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A Hyers-Ulam Stability Analysis for Classes of Bessel Equations

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Abstract. Mathematical modeling helps us to better understand different natural phenomena. Modeling is most of the times based on the consideration of appropriate equations (or systems of equations). Here, differential equations are well-known to be very useful instruments when building mathematical models – specially because that the use of derivatives offers several interpretations associated with real life laws. Differential equations are classified based on several characteristics and, in this way, allow different possibilities of building models. In this paper we will be concentrated in analysing certain stability properties of classes of Bessel differential equations. In fact, the main aim of this work is to seek adequate conditions to derive different kinds of stabilities for the Bessel equation and for the modified Bessel equation by considering a perturbation of the trivial solution. In this way, sufficient conditions are obtained in order to guarantee Hyers-Ulam-Rassias, σ-semi-Hyers-Ulam and Hyers-Ulam stabilities for those equations.

1. Introduction

Mathematical modeling of real problems helps us to better understand the world and the events around us. However, not always models are near enough to the "law" that they are modeling. Therefore, implementing methods to qualify how far are our models from the reality (or from the proper solutions) is an important issue in modeling theory. Moreover, having in mind that typically the models depend on certain parameters (e.g. time and space), sometimes the used approximations can be very good for some values of these parameters and bad for others. In view of this, when we are using mathematical equations to implement a model, it is fundamental to understand what is the stability behaviour of such equations. Such knowledge will give precise information about how far an approximate solution can be from the exact solution (and this can be analysed both in local or global ways upon the consideration of an appropriate setting).

As expected, the variety of examples we could give on mathematical and physical modeling is huge. Here, we simply refer to [10, 20, 30] as some examples where exact and approximate solutions are considered

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within highly nontrivial problems/models associated with wave propagation, thermoelasticity, hybrid nanofluids, nanoparticles and heat transfer.

Realizing that the ways to approximate can be very different, it is easy to understand that we may also have different types of stabilities. In the opposite way, such different types of stabilities allow us to conclude different methods and modeling implementations. Some of them are more restrictive and some others are more flexible in guaranteeing the quality of the mentioned approximations. Additionally, depending on the type of equations, different theories can be proposed and applied. For instance, in differential equations and dynamical systems different types of equilibria can be considered. In particular, it is very useful to identify and classify equilibrium points based on their stability since the dynamics of a system can be significatively different when facing stable and unstable equilibria. In particular, linear stability analysis offers information about how a system behaves near an equilibrium point. However, it does not allows information about what happens farther away from equilibrium. To help on this, we may use phase-plane analysis combined with linear stability analysis to obtain a more global understanding of the dynamics.

Stabilities of Hyers-Ulam and Hyers-Ulam-Rassias have significant interest in different areas of mathematics which involve differential, integral and functional equations. One of the reasons to this has to do with the significant interest on those stabilities when studying concrete problems. In fact, in the sense above explained, the stability is a crucial information to have when considering model situations where although we can not expect to easily obtain the exact solution of the problem we may expect to obtain only an approximate solution which should be stable in a certain specific sense. This is being done for a huge number of different type of equations (e.g. in the cases above indicated). This is also connected with the applicability of those equations in different areas of knowledge like chemical reactions, diffraction theory, elasticity, fluid flow, heat conduction, aerodynamics, population dynamic, polymer rheology and among many others (cf., e.g., [1–13, 16–19, 21–29, 31, 33]).

The first stability results, for functional equations, in particular for the Cauchy multiplicative equation, were originated from a problem presented by S. M. Ulam, in 1940, about to discover when a "perturbation" of the solution of an equation must be somehow near to the solution of the given equation. One year later, D. H. Hyers obtained a partial answer to the question of S. M. Ulam, for Banach spaces, in the case of the additive Cauchy equation $f(x+y) = f(x) + f(y)$, cf. [21], within the framework of (real) Banach spaces. This gave rise to what we now called *Hyers-Ulam stability* of the additive Cauchy equation. A few years later, Th. M. Rassias (see [32]) introduced new ideas giving rise to the so-called *Hyers-Ulam-Rassias stability*. Additional details in such advances can be seen in [4].

