (5.1) and (5.2). The key difference here is that the terminal condition i.e. payoff for for call option at maturity T is,

$$C(T, S(T), J(T)) = (J(T) - S(T))^+ = \max\{J(T) - S(T), 0\}, \tag{5.15}$$

and that of put option is,

$$P(T, S(T), J(T)) = (S(T) - J(T))^+ = \max\{S(T) - J(T), 0\}. \tag{5.16}$$

Set $W(t, S, J) := C(t, S, J) - P(t, S, J)$. Then in $\{0 \le S < \infty, 0 \le J < \infty, 0 \le t \le T\}$, W satisfies following terminal value problem,

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha W}{\partial t^\alpha} + \frac{S-J}{t}\frac{\partial W}{\partial J} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 W}{\partial S^2} + (r-q)S\frac{\partial W}{\partial S} - rW = 0, \tag{5.17}$$

$$W(T, S(T), J(T)) = (J(T) - S(T))^{+} - (S(T) - J(T))^{+} = J(T) - S(T). \tag{5.18}$$

Now arguing on the same lines section 5.1 we can transform the terminal value problem (5.17)-(5.18), using the transformation $\tau = T - t$, and the approximation $\left(1 - \frac{\tau}{T}\right)^{-1} \approx 1 + \frac{\tau}{T}$, into following initial value problem,

$$\begin{cases} \Gamma(2-\alpha)\tau^{\alpha}\frac{\partial^{\alpha} W}{\partial \tau^{\alpha}} = \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\tau\frac{\partial^2 W}{\partial S^2} + \frac{S-J}{T}\left(\tau + \frac{\tau^2}{T}\right)\frac{\partial W}{\partial J} + (r-q)S\tau\frac{\partial W}{\partial S} - r\tau W. \\ W|_{\tau=0} = J - S. \end{cases} \tag{5.19}$$

Now, let us apply the FRDT Algorithm using its properties mentioned in Theorem 3.2 onto the initial value problem (5.44), set of initial value recursive system,

$$\begin{aligned} \frac{\Gamma(2-\alpha)\Gamma(1+m\alpha)}{\Gamma(1+(m-1)\alpha)}\mathcal{W}_m(S,J) &= \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_{m-1}}{\partial S^2}(S,J) + \frac{S-J}{T}\frac{\partial \mathcal{W}_{m-1}}{\partial J}(S,J) + \frac{S-J}{T^2}\frac{\partial \mathcal{W}_{m-2}}{\partial J}(S,J) \\ &\quad + (r-q)S\frac{\partial \mathcal{W}_{m-1}}{\partial S}(S,J) - r\mathcal{W}_{m-1}(S,J) \\ W|_{\tau=0} &= \mathcal{W}_0(S,J) = J - S. \end{aligned} \tag{5.20}$$

For $m = 1$, recurrence relation (5.20) leads to following which may allows us to compute W_1,

$$\Gamma(2-\alpha)\Gamma(1+\alpha)\mathcal{W}_1(S,J) = \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_0}{\partial S^2}(S,J) + \frac{S-J}{T}\frac{\partial \mathcal{W}_0}{\partial J}(S,J) + (r-q)S\frac{\partial \mathcal{W}_0}{\partial S}(S,J) - r\mathcal{W}_0(S,J), \tag{5.21}$$

Since $\mathcal{W}_0(S,J) = (J - S)$, so partial derivatives $\frac{\partial \mathcal{W}_0}{\partial S} = -1$, $\frac{\partial \mathcal{W}_0^2}{\partial S^2} = 0$, and $\frac{\partial \mathcal{W}_0}{\partial J} = 1$. Using these derivatives in equation (5.21) we get,

$$\mathcal{W}_1(S,J) = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{qS - rJ + \frac{S-J}{T}\right\}. \tag{5.22}$$

For $m = 2$, recurrence relation (5.20) gives,

$$\begin{aligned} \Gamma(2-\alpha)\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\mathcal{W}_2(S,J) &= \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_1}{\partial S^2}(S,J) + \frac{S-J}{T}\frac{\partial \mathcal{W}_1}{\partial J}(S,J) + \frac{S-J}{T^2}\frac{\partial \mathcal{W}_0}{\partial J}(S,J) \\ &\quad + (r-q)S\frac{\partial \mathcal{W}_1}{\partial S}(S,J) - r\mathcal{W}_1(S,J). \end{aligned} \tag{5.23}$$

