



Unbounded Order-Norm Continuous and Unbounded Norm Continuous Operator

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Abstract. A continuous operator T between two normed vector lattices E and F is called unbounded order-norm continuous whenever $x_\alpha \xrightarrow{uo} 0$ implies $\|Tx_\alpha\| \rightarrow 0$, for each norm bounded net $(x_\alpha)_\alpha \subseteq E$. Let E and F be two Banach lattices. A continuous operator $T : E \rightarrow F$ is called unbounded norm continuous, if for each norm bounded net $(x_\alpha)_\alpha \subseteq E$, $x_\alpha \xrightarrow{un} 0$ implies $Tx_\alpha \xrightarrow{un} 0$. In this manuscript, we study some properties of these classes of operators and investigate their relationships with the other classes of operators.

1. Introduction

Let E and F be two normed vector lattices. In the second section of this manuscript, we will introduce, study and investigate on continuous operators $T : E \rightarrow F$ which carry every norm bounded and unbounded order convergent net into a norm convergent net. The collection of these operators is called the class of unbounded order-norm continuous operators between two normed vector lattices E and F and will be denoted by $L_{uon}(E, F)$. On the other hand, unbounded order convergence, in general is not topological, so by using an operator between two normed vector lattices we carry unbounded order convergence into norm convergence. In the third section, we will introduce a new classification of operators named as *un*-continuous operators and we will investigate on some properties of them and their relationships with other classifications of operators.

In summary, our motivation to write this article is as follows:

- (a) By Proposition 2.6 of [7] and Theorem 18 of [21], a continuous operator T between two Banach lattices E and F is M -weakly compact iff it is uaw -Dunford-Pettis iff for each norm bounded uo -null sequence $(x_n) \subseteq E$, $T(x_n) \xrightarrow{\|\cdot\|} 0$. Therefore, the concept of unbounded order-norm continuous operators introduced in this article is a new equivalent definition for M -weakly compact operators and uaw -Dunford-Pettis operators by using of the concept of unbounded order convergent.
- (b) It is easy to see that for a normed lattice E , if $I \in L_{uon}(E, E)$, then E has order continuous norm. So, the concept of unbounded order-norm continuous operators can be seen as a generalization of the order continuous norm that has an important role for studying of vector lattices.

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2. (a) If G is a sublattice of Banach lattice E and net $(x_\alpha) \subseteq G$ is *un*-convergent in G , then it is not necessarily *un*-convergent in E . We were interested in defining an operator that maintains *un*-convergence from G to E and thus between two arbitrary Banach lattices E and F .
- (b) By Theorem 2.3 of [13], if Banach lattice E has strong unit, then norm topology is equivalent to *un*-topology. Therefore, the new concept of *un*-continuous operator introduced in this article is similar to the concept of continuous operator but for *un*-convergent nets. In [13], authors introduced *un*-compact operators and therefore we decided to compare them with *un*-continuous operators.

To state our results, we need to fix some notations and recall some definitions. Throughout this paper, the subset $E^+ = \{x \in E : x \geq 0\}$ of vector lattice E is called the positive cone of E and the elements of E^+ are called the positive elements of E . A subset $A \subseteq E$ is called order bounded if there exists $a, b \in E$ such that $A \subseteq [a, b]$ where $[a, b] = \{x \in E : a \leq x \leq b\}$. An operator $T : E \rightarrow F$ between two vector lattices is said to be order bounded if it maps order bounded subsets of E to order bounded subsets of F . If E is a normed space, then E' is the topological dual space of E and $T' : F' \rightarrow E'$ is the topological adjoint of continuous operator $T : E \rightarrow F$ between two normed spaces. In a vector lattice E , two elements x and y are said to be disjoint (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$. If A is a nonempty subset of vector lattice E , then its disjoint complement A^d is defined by $A^d = \{x \in E : x \perp y \text{ for all } y \in A\}$. A sequence $(x_n)_n$ in a vector lattice E is said to be disjoint whenever $n \neq m$ implies $x_n \perp x_m$. An order closed ideal of E is referred to as a band. A band B in a vector lattice E that satisfies $E = B \oplus B^d$ is referred to as a projection band. Let B be a projection band in a vector lattice E . Thus every vector $x \in E$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^d$. Then it is easy to see that a projection $P_B : E \rightarrow E$ is defined via the formula $P_B(x) = x_1$. Clearly, P_B is a positive projection. Any projection of the form P_B is called a band projection.

Let E be a vector lattice and $x \in E$. A net $(x_\alpha)_{\alpha \in A} \subseteq E$ is said to be:

- **order convergent** to x if there is a net $(z_\beta)_{\beta \in B}$ in E such that $z_\beta \downarrow 0$ and for every $\beta \in B$, there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq z_\beta$ whenever $\alpha \geq \alpha_0$. We denote this convergence by $x_\alpha \xrightarrow{o} x$ and write that $(x_\alpha)_\alpha$ is *o*-convergent to x .
- **unbounded order convergent** to x if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$. We denote this convergence by $x_\alpha \xrightarrow{uo} x$ and write that $(x_\alpha)_\alpha$ is *uo*-convergent to x . It was first introduced by Nakano in [17] and was later used by DeMarr in [5].
- **unbounded norm convergent** to x if moreover E is a Banach lattice and $\| |x_\alpha - x| \wedge u \| \rightarrow 0$ for all $u \in E^+$. We denote this convergence by $x_\alpha \xrightarrow{un} x$ and write that $(x_\alpha)_\alpha$ is *un*-convergent to x . It was studied in [6, 13].
- **unbounded absolutely weakly convergent** to x , written as $x_\alpha \xrightarrow{uaw} x$, if $|x_\alpha - x| \wedge u \xrightarrow{w} 0$ for all $u \in E^+$. See, [20].