In the present work we will analyse Hyers-Ulam, Hyers-Ulam-Rassias and the so-called σ-semi-Hyers-Ulam stabilities for the Bessel equation

$$x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) = 0, \tag{1}$$

as well as for the modified Bessel equation

$$x^2y''(x) + xy'(x) - (x^2 + \alpha^2)y(x) = 0, \tag{2}$$

by using initial conditions

$$y(a) = y'(a) = 0, \tag{3}$$

with $y \in C^2([a,b])$, for $x \in [a,b]$ where a and b are fixed positive real numbers and $\alpha \in \mathbb{R}$ or α is a pure imaginary number.

This will allow a step forward in the scientific knowledge by exhibiting a detailed clarification about different types of stabilities available for the above class of differential equations, as well as by identifying conditions under which those stabilities will be in force.

The formal definition of the above mentioned stabilities are now introduced for the Bessel equation (1).

Definition 1.1. *If for each function y satisfying*

$$\left|x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x)\right| \le \theta, \qquad x \in [a,b], \tag{4}$$

where $\theta \geq 0$, there is a solution y_0 of the Bessel equation and a constant $C > 0$ independent of y and y_0 such that

$$\left|y(x) - y_0(x)\right| \leq C\,\theta \tag{5}$$

for all $x \in [a, b]$, then we say that the Bessel equation (1) has the Hyers-Ulam stability.

Definition 1.2. *If for each function y satisfying*

$$\left|x^2 y''(x) + xy'(x) + (x^2 - \alpha^2)y(x)\right| \leq \sigma(x), \qquad x \in [a, b], \tag{6}$$

where σ is a non-negative function, there is a solution y_0 of the Bessel equation and a constant $C > 0$ independent of y and y_0 such that

$$\left|y(x) - y_0(x)\right| \leq C\,\sigma(x) \tag{7}$$

for all $x \in [a, b]$, then we say that the Bessel equation (1) has the Hyers-Ulam-Rassias stability.

We also use a stability in-between the two just mentioned stabilities of Hyers-Ulam-Rassias and Hyers-Ulam, introduced in [15] (see also [14]), in the following way:

Definition 1.3. *Let σ be a non-decreasing function defined on $[a, b]$. If for each function y satisfying*

$$\left|x^2 y''(x) + xy'(x) + (x^2 - \alpha^2)y(x)\right| \leq \theta, \qquad x \in [a, b], \tag{8}$$

where $\theta \geq 0$, there is a solution y_0 of the Bessel equation (1) and a constant $C > 0$ independent of y and y_0 such that

$$\left|y(x) - y_0(x)\right| \leq C\,\sigma(x), \qquad x \in [a, b] \tag{9}$$

then we say that the Bessel equation (1) has the σ-semi-Hyers-Ulam stability.

2. Hyers-Ulam, Hyers-Ulam-Rassias and σ-semi-Hyers-Ulam stabilities for $\alpha \in \mathbb{R}$

The present section is devoted to present sufficient conditions for the Hyers-Ulam stability, the Hyers-Ulam-Rassias stability and the σ-semi-Hyers-Ulam stability of the Bessel equation (1), where $x \in [a, b]$ and $\alpha \in \mathbb{R}$, for some fixed positive real numbers a and b. Note that the parameter θ, which appears bellow in (10), works like a control, allowing the consideration os approximate "solutions" of the original equation (being always possible to consider since it also includes the situation of the exact solution).