Now, let us use (5.22) to compute the partial derivatives required to calculus the W_2,

$$\frac{\partial \mathcal{W}_1}{\partial S} = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left(\frac{1}{T} + q\right), \quad \frac{\partial^2 \mathcal{W}_1}{\partial S^2} = 0, \quad \frac{\partial \mathcal{W}_1}{\partial J} = -\frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left(\frac{1}{T} + r\right).$$

Substituting all above partial derivatives into equation (5.23) we get,

$$\mathcal{W}_2(S,J) = \frac{1}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)}\left\{r^2 J - q^2 S + \frac{J-S}{T^2} - \frac{S\left(r^2 - q^2\right)}{T(r-q)} + \frac{2rJ}{T}\right\} + \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}\left\{\frac{S-J}{T^2}\right\}. \tag{5.24}$$

Using Adomian polynomials $\mathcal{W}_0, \mathcal{W}_1$ and $\mathcal{W}_2$, and employing inverse FRDT algorithm described in Definition 3.1 and section 3.2 the solution of terminal value problem (5.17) can be recovered by substituting the computed functions $\mathcal{W}_0, \mathcal{W}_1$ and $\mathcal{W}_2$ in the following solution series,

$$\begin{aligned} W(\tau,S,J) &= \sum_{m=0}^{\infty}\mathcal{W}_m(S,J)\tau^{\alpha m} = \mathcal{W}_0(S,J) + \mathcal{W}_1(S,J)\tau^{\alpha} + \mathcal{W}_2(S,J)\tau^{2\alpha} + \cdots, \\ &= (J-S) + \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{qS - rJ + \frac{S-J}{T}\right\}\tau^{\alpha} \\ &\quad + \left[\frac{1}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)}\left\{r^2 J - q^2 S + \frac{J-S}{T^2} - \frac{S\left(r^2-q^2\right)}{T\left(r-q\right)} + \frac{2rJ}{T}\right\} + \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}\left\{\frac{S-J}{T^2}\right\}\right]\tau^{2\alpha} + \cdots. \end{aligned}$$

Substituting back $\tau = T - t$, and on quadratic truncation the we get following approximate analytic solution,

$$W(t,S,J) \approx (J-S) + \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{rJ - qS - \frac{S-J}{T}\right\}(T-t)^{\alpha}$$
$$+\left[\frac{1}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)}\left\{r^2J - q^2S + \frac{J-S}{T^2} - \frac{S(r^2-q^2)}{T(r-q)} + \frac{2rJ}{T}\right\} + \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}\left\{\frac{S-J}{T^2}\right\}\right]\tau^{2\alpha}. \tag{5.25}$$

Consider a six-month Asian call and put options option on stock. For $S_0 = K = \$145, r = 6, q = 3$ and $\sigma = 29.5$ following (see next page) graph of solution of (5.14) has been computed using Mathematica over the interval $(t,S) \in [0,200] \times [0,150]$, which describes the evolution of prices of Asian call option. Finally

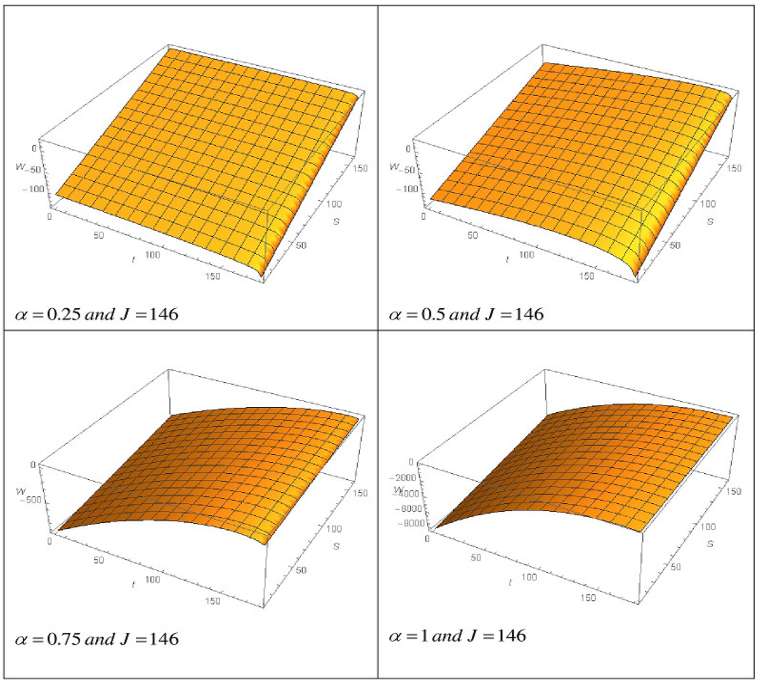

Figure 2: Plot of $W(t,S,J)$ when $S_0 = K = 145, r = 6, q = 3$ and $\sigma = 29.5$