The notion of unbounded order convergence (*uo*-convergence, for short) was firstly introduced by Nakano in [17]. After that, Bahramnezhad and Haghnejad Azar proposed the definition of unbounded order continuous operators in [4]. It is clear that for order bounded nets, *uo*-convergence is equivalent to *o*-convergence. By Corollary 3.6 of [10], every disjoint sequence in vector lattice E is *uo*-null. In [19], Wickstead characterized the spaces in which weak convergence of nets implies *uo*-convergence and vice versa and in [9], Gao characterized the spaces E such that in its topological dual space E' , *uo*-convergence implies weak*-convergence and vice versa.

Let X and Y be two Banach spaces. An operator $T : X \rightarrow Y$, is said to be Dunford-Pettis (or that T has the Dunford-Pettis property) whenever $x_n \xrightarrow{w} 0$ in X implies $\|Tx_n\| \rightarrow 0$. Let E and F be two Banach lattices. An operator $T : E \rightarrow F$, is said to be unbounded absolute weak Dunford-Pettis (or, *uaw*-Dunford-Pettis for short) if for every norm bounded sequence (x_n) in E , $x_n \xrightarrow{uaw} 0$ in E implies $\|Tx_n\| \rightarrow 0$ in F . This class of operators has been introduced in [7]. If E is a Banach lattice and X is a Banach space. An operator $T : E \rightarrow X$, is said to be M -weakly compact if T is continuous and $\lim \|Tx_n\| = 0$ holds for every norm bounded disjoint

sequence $(x_n)_n$ of E . Let E and F be two vector lattices. An operator $T : E \rightarrow F$, is said to be disjointness preserving whenever $x \perp y$ in E implies $Tx \perp Ty$ in F . If for an operator $T : E \rightarrow F$ between two vector lattices, $T \vee (-T)$ exists we say its modulus $|T|$ exists. A vector $e > 0$ in vector lattice E is a strong unit when the ideal I_e (generated by e) is equal to E ; equivalently, for every $x \geq 0$ there exists $n \in \mathbb{N}$ such that $x \leq ne$. A vector $e > 0$ in normed vector lattice E is also a quasi-interior point if the closure of I_e equal with E ; or equivalently, $x \wedge ne \xrightarrow{\|\cdot\|} x$ for every $x \in E^+$. A positive non-zero vector a in a vector lattice E is an atom if the principal ideal I_a generated by a coincides with the vector space generated by a . E is called an atomic vector lattice if it is the band generated by its atoms. A Banach lattice E is said to be KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. Recall that $E^\sim = L_b(E, \mathbb{R})$ is the vector space of all order bounded linear functionals on E and E_n^\sim is the vector space of all order continuous linear functionals on E . A sublattice Y of a vector lattice E is said to be regular if for every subset A of Y , infimum of A is the same in E and in Y , whenever infimum of A exists in Y . A vector lattice E is called laterally complete whenever every subset of pairwise disjoint positive vectors has a supremum. We say that E is laterally σ -complete if this property holds for countable sets. For unexplained notation the reader is referred to [2].

2. Unbounded order-norm continuous operators

Definition 2.1. Let E and F be two normed vector lattices. A continuous operator $T : E \rightarrow F$ is said to be

1. unbounded order-norm continuous (or, uon-continuous for short), if $x_\alpha \xrightarrow{uo} 0$ in E implies $Tx_\alpha \xrightarrow{\|\cdot\|} 0$ in F for each norm bounded net $(x_\alpha)_\alpha \subseteq E$.
2. unbounded σ -order-norm continuous (or, σ -uon-continuous for short), if $x_n \xrightarrow{uo} 0$ in E implies $Tx_n \xrightarrow{\|\cdot\|} 0$ in F for each norm bounded sequence $(x_n)_n \subseteq E$.

The collection of all unbounded order-norm (resp. σ -order-norm) continuous operators between two normed vector lattices E and F will be denoted by $L_{uon}(E, F)$ (resp. $L_{uon}^\sigma(E, F)$). Every continuous operator, in general, is not unbounded order-norm continuous. As example the identity operator $I : c_0 \rightarrow c_0$ is continuous, but is not unbounded order-norm continuous.

Here is an example of uon-continuous and an example of σ -uon-continuous operators. At first, recall that a Banach lattice E is said to have the positive Schur property (the dual positive Schur property) if every positive w -null sequence in E (positive w^* -null sequence in E') is norm null.

Example 2.2. 1. Each positive operator $T : E \rightarrow \ell^1$ defined on reflexive Banach lattice E is uon-continuous. Let $(x_\alpha) \subseteq E$ be norm bounded and $x_\alpha \xrightarrow{uo} 0$ in E . Without loss of generality, assume $(x_\alpha) \subseteq E^+$. Since E has order continuous norm, therefore $x_\alpha \xrightarrow{un} 0$ in E . Since E' has order continuous norm, then by Theorem 6.4 of [6], $x_\alpha \xrightarrow{w} 0$ in E . Because the operator T is continuous, hence $T(x_\alpha) \xrightarrow{w} 0$ in ℓ^1 . By Theorem 5.29(2) of [2], the operator T is weakly compact and therefore $(T(x_\alpha))_\alpha$ is relatively weakly compact. Since ℓ^1 is σ -order complete and has the positive Schur property, by Theorem 3.11 of [11], $T(x_\alpha) \xrightarrow{\|\cdot\|} 0$ in ℓ^1 .