Theorem 2.1. *Let $\alpha \in \mathbb{R}$ and $y \in C^2([a, b])$ with $0 \leq y'(x) \leq ky(x)$ for all $x \in [a, b]$, and some $k > 0$.*
If $y \in C^2([a, b])$ is such that

$$\left|x^2 y''(x) + xy'(x) + (x^2 - \alpha^2)y(x)\right| \leq \theta \tag{10}$$

for all $x \in [a, b]$, where $\theta \geq 0$ and $\alpha^2 - (k^2 + 1)b^2 - 2ab + 2a^2 > 0$, then there is a function $y_0 \in C^2([a, b])$ and a constant $C > 0$ such that

$$|y(x) - y_0(x)| \leq C\,\theta \tag{11}$$

for all $x \in [a, b]$.

This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the Hyers-Ulam stability.

Proof. For every $\theta \geq 0$, by the differential inequality (10), we have

$$-\theta \leq x^2 y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) \leq \theta.$$

Multiplying the above inequality by $y'(x)$, integrating from a to t and using the initial conditions (3) we obtain that

$$\begin{aligned}
&-\theta \int_a^t y'(x)dx \leq \int_a^t x^2 y''(x)y'(x)dx + \int_a^t xy'(x)y'(x)dx + \int_a^t (x^2-\alpha^2)y(x)y'(x)dx \leq \theta \int_a^t y'(x)dx \\
&\Leftrightarrow -\theta y(t) \leq \frac{1}{2}t^2(y'(t))^2 + \frac{1}{2}(t^2-\alpha^2)(y(t))^2 - \int_a^t x(y(x))^2 dx \leq \theta y(t) \\
&\Leftrightarrow -2\theta y(t) \leq t^2(y'(t))^2 + (t^2-\alpha^2)(y(t))^2 - 2\int_a^t x(y(x))^2 dx \leq 2\theta y(t).
\end{aligned}$$

Having in mind the inequality on the left-hand side and taking into account that $y'(x) \leq ky(x)$ for all $x \in [a, b]$ we get that

$$\begin{aligned}
(\alpha^2 - t^2)(y(t))^2 &\leq 2\theta y(t) + t^2(y'(t))^2 - 2\int_a^t x(y(x))^2 dx \\
&\leq 2\theta y(t) + t^2(ky(t))^2 - 2\int_a^t x(y(x))^2 dx \\
&\leq 2\theta y(t) + k^2t^2(y(t))^2 - 2a\int_a^t (y(x))^2 dx \\
&\leq 2\theta y(t) + k^2t^2(y(t))^2 + 2a\int_a^t (y(x))^2 dx.
\end{aligned}$$

Denoting

$$M = \max_{t\in[a,b]} |y(t)|,$$

we can write

$$(\alpha^2 - t^2)M^2 \leq 2\theta M + k^2t^2M^2 + 2a(b-a)M^2 \quad \Leftrightarrow \quad (\alpha^2 - (k^2+1)t^2)M^2 \leq 2\theta M + 2a(b-a)M^2.$$

Taking into account that $\alpha^2 - (k^2 + 1)t^2$ is decreasing, for $t \in [a, b]$, we obtain

$$\begin{aligned}
(\alpha^2 - (k^2+1)b^2)M^2 \leq 2\theta M + 2a(b-a)M^2 \quad &\Leftrightarrow \quad (\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2)M^2 \leq 2\theta M \\
&\Leftrightarrow \quad (\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2)M \leq 2\theta \\
&\Leftrightarrow \quad M \leq \frac{2}{\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2}\theta
\end{aligned}$$

if $\alpha^2 - (k^2 + 1)b^2 - 2ab + 2a^2 > 0$ and $\alpha \in \mathbb{R}$. Thus,

$$|y(x)| \leq \frac{2}{\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2}\theta.$$

The trivial solution $y_0(x) = 0$ ensures that $|y(x) - y_0(x)| \leq C\theta$ with $C = \frac{2}{\alpha^2-(k^2+1)b^2-2ab+2a^2}$. This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the Hyers-Ulam stability. □

Theorem 2.2. *Let σ be a non-negative continuous function defined on $[a, b]$. In addition, suppose that there is $\beta \in \mathbb{R}$ such that*

$$\int_a^x \sigma(t)dt \leq \beta\sigma(x) \tag{12}$$

for all $x \in [a, b]$.