The set of surfaces in Fig. 1, computed through Mathematica, show the evolution of $W(t,S,J)$ when time t and stock price S are changing. The results are consistent with difference of premiums computed in [33] (See chapter 9).

the required following form of put-call parity for arithmetic average Asian options with Fixed strike price has been explicitly obtained by substituting the $W(t,S,J)$ into (5.3),

$$Se^{-r(T-t)} - S = (J-S) + \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{rJ - qS - \frac{S-J}{T}\right\}(T-t)^{\alpha}$$
$$+\left[\frac{1}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)}\left\{r^2J - q^2S + \frac{J-S}{T^2} - \frac{S(r^2-q^2)}{T(r-q)} + \frac{2rJ}{T}\right\} + \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}\left\{\frac{S-J}{T^2}\right\}\right]\tau^{2\alpha}. \tag{5.26}$$

5.3. Call-Put Parity for Time Fractional Geometric Average Asian Option with Fixed Strike Price

Let $C(t,S,J)$ be premium of the Asian call option whose payoff at expiry T depends on geometric average of underlying asset on $[0,T]$. Moreover the strike price K is fixed. Then we shown by no arbitrage argument in section 4 that any instant $t \in [0, T]$, satisfies the following fractional terminal value problem in the domain domain $\{0 \leq S < \infty, 0 \leq J < \infty, 0 \leq t \leq T\}$ and for $0 < \alpha \leq 1$: ,

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha C}{\partial t^\alpha} + J\frac{\ln S - \ln J}{t}\frac{\partial C}{\partial J} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC = 0, \tag{5.27}$$

subject to terminal condition (i.e. payoff of call option),

$$C(T, S(T), J(T)) = (J(T) - K)^+ = \max\{J(T) - K, 0\}.$$

The terminal condition can financially be interpreted as the payoff of the option at maturity whether it is exercises or not. Now similarly, let $P(t, S, J)$ be premium of the Asian put option, contingent upon the same underlying asset and with the same expiry T and same fixed strike price K that of call option described above, at the instant t prior to expiry, satisfied the following terminal value problem in the domain domain $\{0 \leq S < \infty, 0 \leq J < \infty, 0 \leq t \leq T\}$ and for $0 < \alpha \leq 1$:

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha P}{\partial t^\alpha} + J\frac{\ln S - \ln J}{t}\frac{\partial P}{\partial J} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial P}{\partial S} - rP = 0, \tag{5.28}$$

subject to terminal condition (i.e. payoff of put option),

$$P(T, S(T), J(T)) = (K - J(T))^+ = \max\{K - J(T), 0\}.$$

Set $W(t, S, J) := C(t, S, J) - P(t, S, J)$. Therefore, in $\{0 \leq S < \infty, 0 \leq J < \infty, 0 \leq t \leq T\}$. Then W satisfies (4.16)

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha W}{\partial t^\alpha} + J\frac{\ln S - \ln J}{t}\frac{\partial W}{\partial J} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 W}{\partial S^2} + (r-q)S\frac{\partial W}{\partial S} - rW = 0, \tag{5.29}$$

$$W(T, S, J) = (J - K)^+ - (K - J)^+ = J - K. \tag{5.30}$$

Set $\tau = T - t$ and using approximation $\left(1 - \frac{\tau}{T}\right)^{-1} \approx 1 + \frac{\tau}{T}$, we will lead to the following evolution equation,

$$\begin{aligned}
\Gamma(2-\alpha)\tau^\alpha\frac{\partial^\alpha W}{\partial \tau^\alpha} &= \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\tau\frac{\partial^2 W}{\partial S^2} + J\frac{\ln S - \ln J}{T}\tau\frac{\partial W}{\partial J} + J\frac{\ln S - \ln J}{T^2}\tau^2\frac{\partial W}{\partial J} \\
&\quad + (r-q)S\tau\frac{\partial W}{\partial S} - r\tau W, &(5.31)\\
W|_{\tau=0} &= J - K. &(5.32)
\end{aligned}$$