As a special case, each positive operator $T : \ell^p \rightarrow \ell^1$ that $1 < p < \infty$ is uon-continuous.

2. Let $T : E \rightarrow F$ be a positive operator where E has the dual positive Schur property and F has order continuous norm. Then $T' : F' \rightarrow E'$ is a σ -uon-continuous operator. Let $(f_n)_n \subseteq F'$, norm bounded and $f_n \xrightarrow{uo} 0$ in F' . Without loss of generality, assume $0 \leq f_n$. Note that $0 \leq T'f_n$. Now since F has order continuous norm, by Theorem 2.1 from [9], $f_n \xrightarrow{w^*} 0$ in F' . Since T' is w^* -to- w^* continuous, hence $T'f_n \xrightarrow{w^*} 0$ in E' . Since E has the dual positive Schur property, hence $T'f_n \xrightarrow{\|\cdot\|} 0$ in $(E)'$. Specifically, since $C[0, 1]$ has the dual positive Schur property and c_0 has order continuous norm, each positive operator $T : C[0, 1] \rightarrow c_0$ is σ -uon-continuous.

Let E, G and F be normed vector lattices. Then, it is obvious that for each uon-continuous operator $T : E \rightarrow G$ and continuous operator $S : G \rightarrow F$, $ST : E \rightarrow F$ is a uon-continuous operator.

Recall that an operator T from vector lattice E into normed vector lattice F is said to be an order-norm

continuous operator whenever $(x_\alpha)_\alpha \subseteq E$ and $x_\alpha \xrightarrow{o} 0$ implies $Tx_\alpha \xrightarrow{\|\cdot\|} 0$. This classification of operators has been introduced and studied by Jalili, Haghnejad Azar and Farshbaf Moghimi, see [12]. The following example shows that the classification of order-norm continuous operators differ from classification of *uon*-continuous operators.

Example 2.3. The operator $T : \ell^1 \rightarrow \ell^\infty$ defined by

$$T(x_1, x_2, \dots) = \left(\sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \dots \right)$$

is an order-norm continuous operator (ℓ^1 has order continuous norm and T is a continuous operator). Now, if $(e_n)_n$ is the standard basis of ℓ^1 , then (e_n) is *uo*-null and norm bounded in ℓ^1 and $T(e_n) = (1, 1, 1, \dots)$. Therefore, $\|T(e_n)\| \rightarrow 0$ in ℓ^∞ . Thus, T is not *uon*-continuous.

If a normed vector lattice E is Dedekind σ -complete and laterally σ -complete, then every order-norm continuous operator from E into normed vector lattice F is σ -*uon*-continuous. Namely, if $(x_n)_n$ is a norm bounded sequence in E such that $x_n \xrightarrow{uo} 0$, then by Theorem 3.2 of [14], $(x_n)_n$ is order bounded and therefore $x_n \xrightarrow{o} 0$ in E and so $T(x_n) \xrightarrow{\|\cdot\|} 0$ in F . Hence T is a σ -*uon*-continuous operator. If E has strong unit, since each norm bounded subset of E is order bounded, then each order-norm continuous operator $T : E \rightarrow F$ is *uon*-continuous.

Let $T : E \rightarrow F$ be a positive operator between two vector lattices. We say that an operator $S : E \rightarrow F$ is dominated by T (or that T dominates S) whenever $|Sx| \leq T|x|$ holds for each $x \in E$. If T is a *uon*-continuous operator between two normed vector lattices E and F , then it is obvious that $S : E \rightarrow F$ is *uon*-continuous whenever S is dominated by T .

Theorem 2.4. Let $T : E \rightarrow F$ be an order bounded *uon*-continuous operator between two normed vector lattices. If,

1. F is Archimedean and T preserves disjointness, then $|T|$ exists and belongs to $L_{uon}(E, F)$.
2. E is Dedekind σ -complete and laterally σ -complete and F is an atomic Banach lattice with order continuous norm, then $|T|$ exists and belongs to $L_{uon}^\sigma(E, F)$.

Proof. 1. By Theorem 2.40 of [2], $|T|$ exists and for all x , we have $|T|(|x|) = |T(|x|)| = |T(x)|$. If $(x_\alpha)_\alpha \subseteq E$ is norm bounded and $x_\alpha \xrightarrow{uo} 0$, then for each α , $|T|(|x_\alpha|) = |T(|x_\alpha|)| = |T(x_\alpha)| \xrightarrow{\|\cdot\|} 0$ in F . Now by inequality $\| |T|(|x_\alpha)| \| \leq \|T\| \|x_\alpha\|$, we have $|T|(x_\alpha) \xrightarrow{\|\cdot\|} 0$.

2. First we show that T is a σ -order continuous operator. Let $(x_n)_n \subseteq E$ and $x_n \xrightarrow{o} 0$ in E . Then $x_n \xrightarrow{uo} 0$ and by Theorem 3.2 of [14], $(x_n)_n$ is order bounded in E . So $(x_n)_n$ is norm bounded. Hence by the assumption, $T(x_n) \xrightarrow{\|\cdot\|} 0$ in F . Note that $(T(x_n))_n$ is order bounded. Now by Lemma 5.1 of [6], $T(x_n) \xrightarrow{o} 0$ in F . Hence T is a σ -order continuous operator. Note that by Theorem 4.10 of [2], F is Dedekind complete, therefore similar to Theorem 1.56 of [2], $|T|$ exists and it is a σ -order continuous operator. Now, assume that $(x_n)_n \subseteq E$ is norm bounded and $x_n \xrightarrow{uo} 0$ in E , since E is Dedekind σ -complete and laterally σ -complete, $(x_n)_n$ is order bounded. It follows that $x_n \xrightarrow{o} 0$ in E and $|T|(x_n) \xrightarrow{o} 0$ in F . Since F has order continuous norm, we have $|T|(x_n) \xrightarrow{\|\cdot\|} 0$.