Let $\alpha \in \mathbb{R}$ and $y \in C^2([a, b])$ with $0 \leq y'(x) \leq ky(x)$ for all $x \in [a, b]$, and some $k > 0$.

If $y \in C^2([a, b])$ is such that

$$\left|x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x)\right| \leq \sigma(x) \tag{13}$$

for all $x \in [a,b]$ and $\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2 > 0$, then there is a function $y_0 \in C^2([a,b])$ and a constant $C > 0$ such that

$$|y(x) - y_0(x)| \le C\,\sigma(x) \tag{14}$$

for all $x \in [a,b]$.

This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the Hyers-Ulam-Rassias stability.

Proof. By the differential inequality (13), we have

$$-\sigma(x) \le x^2 y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) \le \sigma(x).$$

Multiplying the above inequality by $y'(x)$, integrating from a to t and using the initial conditions (3) we obtain that

$$-2\int_a^t \sigma(x)y'(x)dx \le t^2(y'(t))^2 + (t^2-\alpha^2)(y(t))^2 - 2\int_a^t x(y(x))^2dx \le 2\int_a^t \sigma(x)y'(x)dx.$$

Having in mind the inequality on the left-hand side and taking into account that y is a non negative function with $y'(x) \le ky(x)$ for all $x \in [a,b]$ we get that

$$\begin{aligned}
(\alpha^2 - t^2)(y(t))^2 &\le 2\int_a^t \sigma(x)y'(x)dx + t^2(y'(t))^2 - 2\int_a^t x(y(x))^2dx \\
&\le 2\int_a^t \sigma(x)y'(x)dx + t^2(ky(t))^2 - 2\int_a^t x(y(x))^2dx \\
&\le 2k\int_a^t \sigma(x)y(x)dx + k^2t^2(y(t))^2 - 2a\int_a^t (y(x))^2dx \\
&\le 2k\int_a^t \sigma(x)y(x)dx + k^2t^2(y(t))^2 + 2a\int_a^t (y(x))^2dx.
\end{aligned}$$

Considering once again

$$M = \max_{t\in[a,b]} |y(t)|$$

it follows

$$\begin{aligned}
(\alpha^2 - t^2)M^2 &\le 2kM\int_a^t \sigma(x)dx + k^2t^2M^2 + 2a(b-a)M^2 \\
&\le 2kM\beta\sigma(t) + k^2t^2M^2 + 2a(b-a)M^2 \quad \Leftrightarrow \quad (\alpha^2 - (k^2+1)t^2)M^2 \le 2k\beta\sigma(t)M + 2a(b-a)M^2.
\end{aligned}$$

Taking into account that $\alpha^2 - (k^2+1)t^2$ is decreasing for $t \in [a,b]$, we obtain

$$\begin{aligned}
(\alpha^2 - (k^2+1)b^2)M^2 \le 2k\beta\sigma(t)M + 2a(b-a)M^2 \quad &\Leftrightarrow \quad (\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2)M^2 \le 2k\beta\sigma(t)M \\
&\Leftrightarrow \quad (\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2)M \le 2k\beta\sigma(t) \\
&\Leftrightarrow \quad M \le \frac{2k\beta}{\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2}\sigma(t)
\end{aligned}$$

if $\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2 > 0$ and $\alpha \in \mathbb{R}$. Thus,

$$|y(x)| \le \frac{2k\beta}{\alpha^2 - (k^2+1)b^2 - 2ab + 2a^2}\sigma(x)$$

for all $x \in [a,b]$.