Next, the application of FRDT Algorithm using its properties mentioned in Theorem 3.2 onto the initial value problem (5.31)-(5.32) transforms into set of initial value recursive system,

$$\begin{aligned}
\frac{\Gamma(2-\alpha)\Gamma(1+m\alpha)}{\Gamma(1+(m-1)\alpha)}\mathcal{W}_m(S, J) &= \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_{m-1}}{\partial S^2}(S, J) + J\frac{\ln S - \ln J}{T}\frac{\partial \mathcal{W}_{m-1}}{\partial J}(S, J) \\
&\quad + J\frac{\ln S - \ln J}{T^2}\frac{\partial \mathcal{W}_{m-2}}{\partial J}(S, J) + (r-q)S\frac{\partial \mathcal{W}_{m-1}}{\partial S}(S, J) - r\mathcal{W}_{m-1}(S, J), \\
W|_{\tau=0} &= \mathcal{W}_0(S, J) = J - K. &(5.33)
\end{aligned}$$

For $m = 1$, recurrence relation (5.33) leads to following which may allows us to compute W_1,

$$\begin{aligned}
\Gamma(2-\alpha)\Gamma(1+\alpha)\mathcal{W}_1(S, J) &= \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_0}{\partial S^2}(S, J) + J\frac{\ln S - \ln J}{T}\frac{\partial \mathcal{W}_0}{\partial J}(S, J) + 0 \\
&\quad + (r-q)S\frac{\partial \mathcal{W}_0}{\partial S}(S, J) - r\mathcal{W}_0(S, J). &(5.34)
\end{aligned}$$

Using $\mathcal{W}_0(S,J) = (J-K)$, we compute following,

$$\frac{\partial \mathcal{W}_0}{\partial S} = \frac{\partial^2 \mathcal{W}_0}{\partial S^2} = 0, \quad \frac{\partial \mathcal{W}_0}{\partial J} = 1.$$

And therefore,

$$\mathcal{W}_1(S,J) = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{J\frac{\ln S - \ln J}{T} - r(J-K)\right\}.$$

By setting $m = 2$, in recursive equation (5.33) we get,

$$\begin{aligned}\frac{\Gamma(2-\alpha)\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\mathcal{W}_2(S,J) \quad &= \quad \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_1}{\partial S^2}(S,J) + J\frac{\ln S - \ln J}{T}\frac{\partial \mathcal{W}_1}{\partial J}(S,J) + J\frac{\ln S - \ln J}{T^2}\frac{\partial \mathcal{W}_0}{\partial J}(S,J) \\ &\quad + (r-q)S\frac{\partial \mathcal{W}_1}{\partial S}(S,J) - r\mathcal{W}_1(S,J).\end{aligned} \tag{5.35}$$

Now using $\mathcal{W}_1(S,J)$ from above, we compute following,

$$\frac{\partial \mathcal{W}_1}{\partial S} = \frac{J}{\Gamma(2-\alpha)\Gamma(1+\alpha)\,TS}, \quad \frac{\partial^2 \mathcal{W}_1}{\partial S^2} = -\frac{J}{\Gamma(2-\alpha)\Gamma(1+\alpha)\,TS^2},$$

$$\frac{\partial \mathcal{W}_1}{\partial J} = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{\frac{\ln S - \ln J}{T} - \left(r + \frac{1}{T}\right)\right\}.$$

Using these partial derivatives in equation (5.35), if follows that,

$$\begin{aligned}\mathcal{W}_2(S,J) \quad &= \quad \frac{1}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)}\left[\begin{array}{c}\frac{J}{T}\left\{-\frac{\sigma^2}{2\Gamma^2(1+\alpha)} + \frac{(\ln S - \ln J)^2}{T} - \frac{\ln S - \ln J}{T} - 2r(\ln S - \ln J) + (r-q)\right\} \\ +r^2(J-K)\end{array}\right] \\ &\quad + \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}J\frac{\ln S - \ln J}{T^2}.\end{aligned} \tag{5.36}$$