□

Recall that a vector lattice E is said to be perfect whenever the natural embedding $x \mapsto \hat{x}$ from E to $(E_n^\sim)_n$ is one-to-one and onto. By Exercise 3 of page 74 of [2], if F is a perfect vector lattice, then $L_b(E, F)$ is likewise a perfect vector lattice for each vector lattice E .

Theorem 2.5. Let E and F be two normed vector lattices.

1. If F has strong unit, then $L_{uon}(E, F)$ is a subspace of $L_b(E, F)$.
Moreover, if the condition (2) from Theorem 2.4 holds, then we have the following assertions.

2. An order bounded operator $T : E \rightarrow F$ is a σ -order continuous operator if and only if it is σ -uon-continuous. It follows that $L_{uon}^\sigma(E, F) \cap L_b(E, F)$ is a band of $L_b(E, F)$.
3. If F is perfect, then $L_{uon}^\sigma(E, F) \cap L_b(E, F)$ is likewise a perfect vector lattice for each vector lattice E .

Proof. 1. Let $T : E \rightarrow F$ be a uon-continuous operator and let $x \in E^+$. similar to Lemma 1.54 of [1], we consider the order interval $[0, x]$ as a net $(x_\alpha)_\alpha$, where $x_\alpha = \alpha$ for each $\alpha \in [0, x]$, then $x_\alpha \downarrow 0$. It is clear that (x_α) is norm bounded and $x_\alpha \xrightarrow{uo} 0$ in E . By assumption $T(x_\alpha) \xrightarrow{\|\cdot\|} 0$ and therefore $(T(x_\alpha))_\alpha$ is norm bounded in F . Because F has strong unit, therefore $(T(x_\alpha))_\alpha$ is order bounded. Hence $T \in L_b(E, F)$.

2. Follows from Theorem 1.57 of [2].

3. We will show that if E is a perfect vector lattice and B is a band of E , then B is perfect vector lattice in its own right. Suppose $x, y \in B$ and $x \neq y$, by Theorem 1.71 of [2], there exists $f \in E_n^\sim$ such that $f(x) \neq f(y)$. The restriction of f to B is order continuous and $f|_B(x) \neq f|_B(y)$. Therefore B_n^\sim separates the points of B . Let $(x_\alpha)_\alpha \subseteq B$, $0 \leq x_\alpha \uparrow$ and $\sup\{f(x_\alpha)\} < \infty$ for each $0 \leq f \in B_n^\sim$. It is clear that $(x_\alpha)_\alpha \subseteq E$ and $0 \leq x_\alpha \uparrow$ in E . On the other hand, for each $f \in E_n^\sim$, $f|_B \in B_n^\sim$ and $f|_B(x_\alpha) = f(x_\alpha)$ for all α , therefore $\sup\{f(x_\alpha)\} < \infty$ for each $0 \leq f \in E_n^\sim$. Thus by Theorem 1.71 of [2], there exists some $x \in E$ satisfying $0 \leq x_\alpha \uparrow x$. Since B is a band of E , hence $x \in B$. Therefore by Theorem 1.71 of [2], B is a perfect vector lattice. Similarly $L_{uon}^\sigma(E, F) \cap L_b(E, F)$ is a perfect vector lattice.

□

Recall that a Banach lattice E is said to be AL-space, if $\|x + y\| = \|x\| + \|y\|$ holds for all $x, y \in E^+$ with $x \wedge y = 0$.

Theorem 2.6. Let E be an AL-space, F be a normed lattice and let $T : E \rightarrow F$ be a positive operator. Then for the following assertions:

1. T is a uon-continuous operator.
2. T is a Dunford-Pettis operator.
3. For every relatively weakly compact net $(x_\alpha)_\alpha \subseteq E$, $x_\alpha \xrightarrow{w} 0$ implies $Tx_\alpha \xrightarrow{\|\cdot\|} 0$.
4. For every relatively weakly compact net $(x_\alpha)_\alpha \subseteq E$, $x_\alpha \xrightarrow{uo} 0$ implies $Tx_\alpha \xrightarrow{\|\cdot\|} 0$.

We have

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

Proof. (1) \Rightarrow (2) Let $(x_n)_n \subseteq E$ be a norm bounded and disjoint sequence. Since $x_n \xrightarrow{uo} 0$ and it is norm bounded in E , therefore by assumption we have $Tx_n \xrightarrow{\|\cdot\|} 0$ in F . Hence T is a M -weakly compact operator. Now by Theorem 5.61 of [2], T is weakly compact and therefore by Theorems 5.85 and 5.82 of [2], T is a Dunford-Pettis operator.

(2) \Rightarrow (3) Suppose $(x_\alpha)_\alpha$ is relatively weakly compact and $x_\alpha \xrightarrow{w} 0$. Suppose also $(Tx_\alpha)_\alpha$ does not converge to 0 in norm. Then there exists $\epsilon > 0$ such that for any α , there exists $\beta(\alpha) \geq \alpha$ satisfying $\|Tx_{\beta(\alpha)}\| \geq \epsilon$. Thus by passing to the subnet $(Tx_{\beta(\alpha)})$, we may assume $\inf \|Tx_\alpha\| > 0$. Since $0 \in \overline{(x_\alpha)}^w$, By Theorem 4.50 of [8] there exists a sequence $(y_n)_n \subseteq \{x_\alpha : \alpha\}$ such that $y_n \xrightarrow{w} 0$. The assumption (2) now implies $Ty_n \xrightarrow{\|\cdot\|} 0$, which is a contradiction.