The trivial solution $y_0(x) = 0$ guarantees that $|y(x) - y_0(x)| \le C\,\sigma(x)$ with $C = \frac{2k\beta}{\alpha^2-(k^2+1)b^2-2ab+2a^2}$. This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the Hyers-Ulam-Rassias stability. □

Theorem 2.3. *Let $\alpha \in \mathbb{R}$ and $y \in C^2([a,b])$ with $0 \le y'(x) \le ky(x)$ for all $x \in [a,b]$, and some $k > 0$.*

If $y \in C^2([a,b])$ is such that

$$\left|x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x)\right| \le \theta \tag{15}$$

for all $x \in [a,b]$, where $\theta \ge 0$, $\alpha^2 - (k^2+1)x^2 - 2ab + 2a^2 > 0$ and is decreasing on $[a,b]$, then there is a function $y_0 \in C^2([a,b])$, a constant $C > 0$ and a non-decreasing continuous function σ defined on $[a,b]$ such that

$$|y(x) - y_0(x)| \le C\,\sigma(x) \tag{16}$$

for all $x \in [a,b]$.

This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the σ-semi-Hyers-Ulam stability.

Proof. For every $\theta \ge 0$, by the differential inequality (15), we have

$$-\theta \le x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) \le \theta.$$

Multiplying the above inequality by $y'(x)$, integrating from a to t and using the initial conditions (3) we obtain that

$$\begin{aligned}
&-\theta\int_a^t y'(x)dx \le \int_a^t x^2y''(x)y'(x)dx + \int_a^t xy'(x)y'(x)dx + \int_a^t (x^2-\alpha^2)y(x)y'(x)dx \le \theta\int_a^t y'(x)dx\\
&\Leftrightarrow -\theta y(t) \le \frac{1}{2}t^2(y'(t))^2 + \frac{1}{2}(t^2-\alpha^2)(y(t))^2 - \int_a^t x(y(x))^2dx \le \theta y(t)\\
&\Leftrightarrow -2\theta y(t) \le t^2(y'(t))^2 + (t^2-\alpha^2)(y(t))^2 - 2\int_a^t x(y(x))^2dx \le 2\theta y(t).
\end{aligned}$$

Considering the inequality on the left-hand side and taking into account that $y'(x) \le ky(x)$ for all $x \in [a,b]$ we get that

$$\begin{aligned}
(\alpha^2 - t^2)(y(t))^2 &\le 2\theta y(t) + t^2(y'(t))^2 - 2\int_a^t x(y(x))^2dx\\
&\le 2\theta y(t) + t^2(ky(t))^2 - 2\int_a^t x(y(x))^2dx\\
&\le 2\theta y(t) + k^2t^2(y(t))^2 - 2a\int_a^t (y(x))^2dx\\
&\le 2\theta y(t) + k^2t^2(y(t))^2 + 2a\int_a^t (y(x))^2dx.
\end{aligned}$$

Taking

$$M = \max_{t\in[a,b]} |y(t)|,$$

we obtain

$$\begin{aligned}
(\alpha^2 - t^2)M^2 \le 2\theta M + k^2t^2M^2 + 2a(b-a)M^2 \quad &\Leftrightarrow \quad (\alpha^2 - (k^2+1)t^2 - 2ab + 2a^2)M^2 \le 2\theta M\\
&\Leftrightarrow \quad (\alpha^2 - (k^2+1)t^2 - 2ab + 2a^2)M \le 2\theta\\
&\Leftrightarrow \quad M \le \frac{2}{\alpha^2 - (k^2+1)t^2 - 2ab + 2a^2}\theta
\end{aligned}$$

if $\alpha^2 - (k^2+1)t^2 - 2ab + 2a^2 > 0$ for all $t \in [a,b]$ and $\alpha \in \mathbb{R}$. Thus,

$$|y(x)| \le \frac{2}{\alpha^2 - (k^2+1)x^2 - 2ab + 2a^2}\theta$$

for all $x \in [a,b]$.