By using inverse fractional differential reduced transform and then substituting previously computed $\mathcal{W}_0(S,J), \mathcal{W}_1(S,J)$ and $\mathcal{W}_2(S,J)$ in solution series, leads to following solution,

$$\begin{aligned}W(\tau,S,J) \quad &= \quad \sum_{m=0}^{\infty}\mathcal{W}_m(S,J)\,\tau^{\alpha m} = \mathcal{W}_0(S,J) + \mathcal{W}_1(S,J)\,\tau^{\alpha} + \mathcal{W}_2(S,J)\,\tau^{2\alpha} + \cdots, \\ &= \quad (J-K) + \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{J\frac{\ln S - \ln J}{T} - r(J-K)\right\}\tau^{\alpha} + \frac{1}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)}\times \\ &\left[\frac{J}{T}\left\{-\frac{\sigma^2}{2\Gamma^2(1+\alpha)} + \frac{(\ln S - \ln J)^2}{T} - \frac{\ln S - \ln J}{T} - 2r(\ln S - \ln J) + (r-q)\right\} + r^2(J-K)\right]\tau^{2\alpha} \\ &+ \frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}J\frac{\ln S - \ln J}{T^2}\tau^{2\alpha} + \cdots.\end{aligned} \tag{5.37}$$

On replacing back $\tau = T - t$, and ignoring higher power that 2 of t we get following explicit approximate solution,

$$\begin{aligned}W(t,S,J) \quad &\approx \quad (J-K) + \frac{(T-t)^{\alpha}}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{J\frac{\ln S - \ln J}{T} - r(J-K)\right\} + \frac{(T-t)^{2\alpha}}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)}\times \\ &\quad \left[\frac{J}{T}\left\{-\frac{\sigma^2}{2\Gamma^2(1+\alpha)} + \frac{(\ln S - \ln J)^2}{T} - \frac{\ln S - \ln J}{T} - 2r(\ln S - \ln J) + (r-q)\right\} + r^2(J-K)\right] \\ &\quad + \frac{\Gamma(1+\alpha)(T-t)^{2\alpha}}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}J\frac{\ln S - \ln J}{T^2}.\end{aligned} \tag{5.38}$$

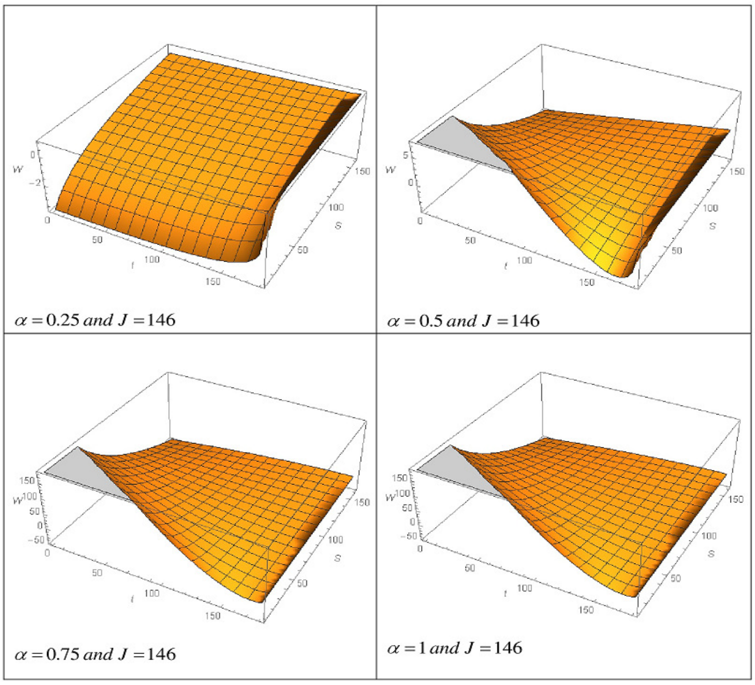

Figure 3: Plot of $W(t, S, J)$ when $S_0 = K = 145, r = 6, q = 3$ and $\sigma = 29.5$

The set of surfaces in Fig. 3, computed through Mathematica, show the evolution of $W(t, S, J)$ when time t and stock price S are changing. The results are consistent with difference of premiums computed in [33] (See chapter 9).