(3) \Rightarrow (4) Suppose $(x_\alpha)_\alpha$ is relatively weakly compact and $x_\alpha \xrightarrow{uo} 0$. By Proposition 3.9 of [11], $|x_\alpha| \xrightarrow{w} 0$. Thus by assumption (3), $T|x_\alpha| \xrightarrow{\|\cdot\|} 0$. Now by inequality $|Tx_\alpha| \leq T|x_\alpha|$ we have $Tx_\alpha \xrightarrow{\|\cdot\|} 0$.

□

Remark 2.7. One should note that statements in Theorem 2.6 are not equivalent. For example, the identity operator of ℓ^1 , I_{ℓ^1} , is Dunford-Pettis, since ℓ^1 has the Schur property. On the other hand, for (e_n) the standard basis of ℓ^1 we have $e_n \xrightarrow{uo} 0$ and it is norm bounded. As $\|I_{\ell^1}e_n\| = \|e_n\| = 1$ for each n , hence I_{ℓ^1} is not uon-continuous.

Remark 2.8. A continuous operator T between two Banach lattices E and F is σ -uon-continuous iff it is M -weakly compact iff it is uaw-Dunford-Pettis (see [7, Proposition 2.6] and [21, Theorem 18]).

3. Unbounded norm continuous operators

A continuous operator T between two Banach lattices E and F is said to be unbounded norm continuous (or, *un*-continuous for short) whenever $x_\alpha \xrightarrow{un} 0$ implies $Tx_\alpha \xrightarrow{un} 0$, for each norm bounded net $(x_\alpha)_\alpha \subseteq E$. For norm bounded sequence $(x_n)_n \subseteq E$, if $x_n \xrightarrow{un} 0$ implies $Tx_n \xrightarrow{un} 0$, then T is called a σ -unbounded norm continuous operator (or, σ -*un*-continuous for short). The collection of all unbounded norm continuous operators of $L(E, F)$ (the class of linear operators from E to F) will be denoted by $L_{un}(E, F)$. That is,

$$L_{un}(E, F) = \{T \in L(E, F) : T \text{ is unbounded norm continuous} \}.$$

Similarly, $L_{un}^\sigma(E, F)$ will denote the collection of all operators from E to F that are σ -unbounded norm continuous. That is,

$$L_{un}^\sigma(E, F) = \{T \in L(E, F) : T \text{ is } \sigma\text{-unbounded norm continuous} \}.$$

Every continuous operator in general is not unbounded norm continuous operator. As example, the inclusion mapping from ℓ^1 into ℓ^∞ is continuous, but is not unbounded norm continuous operator. Now in the following, we give some examples of *un*-continuous operators.

Example 3.1. 1. Let B be a projection band of Banach lattice E and P_B the corresponding band projection. It follows easily from $0 \leq P_B \leq I$ (see Theorem 1.44 of [2]) that if $x_\alpha \xrightarrow{un} 0$ in E then $P_B x_\alpha \xrightarrow{un} 0$ in B . Therefore P_B is a *un*-continuous operator.

2. Let E and F be two Banach lattices such that E has a strong unit. Then by Theorem 2.3 of [13], each continuous operator $T : E \rightarrow F$ is *un*-continuous.

Since $C[0, 1]$ has a strong unit and operator $T : C[0, 1] \rightarrow c_0$, given by

$$T(f) = \left(\int_0^1 f(x) \sin x dx, \int_0^1 f(x) \sin 2x dx, \dots \right),$$

is a continuous operator, therefore T is *un*-continuous.

3. Let E and F be two Banach lattices such that E' has order continuous norm and F is atomic with order continuous norm, then each continuous operator $T : E \rightarrow F$ is σ -*un*-continuous. Let $(x_n)_n \subseteq E$ be norm bounded and $x_n \xrightarrow{un} 0$ in E , then by Theorem 6.4 of [6] we have $x_n \xrightarrow{w} 0$ and therefore $Tx_n \xrightarrow{w} 0$ in F . By Proposition 6.2 of [6], $Tx_n \xrightarrow{un} 0$ in F .

For example operator $T : C[0, 1] \rightarrow \ell^1$, given by

$$T(f) = \left(\frac{\int_0^1 f(x) \sin x dx}{1^2}, \frac{\int_0^1 f(x) \sin 2x dx}{2^2}, \dots \right),$$

is σ -*un*-continuous.

It is clear that if E' has order continuous norm and F has the Schur property, then each continuous operator $T : E \rightarrow F$ is σ -*un*-continuous.

Recall from Definition 1 of [21] that a continuous operator $T : E \rightarrow F$ between two Banach lattices is said to be *uaw*-continuous if it maps every norm bounded *uaw*-null sequence into a *uaw*-null sequence.

Remark 3.2. Let $T : E \rightarrow F$ be an operator between two Banach lattices.

1. If E has order continuous norm and T is σ -*un*-continuous, then T is *uaw*-continuous.
2. If F has order continuous norm and T is *uaw*-continuous, then T is σ -*un*-continuous.

Now in the following Proposition, by using Theorem 5.3 from [6] and Theorem 2.3 from [13], we show the relationship between the classifications of unbounded norm continuous and unbounded order-norm continuous operators.

Proposition 3.3. Assume that $T : E \rightarrow F$ is an operator between two Banach lattices and E has order continuous norm. Then we have the following assertions.