The trivial solution $y_0(x) = 0$ allows us to realize that $|y(x) - y_0(x)| \leq C\,\sigma(x)$ with $\sigma(x) = \frac{2}{\alpha^2-(k^2+1)x^2-2ab+2a^2}$ a non-decreasing continuous function and $C = \theta$. This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the σ-semi-Hyers-Ulam stability. □

3. Hyers-Ulam, Hyers-Ulam-Rassias and σ-semi-Hyers-Ulam stabilities for $\alpha \in i\mathbb{R}$

Until recently, the Bessel equation with complex indices was considered an equation with little applicability in the applied sciences. However, the normal and modified Bessel equation with pure imaginary index have proved to be useful for the study of the solution governing the motion of charged particles in isotrajectory quadrupoles. They also play an important role in the study of solutions of the Lame's equation, which characterizes the displacement and deformation field distribution in semiconductor nanostructures. Therefore, the present section is devoted to present sufficient conditions for the Hyers-Ulam stability, the Hyers-Ulam-Rassias stability and the σ-semi-Hyers-Ulam stability of the Bessel equation (1), where $x \in [a,b]$ and $\alpha \in i\mathbb{R}$, for some fixed positive real numbers a and b.

Theorem 3.1. *Let $\alpha \in i\mathbb{R}$ and $y \in C^2([a,b])$ with $y'(x) \geq 0$ for all $x \in [a,b]$.*
If $y \in C^2([a,b])$ is such that

$$\left|x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x)\right| \leq \theta \tag{17}$$

for all $x \in [a,b]$, where $\theta \geq 0$ and $a^2 - \alpha^2 - 2b^2 + 2ab > 0$, then there is a function $y_0 \in C^2([a,b])$ and a constant $C > 0$ such that

$$|y(x) - y_0(x)| \leq C\,\theta \tag{18}$$

for all $x \in [a,b]$.
This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the Hyers-Ulam stability.

Proof. For every $\theta \geq 0$, by the differential inequality (17), we have

$$-\theta \leq x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) \leq \theta.$$

Multiplying the above inequality by $y'(x)$, integrating from a to t and using the initial conditions (3) we obtain that

$$\begin{aligned}
&-\theta \int_a^t y'(x)dx \leq \int_a^t x^2y''(x)y'(x)dx + \int_a^t xy'(x)y'(x)dx + \int_a^t (x^2 - \alpha^2)y(x)y'(x)dx \leq \theta \int_a^t y'(x)dx \\
&\Leftrightarrow -\theta y(t) \leq \frac{1}{2}t^2(y'(t))^2 + \frac{1}{2}(t^2 - \alpha^2)(y(t))^2 - \int_a^t x(y(x))^2dx \leq \theta y(t) \\
&\Leftrightarrow -2\theta y(t) \leq t^2(y'(t))^2 + (t^2 - \alpha^2)(y(t))^2 - 2\int_a^t x(y(x))^2dx \leq 2\theta y(t).
\end{aligned}$$

Having in mind the inequality on the right-hand side we derive that

$$\begin{aligned}
(t^2 - \alpha^2)(y(t))^2 &\leq 2\theta y(t) - t^2(y'(t))^2 + 2\int_a^t x(y(x))^2dx \\
&\leq 2\theta y(t) + 2\int_a^t x(y(x))^2dx \\
&\leq 2\theta y(t) + 2b\int_a^t (y(x))^2dx.
\end{aligned}$$

Let us consider

$$M = \max_{t\in[a,b]} |y(t)|.$$

Taking into account that $(t^2 - \alpha^2)$ is non-decreasing for $t \in [a, b]$, we obtain

$$\begin{aligned}(a^2 - \alpha^2)M^2 \le 2\theta M + 2b(b-a)M^2 \quad &\Leftrightarrow \quad (a^2 - \alpha^2 - 2b^2 + 2ab)M^2 \le 2\theta M \\ &\Leftrightarrow \quad (a^2 - \alpha^2 - 2b^2 + 2ab)M \le 2\theta \\ &\Leftrightarrow \quad M \le \frac{2}{a^2 - \alpha^2 - 2b^2 + 2ab}\theta\end{aligned}$$

if $a^2 - \alpha^2 - 2b^2 + 2ab > 0$ and $\alpha \in i\mathbb{R}$. Thus,

$$|y(x)| \le \frac{2}{a^2 - \alpha^2 - 2b^2 + 2ab}\theta.$$

The trivial solution $y_0(x) = 0$ assures us that $|y(x) - y_0(x)| \le C\theta$ with $C = \frac{2}{a^2-\alpha^2-2b^2+2ab}$. This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the Hyers-Ulam stability. □

Theorem 3.2. *Let σ be a non-negative continuous function defined on $[a, b]$. In addition, suppose that there is $\beta \in \mathbb{R}$ such that*

$$\int_a^x \sigma(t)dt \le \beta\sigma(x) \tag{19}$$

for all $x \in [a, b]$.