Consider a six-month Asian call and put options option on stock. For $S_0 = K = \$145, r = 6, q = 3$ and $\sigma = 29.5$ following graph of solution of (5.38) has been computed using Mathematica over the interval $(t, S) \in [0, 200] \times [0, 150]$, which describes the evolution of prices of Asian call option. Finally the required following form of put-call parity for geometric average Asian options with Fixed strike price has been explicitly obtained by substituting the $W(t, S, J)$ into (5.3),

$$\begin{aligned} Ke^{-r(T-t)} - S &= (J-K) + \frac{(T-t)^{\alpha}}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{J\frac{\ln S - \ln J}{T} - r(J-K)\right\} + \frac{(T-t)^{2\alpha}}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)} \times \\ &\left[\ \frac{J}{T}\left\{-\frac{\sigma^2}{2\Gamma^2(1+\alpha)} + \frac{(\ln S - \ln J)^2}{T} - \frac{\ln S - \ln J}{T} - 2r(\ln S - \ln J) + (r-q)\right\} + r^2(J-K)\ \right] \\ &+ \frac{\Gamma(1+\alpha)(T-t)^{2\alpha}}{\Gamma(2-\alpha)\Gamma(1+2\alpha)} J \frac{\ln S - \ln J}{T^2}. \end{aligned} \tag{5.39}$$

The violation of last inequality will reflect there is an arbitrage opportunities available that can be exploited to create zero investment strategies to earn *free lunch*.

5.4. Call-Put Parity for Time Fractional Geometric Average Asian Option with Floating Strike Price

Let $C(t, S, J)$ and $P(t, S, J)$ be premium of the Asian call and put options respectively, with share expiry T and shared floating strike price $S(T)$, and J is the geometric average of underling asset from inception of contract to its expiry. Indeed, $C(t, S, J)$ and $P(t, S, J)$ satisfies the fractional evolution equations respectively equations (5.26) and (5.27) subject to following payoff for call option at maturity T is,

$$C(T, S(T), J(T)) = (J(T) - S(T))^{+} = \max\{J(T) - S(T), 0\}, \tag{5.40}$$

and subject to following floating payoff of put option at expiry ,

$$P(T, S(T), J(T)) = (S(T) - J(T))^+ = \max\{S(T) - J(T), 0\}. \tag{5.41}$$

Set $W(t,S,J) := C(t,S,J) - P(t,S,J)$. Then in $\{0 \le S < \infty, 0 \le J < \infty, 0 \le t \le T\}$, W satisfies following fractional terminal value problem,

$$\Gamma(2-\alpha)t^{\alpha-1}\frac{\partial^\alpha W}{\partial t^\alpha} + \frac{S-J}{t}\frac{\partial W}{\partial J} + \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 W}{\partial S^2} + (r-q)S\frac{\partial W}{\partial S} - rW = 0, \tag{5.42}$$

$$W(T,S(T),J(T)) = (J(T) - S(T))^+ - (S(T) - J(T))^+ = J(T) - S(T). \tag{5.43}$$

Now arguing on the same lines section 5.3 we can transform the terminal value problem (5.17)-(5.18), using the transformation $\tau = T - t$, and the approximation $\left(1 - \frac{\tau}{T}\right)^{-1} \approx 1 + \frac{\tau}{T}$, into following initial value problem,

$$\begin{cases} \Gamma(2-\alpha)\tau^\alpha \frac{\partial^\alpha W}{\partial \tau^\alpha} = \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\tau\frac{\partial^2 W}{\partial S^2} + \frac{\ln S - \ln J}{T}\left(\tau + \frac{\tau^2}{T}\right)\frac{\partial W}{\partial J} + (r-q)S\tau\frac{\partial W}{\partial S} - r\tau W. \\ W|_{\tau=0} = J - S. \end{cases} \tag{5.44}$$

Taking Differential Reduced Transform gives the following recursive relation,

$$\begin{cases} \Gamma(2-\alpha)\frac{\Gamma(1+m\alpha)}{\Gamma(1+(m-1)\alpha)}\mathcal{W}_m(S,J) = \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_{m-1}}{\partial S^2}(S,J) + J\frac{\ln S - \ln J}{T}\frac{\partial \mathcal{W}_{m-1}}{\partial J}(S,J) \\ +J\frac{\ln S - \ln J}{T^2}\frac{\partial \mathcal{W}_{m-2}}{\partial J}(S,J) + (r-q)S\frac{\partial \mathcal{W}_{m-1}}{\partial S}(S,J) - r\mathcal{W}_{m-1}(S,J), \text{ for } m \ge 1, \text{ else } 0. \\ W|_{\tau=0} = \mathcal{W}_0(S,J) = S - J. \end{cases} \tag{5.45}$$