(a) If E is atomic, then $L_{uon}(E, F)$ is a subspace of $L_{un}(E, F)$.

(b) If F has a strong unit, then each $T \in L_{un}(E, F)$ is uaw -Dunford-Pettis and by one of the following conditions $T \in L_{uon}(E, F)$.

1. $T \in L_{un}(E, F)$.
2. T is onto homomorphism.

Proof. (a) By using Theorem 5.3 from [6], proof is clear.

(b) Assume that $(x_n) \subseteq E$ and norm bounded with $x_n \xrightarrow{uaw} 0$ in E . By using Theorem 4 of [20], $x_n \xrightarrow{un} 0$ in E , and so by assumption we have $Tx_n \xrightarrow{un} 0$ in F . Then by Theorem 2.3 from [13], proof follows.

1. Assume that norm bounded net $(x_\alpha) \subseteq E$ with $x_\alpha \xrightarrow{uo} 0$ in E and $T \in L_{un}(E, F)$. It is clear that $x_\alpha \xrightarrow{un} 0$. By assumption $T(x_\alpha) \xrightarrow{un} 0$ and therefore by Theorem 2.3 from [13], $T(x_\alpha) \xrightarrow{\|\cdot\|} 0$.
2. If T is homomorphism, then T is continuous, and so $\|T(|x_\alpha| \wedge u)\| \rightarrow 0$ whenever $\| |x_\alpha| \wedge u \| \rightarrow 0$ for all $u \in E^+$. On the other since T is onto, it is clear that T is un -continuous and therefore it is uon -continuous.

□

With notice Example 6 from [7], the inclusion mapping $\ell_2 \rightarrow \ell_\infty$ is not uaw -Dunford-Pettis, and so by proceeding proposition is not un -continuous.

Corollary 3.4. Let E be a Banach lattice with order continuous norm and F has strong unit. Then for the following assertions:

1. Each positive Dunford-Pettis operator $T : E \rightarrow F$ is σ -unbounded norm continuous.
2. Each positive compact operator $T : E \rightarrow F$ is σ -unbounded norm continuous.
3. One of the following conditions is valid:
 - (a) The norm of E' is order continuous.
 - (b) $F = \{0\}$.

We have $1 \Rightarrow 2 \Rightarrow 3$.

Proof. $1 \Rightarrow 2$ Note that each compact operator is Dunford-Pettis, so the proof is complete.

$2 \Rightarrow 3$ By assumption and Proposition 3.3, each compact operator is uaw -Dunford-Pettis. Then by Theorem 3.1 of [15] the proof is complete. □

By using Theorem 4.3 of [13], we also have the following proposition.

Proposition 3.5. Let E and F be two Banach lattices and Let G be a sublattice of E and $T \in L_{un}(E, F)$. Each of the following conditions implies that $T|_G \in L_{un}(G, F)$.

1. G is majorizing in E ;
2. G is norm dense in E ;
3. G is a projection band in E .

The preceding Proposition implies that if $T \in L_{un}(E^\delta, F)$, then $T|_E \in L_{un}(E, F)$ whenever E and F are Banach lattices and E^δ is a Dedekind completion of E .

Let E, G and F be Banach lattices. If $E \xrightarrow{T} G \xrightarrow{S} F$ are un -continuous operators, clearly ST is likewise a un -continuous operator. Also, by using Theorem 2.40 of [2], if $T : E \rightarrow F$ preserves disjointness and un -continuous operator, then $|T|$ exists and is a un -continuous operator.

Deng, Brien and Troitsky in [6], show that un -convergence is topological. For each $\epsilon > 0$ and $x \in E^+$ the collections $V_{\epsilon, x} = \{y \in E : \| |y| \wedge x \| < \epsilon\}$ is a base of zero neighborhoods for a topology, and convergence in this topology agrees with un -convergence.

Recall that a topological space is said to be sequentially compact if every sequence has a convergent subsequence. An operator $T : E \rightarrow F$ between two Banach lattices is said to be (sequentially) *un*-compact if TB_E (B_E is closed unit ball of E) is relatively (sequentially) *un*-compact in E . Equivalently, for every bounded net $(x_\alpha)_\alpha$ (respectively, every bounded sequence $(x_n)_n$) its image has a subnet (respectively, subsequence), which is *un*-convergent. A net $(x_\alpha)_\alpha$ is *un*-Cauchy if for every *un*-neighborhood U of zero there exists α_0 such that $x_\alpha - x_\beta \in U$ whenever $\alpha, \beta \geq \alpha_0$. The order continuous Banach lattice E is *un*-complete if each *un*-Cauchy net $(x_\alpha)_\alpha$ of E is *un*-convergent to $x \in E$. These concepts have been introduced by Kandic, Marabeh and Troitsky, see [13].

Clearly, every compact operator is both *un*-compact and sequentially *un*-compact. In general *un*-compact and sequentially *un*-compact operators are not *un*-continuous and vice versa as shown by the following example.

Example 3.6. 1. The operator $T : \ell^1 \rightarrow \ell^\infty$ defined by

$$T(x_1, x_2, \dots) = \left(\sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \dots \right),$$

is clearly rank one, and so T is a compact operator. It follows that T is *un*-compact and sequentially *un*-compact. If $(e_n)_n$ is the standard basis of ℓ^1 , by Proposition 3.5 of [13], $e_n \xrightarrow{un} 0$ in ℓ^1 . We have $T(e_n) = (1, 1, 1, \dots)$, therefore $(T(e_n))_n$ is not *un*-convergent to 0. Hence T is *un*-compact but is not a *un*-continuous operator.