Let $\alpha \in i\mathbb{R}$ and $y \in C^2([a, b])$ with $0 \le y'(x) \le ky(x)$ for all $x \in [a, b]$ and some $k > 0$.

If $y \in C^2([a, b])$ is such that

$$\left|x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x)\right| \le \sigma(x) \tag{20}$$

for all $x \in [a, b]$ and $a^2 - \alpha^2 - 2b^2 + 2ab > 0$, then there is a function $y_0 \in C^2([a, b])$ and a constant $C > 0$ such that

$$|y(x) - y_0(x)| \le C\,\sigma(x) \tag{21}$$

for all $x \in [a, b]$.

This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the Hyers-Ulam-Rassias stability.

Proof. By the differential inequality (20), we have

$$-\sigma(x) \le x^2y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) \le \sigma(x).$$

Multiplying the above inequality by $y'(x)$, integrating from a to t and using the initial conditions (3) we obtain that

$$\begin{aligned}&-\int_a^t \sigma(x)y'(x)dx \le \int_a^t x^2y''(x)y'(x)dx + \int_a^t xy'(x)y'(x)dx + \int_a^t (x^2-\alpha^2)y(x)y'(x)dx \le \int_a^t \sigma(x)y'(x)dx \\ &\Leftrightarrow -\int_a^t \sigma(x)y'(x)dx \le \frac{1}{2}t^2(y'(t))^2 + \frac{1}{2}(t^2-\alpha^2)(y(t))^2 - \int_a^t x(y(x))^2dx \le \int_a^t \sigma(x)y'(x)dx \\ &\Leftrightarrow -2\int_a^t \sigma(x)y'(x)dx \le t^2(y'(t))^2 + (t^2-\alpha^2)(y(t))^2 - 2\int_a^t x(y(x))^2dx \le 2\int_a^t \sigma(x)y'(x)dx\end{aligned}$$

Having in mind the inequality on the right-hand side and taking into account that y is a non negative function with $y'(x) \le ky(x)$ for all $x \in [a, b]$ we get that

$$\begin{aligned}(t^2-\alpha^2)(y(t))^2 &\le 2\int_a^t \sigma(x)y'(x)dx + 2\int_a^t x(y(x))^2dx - t^2(y'(t))^2 \\ &\le 2\int_a^t \sigma(x)y'(x)dx + 2\int_a^t x(y(x))^2dx \\ &\le 2k\int_a^t \sigma(x)y(x)dx + 2b\int_a^t (y(x))^2dx.\end{aligned}$$

Fixing

$$M = \max_{t\in[a,b]} |y(t)|,$$

we have

$$\begin{aligned}(t^2-\alpha^2)M^2 &\le 2kM\int_a^t \sigma(x)dx + 2b(b-a)M^2\\ &\le 2kM\beta\sigma(t) + 2b(b-a)M^2.\end{aligned}$$