For $m = 1$ above recursive relation (5.43), we get,

$$\begin{aligned} \Gamma(2-\alpha)\Gamma(1+\alpha)\mathcal{W}_1(S,J) &= \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_0}{\partial S^2}(S,J) + J\frac{\ln S - \ln J}{T}\frac{\partial \mathcal{W}_0}{\partial J}(S,J) \\ &\quad +(r-q)S\frac{\partial \mathcal{W}_0}{\partial S}(S,J) - r\mathcal{W}_0(S,J). \end{aligned} \tag{5.46}$$

Using $\mathcal{W}_0(S,J) = J - S$, we have the following partial derivatives,

$$\frac{\partial \mathcal{W}_0}{\partial S} = -1, \quad \frac{\partial^2 \mathcal{W}_0}{\partial S^2} = 0, \quad \frac{\partial \mathcal{W}_0}{\partial J} = 1. \tag{5.47}$$

On substituting the above partial derivatives in relation (5.48),

$$\mathcal{W}_1(S,J) = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{qS - rJ + J\frac{\ln S - \ln J}{T}\right\}.$$

On plugging $m = 2$ into equation (5.45), we have following relation,

$$\begin{aligned} \frac{\Gamma(2-\alpha)\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\mathcal{W}_2(S,J) &= \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 \mathcal{W}_1}{\partial S^2}(S,J) + J\frac{\ln S - \ln J}{T}\frac{\partial \mathcal{W}_1}{\partial J}(S,J) \\ &\quad +J\frac{\ln S - \ln J}{T^2}\frac{\partial \mathcal{W}_0}{\partial J}(S,J) + (r-q)S\frac{\partial \mathcal{W}_1}{\partial S}(S,J) - r\mathcal{W}_1(S,J). \end{aligned} \tag{5.48}$$

Using expression for $\mathcal{W}_1$ from (5.47), we can compute the following required partial derivatives, to compute $\mathcal{W}_2$,

$$\frac{\partial \mathcal{W}_1}{\partial S} = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{q + \frac{J}{ST}\right\}, \quad \frac{\partial^2 \mathcal{W}_1}{\partial S^2} = -\frac{J}{\Gamma(2-\alpha)\Gamma(1+\alpha)TS^2},$$

$$\frac{\partial \mathcal{W}_1}{\partial J} = \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{\frac{\ln S - \ln J}{T} - \left(r + \frac{1}{T}\right)\right\}.$$

Using these partial derivatives in (5.48),

$$\begin{aligned}\mathcal{W}_2(S,J) &= \frac{J}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)\,T}\left\{-\frac{\sigma^2}{2\Gamma^2(1+\alpha)}+\frac{(\ln S-\ln J)^2}{T}-\frac{\ln S-\ln J}{T}-2r(\ln S-\ln J)+r-q\right\}\\ &\quad \frac{1}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)}(+r^2J-q^2S)-\frac{\Gamma(1+\alpha)}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}J\frac{\ln S-\ln J}{T^2}. \end{aligned} \tag{5.49}$$

By using inverse Differential Reduced Transform and on replacing $\tau = T - t$, an approximate solution is

$$\begin{aligned} W(S,J,t) &\approx (J-S)+\frac{(T-t)^{\alpha}}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{qS-rJ+J\frac{\ln S-\ln J}{T}\right\}+\frac{(T-t)^{2\alpha}}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)}\\ &\quad\times\left[\ \frac{J}{T}\left\{-\frac{\sigma^2}{2\Gamma^2(1+\alpha)}+\frac{(\ln S-\ln J)^2}{T}+\frac{\ln S-\ln J}{T}+2r(\ln S-\ln J)-(r-q)\right\}-r^2J+q^2S\ \right]\\ &\quad-\frac{\Gamma(1+\alpha)(T-t)^{2\alpha}}{\Gamma(2-\alpha)\Gamma(1+2\alpha)}J\frac{\ln S-\ln J}{T^2}. \end{aligned} \tag{5.50}$$

Consider a six-month Asian call and put options option on stock. For $S_0 = K = \$145, r = 6, q = 3$ and $\sigma = 29.5$ following graph of solution of (5.50) has been computed using Mathematica over the interval $(t,S) \in [0,200]\times[0,150]$, which describes the evolution of prices of Asian call option. Finally the required