2. Let $E = L_1[0, 1]$. Clearly, the identity operator $I : E \rightarrow E$ is *un*-continuous. Since E is a KB-space, by Theorem 6.4 of [13], B_E is *un*-complete. But since E is not atomic, by Theorem 7.5 of [13], B_E is not *un*-compact. Hence I is not *un*-compact.

The operator T in Example 3.6 is *un*-compact but it is not *un*-continuous. In the following theorem, we want to show that under certain conditions, each positive *un*-continuous operator T is a *un*-compact operator.

Theorem 3.7. Let F be a Banach lattice. If $T : \ell^1 \rightarrow F$ is a *un*-continuous operator, then T is *un*-compact.

Proof. Let $(x_n)_n$ be a norm bounded sequence in ℓ^1 , then by Theorem 7.5 of [13], there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} \xrightarrow{un} x$ for some $x \in E$. Since *un*-convergence and *un*-convergence agree on ℓ^1 , so by assumption we have $Tx_{n_k} \rightarrow Tx$ in F . Therefore T is compact and so it is *un*-compact. \square

A subset A of Banach lattice E is said to be *un*-bounded, if A is bounded with respect to *un*-topology. An operator $T : E \rightarrow F$ between two Banach lattices is *un*-bounded if $T(A)$ is *un*-bounded in F for each *un*-bounded subset A of E .

Proposition 3.8. Assume that E and F are two Banach lattices and E has quasi-interior point. If $T : E \rightarrow F$ is a *un*-bounded operator, then T is σ -*un*-continuous.

Proof. Let $(x_n)_n \subseteq E$ be norm bounded and $x_n \xrightarrow{un} 0$. By Theorem 3.2 from [13], E is metrizable, and so similar to Theorem 1.28(b) of [18], there exists a sequence $(\alpha_n) \subseteq \mathbb{R}^+$ such that $\alpha_n \rightarrow +\infty$ and $\alpha_n x_n \xrightarrow{un} 0$. Obviously, $(\alpha_n x_n)$ is *un*-bounded. Hence $(T(\alpha_n x_n))$ is *un*-bounded. Thus similar to Theorem 1.30 of [18], we have $Tx_n = \frac{1}{\alpha_n} T(\alpha_n x_n) \xrightarrow{un} 0$. That is, T is σ -*un*-continuous. \square

Remark 3.9. There is a *un*-compact operator which is not *un*-bounded. Let $T : \ell^1 \rightarrow \ell^\infty$ be the *un*-compact operator in Example 3.6. But, since ℓ^1 has quasi-interior point and T is not a σ -*un*-continuous operator, by Proposition 3.8, T is not *un*-bounded.

Proposition 3.10. Assume that E and F are two Banach lattices. If F has a strong unit and let $T : E \rightarrow F$ be a sequentially *un*-compact operator. Then the adjoint operator T' is both sequentially *un*-compact and *un*-compact.

Proof. Let $(x_n)_n$ be a bounded sequence in E . Then by the assumption $(T(x_n))_n$ has a subsequence, which is *un*-convergent. Now by Theorem 2.3 of [13], $(T(x_n))_n$ is norm-convergent and therefore T is a compact operator. Now by Theorem 5.2 of [2], T' is compact and therefore it is both sequentially *un*-compact and *un*-compact. \square

Recall that, for every ideal I of a vector lattice E , the vector space $\frac{E}{I}$ is a vector lattice (see page 99 from [2]).

Theorem 3.11. *Let $T : E \rightarrow F$ be an operator between two Banach lattices and $T(E^+) = F^+$. If $\ker(T)$ is an ideal of E , then T is *un*-continuous.*

Proof. At first it is clear that T is surjective. On the other hand, as kernel of T is an ideal of E , by Theorem 2.22 of [2], the quotient vector space $\frac{E}{\ker(T)}$ is a vector lattice and the operator $S : E \rightarrow \frac{E}{\ker(T)}$ defined by $S(x) = x + \ker(T)$ is a Riesz homomorphism. Now we define the operator $K : \frac{E}{\ker(T)} \rightarrow F$ via $K(x + \ker(T)) = Tx$. It is clear that K is well defined, one-to-one operator and $T = KS$. Let $y \in F$ and $x \in E$ such that $y = T(x) = K(x + \ker(T))$. It follows that K is a surjective operator. Now if $x + \ker(T) \geq 0$, then $|x + \ker(T)| = x + \ker(T)$. On the other hand, since S is Riesz homomorphism, by Theorem 2.14 of [2], we have $S(|x|) = |S(x)|$. Therefore $|x| + \ker(T) = |x + \ker(T)| = x + \ker(T)$ and hence $K(x + \ker(T)) = K(|x| + \ker(T)) = T(|x|) \geq 0$. Thus K is a positive operator. Let $y \in F^+$. Since $T(E^+) = F^+$, it follows that there exists $x \in E^+$ that $y = T(x) = K(x + \ker(T))$. It follows from $K^{-1}(y) = K^{-1}(K(x + \ker(T))) = x + \ker(T)$ and $|x + \ker(T)| = |x| + \ker(T) = x + \ker(T)$ that K^{-1} is a positive operator. Now by Theorem 2.15 of [2], K is a Riesz homomorphism. It follows that T is Riesz homomorphism. Now, the proof is complete by this fact that each surjective Riesz homomorphism is *un*-continuous. \square

Theorem 3.12. *Let E, F and G be Banach lattices. Then we have the following assertions.*

1. *If $T : E \rightarrow F$ and $Q : F \rightarrow G$ are two positive operators that E' and G have order continuous norm, then QT is *un*-continuous.*
2. *If there exists a positive operator $Q : \ell^1 \rightarrow F$, then there exists a positive *un*-continuous operator $T : c_0 \rightarrow F$.*

Proof. 1. Let $(x_\alpha)_\alpha \subseteq E$ be norm bounded and $x_\alpha \xrightarrow{un} 0$ in E . Since E' has order continuous norm, by Theorem 6.4 of [6], $x_\alpha \xrightarrow{w} 0$ in E . Therefore $Tx_\alpha \xrightarrow{w} 0$ in F and so $QT(x_\alpha) \xrightarrow{w} 0$ in G . Since G has order continuous norm, then by Proposition 6.3 of [6], $QT(x_\alpha) \xrightarrow{un} 0$ in G .

2. Let $S : c_0 \rightarrow \ell^1$ be a positive operator. Let $(x_\alpha)_\alpha \subseteq c_0$ is norm bounded and *un*-null. Since c'_0 has order continuous norm, hence by Theorem 6.4 of [6], $x_\alpha \xrightarrow{w} 0$ in c_0 . So, $S(x_\alpha) \xrightarrow{w} 0$ in ℓ^1 . By Theorem 5.29(2) of [2], S is weakly compact. Therefore $(S(x_\alpha))_\alpha$ is relatively weakly compact in ℓ^1 . Now by Theorem 3.11 of [11], $S(x_\alpha) \xrightarrow{\|\cdot\|} 0$ in ℓ^1 . We put $T = QS$. It is clear that T is a positive *un*-continuous operator.

\square

Let E be a vector lattice and $E^{\sim\sim}$ be the bidual of E . Recall that a subset A of E is *b*-order bounded in E if A is order bounded in $E^{\sim\sim}$.

Theorem 3.13. *Let E be a Banach lattice with order continuous norm and F be a Dedekind σ -complete Banach lattice such that the norm of F is not order continuous. If each operator $T : E \rightarrow F$ is *un*-continuous, then E is KB-space.*

Proof. By way of contradiction, suppose that E is not a KB-space. Then by Lemmas 2.1 and 3.4 of [3], there exists a *b*-order bounded disjoint sequence $(x_n)_n$ of E^+ such that $\|x_n\| = 1$ for all n , and there exists a positive disjoint $(g_n)_n$ of E' with $\|g_n\| \leq 1$ such that $g_n(x_n) = 1$ for all n , and $g_n(x_m) = 0$ for $n \neq m$. Now we consider the operator $S : E \rightarrow \ell^\infty$ defined by $S(x) = (g_k(x))_{k=1}^\infty$ for all $x \in E$. Since $(x_n)_n$ is disjoint, by Corollary 3.6 of [10], it is *uo*-null. Hence $x_n \xrightarrow{uo} 0$ in E and therefore $x_n \xrightarrow{un} 0$ in E . It follows that $\|S(x_n)\| = \|(g_k(x_n))_{k=1}^\infty\| \geq g_n(x_n) = 1$. By Theorem 2.3 of [13] $S(x_n) \not\rightarrow 0$ with respect to *un*-topology in ℓ^∞ . Therefore S is not *un*-continuous. On the other hand, since the norm of F is not order continuous and F is

Dedekind σ -complete, by Corollary 2.4.3 of [16], F contains a complemented copy of ℓ^∞ . Now we consider the composed operator $T = I \circ S : E \rightarrow \ell^\infty \rightarrow F$, where I is the canonical injection of ℓ^∞ into F . This operator is not *un*-continuous which is impossible, and so the proof follows. \square

In the following, we show that under certain conditions, each *un*-continuous operator is a combination of two *un*-continuous operators.

Theorem 3.14. *Let F be a Banach lattice. For the following statements:*

1. For any Banach lattice E such that E' has order continuous norm, each positive operator $T : E \rightarrow F$ is *un*-continuous.
2. There exist a reflexive Banach lattice G and two *un*-continuous operators $Q : E \rightarrow G$ and $S : G \rightarrow F$ such that $T = SQ$.

We have, (1) implies (2).

Proof. Since $c'_0 = \ell^1$ has order continuous norm, therefore by assumption each positive operator $T : c_0 \rightarrow F$ is *un*-continuous. We want to show that F is a *KB*-space. Assume by way of contradiction that F is not *KB*-space. Then by Theorem 4.60 of [2], c_0 is embeddable in F . Let $T : c_0 \rightarrow F$ be this embedding. Then there exist two positive constants K and M satisfying

$$K\|x_n\|_\infty \leq \|Tx_n\| \leq M\|x_n\|_\infty \quad \text{for all } (x_n) \subseteq c_0.$$

Since (e_n) is norm bounded and *uo*-null in c_0 , we have $\|T(e_n)\| \geq K\|e_n\|_\infty = K > 0$. Therefore, T is not *un*-continuous, which contradicts. That is, F is *KB*-space. Since T is *un*-continuous, it is M -weakly compact. Therefore by Exercise 10 of page 338 of [2], there exists a reflexive Banach lattice G and positive operators $Q : E \rightarrow G$ and $S : G \rightarrow F$ such that $T = SQ$. We show that Q and S are *un*-continuous. Let $(x_\alpha)_\alpha \subseteq E$ be norm bounded and *un*-null. Since E' has order continuous, then by Theorem 6.4 of [6], $x_\alpha \xrightarrow{w} 0$ and therefore $Qx_\alpha \xrightarrow{w} 0$ in G . By Proposition 6.3 of [6], $Q(x_\alpha) \xrightarrow{wm} 0$. Since G' has order continuous norm and F is a *KB*-space, a similar argument shows that S is *un*-continuous. \square

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