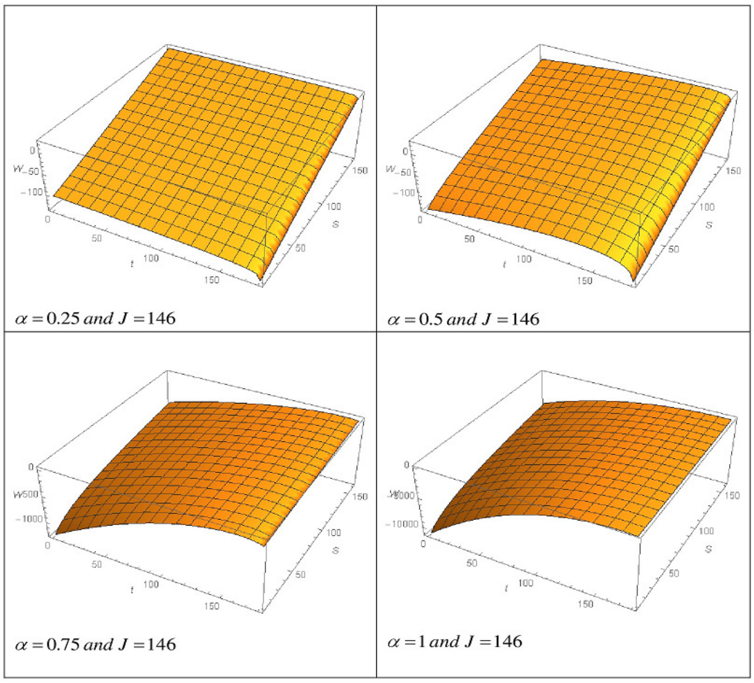

Figure 4: Plot of $W(t,S,J)$ when $S_0 = K = 145, r = 6, q = 3$ and $\sigma = 29.5$

The set of surfaces in Fig. 4, computed through Mathematica, show the evolution of $W(t,S,J)$ when time t and stock price S are changing. The results are consistent with difference of premiums computed in [33] (See chapter 9).

following form of put-call parity for geometric average Asian options with Fixed strike price has been explicitly obtained by substituting the $W(t, S, J)$ into (5.3),

$$\begin{aligned} Se^{-r(T-t)} - S &= (J-S) + \frac{(T-t)^{\alpha}}{\Gamma(2-\alpha)\Gamma(1+\alpha)}\left\{qS - rJ + J\frac{\ln S - \ln J}{T}\right\} + \frac{(T-t)^{2\alpha}}{\Gamma^2(2-\alpha)\Gamma(1+2\alpha)} \\ &\times\left[\ \frac{J}{T}\left\{-\frac{\sigma^2}{2\Gamma^2(1+\alpha)} + \frac{(\ln S-\ln J)^2}{T} + \frac{\ln S-\ln J}{T} + 2r(\ln S - \ln J) - (r-q)\right\} - r^2 J + q^2 S\ \right] \\ &-\frac{\Gamma(1+\alpha)(T-t)^{2\alpha}}{\Gamma(2-\alpha)\Gamma(1+2\alpha)} J\frac{\ln S - \ln J}{T^2}. \end{aligned} \tag{5.51}$$

The violation of last inequality will reflect there is an arbitrage opportunities available that can be exploited to create zero investment strategies to earn *free lunch*.

6. Conclusion

In this paper, for the first time we have applied fractional differential transform algorithm to price the put-call parities for the Asian style exotic options, with both geometric and arithmetic averages, by solving effectively the governing fractional evolution equations. Based on the No-arbitrage principle (fundamental theorem of asset pricing), we have derived the governing equation to price the Asian style options, where the underlying asset price is driven fractional stochastic differential equation. Another significant theoretical contribution is that we have proved a new result (Theorem 3.3) for the convergence of the fractional reduced differential transform algorithm for the two independent variables.
There can several future directions in which work this work can be extended, for instance,
1. A similar derivation of fractional evolution equations can be done for more exotic options such as Rainbow options, Quanto options, Barrier options, etc, under the Black-Scholes framework, contingent upon underlying asset is driven fractional stochastic differential equation.
2. Similarly to this study, put-call parities for more exotic options can be developed.
3. An empirical study can be done for real derivative market data through our developed formula to crunch that whether arbitrage exists or not in the derivative market.

Conflict of interest The authors declare no conflict of interest.

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