Taking into account that $(t^2-\alpha^2)$ is non-decreasing for $t\in[a,b]$, we obtain

$$\begin{aligned}(a^2-\alpha^2)M^2 \le 2k\beta\sigma(t)M + 2b(b-a)M^2 \quad &\Leftrightarrow \quad (a^2-\alpha^2-2b^2+2ab)M^2 \le 2k\beta\sigma(t)M\\ &\Leftrightarrow \quad (a^2-\alpha^2-2b^2+2ab)M \le 2k\beta\sigma(t)\\ &\Leftrightarrow \quad M \le \frac{2k\beta}{a^2-\alpha^2-2b^2+2ab}\sigma(t).\end{aligned}$$

if $a^2-\alpha^2-2b^2+2ab>0$ and $\alpha\in i\mathbb{R}$. Thus,

$$|y(x)| \le \frac{2k\beta}{a^2-\alpha^2-2b^2+2ab}\sigma(t).$$

The trivial solution $y_0(x)=0$ enable us to see that $|y(x)-y_0(x)|\le C\,\sigma(x)$ with $C=\frac{2k\beta}{a^2-\alpha^2-2b^2+2ab}$. This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the Hyers-Ulam-Rassias stability. □

Theorem 3.3. *Let $\alpha\in i\mathbb{R}$ and $y\in C^2([a,b])$ with $y'(x)\ge 0$ for all $x\in[a,b]$.*
If $y\in C^2([a,b])$ is such that

$$\left|x^2y''(x)+xy'(x)+(x^2-\alpha^2)y(x)\right| \le \theta \tag{22}$$

for all $x\in[a,b]$, where $\theta\ge 0$ and $x^2-\alpha^2-2b^2+2ab>0$ is decreasing on $[a,b]$, then there is a function $y_0\in C^2([a,b])$, a constant $C>0$ and a non-decreasing continuous function σ defined on $[a,b]$ such that

$$|y(x)-y_0(x)| \le C\,\sigma(x) \tag{23}$$

for all $x\in[a,b]$.

This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the σ-semi-Hyers-Ulam stability.

Proof. For every $\theta\ge 0$, by the differential inequality (22), we have

$$-\theta \le x^2y''(x)+xy'(x)+(x^2-\alpha^2)y(x)\le\theta.$$

Multiplying the above inequality by $y'(x)$, integrating from a to t and using the initial conditions (3) we obtain that

$$\begin{aligned}&-\theta\int_a^t y'(x)dx \le \int_a^t x^2y''(x)y'(x)dx + \int_a^t xy'(x)y'(x)dx + \int_a^t (x^2-\alpha^2)y(x)y'(x)dx \le \theta\int_a^t y'(x)dx\\ &\Leftrightarrow -\theta y(t) \le \frac{1}{2}t^2(y'(t))^2 + \frac{1}{2}(t^2-\alpha^2)(y(t))^2 - \int_a^t x(y(x))^2dx \le \theta y(t)\\ &\Leftrightarrow -2\theta y(t) \le t^2(y'(t))^2 + (t^2-\alpha^2)(y(t))^2 - 2\int_a^t x(y(x))^2dx \le 2\theta y(t).\end{aligned}$$

Using the inequality on the right-hand side we get that

$$\begin{aligned}(t^2-\alpha^2)(y(t))^2 &\leq 2\theta y(t) - t^2(y'(t))^2 + 2\int_a^t x(y(x))^2dx\\ &\leq 2\theta y(t) + 2\int_a^t x(y(x))^2dx\\ &\leq 2\theta y(t) + 2b\int_a^t (y(x))^2dx.\end{aligned}$$

Taking

$$M = \max_{t\in[a,b]} |y(t)|,$$

it follows

$$\begin{aligned}(t^2-\alpha^2)M^2 \leq 2\theta M + 2b(b-a)M^2 \quad &\Leftrightarrow \quad (t^2-\alpha^2-2b^2+2ab)M^2 \leq 2\theta M\\ &\Leftrightarrow \quad (t^2-\alpha^2-2b^2+2ab)M \leq 2\theta\\ &\Leftrightarrow \quad M \leq \frac{2}{t^2-\alpha^2-2b^2+2ab}\theta\end{aligned}$$

if $t^2-\alpha^2-2b^2+2ab>0$ for all $t\in[a,b]$ and $\alpha\in i\mathbb{R}$. Thus,

$$|y(x)| \leq \frac{2}{x^2-\alpha^2-2b^2+2ab}\theta.$$

The trivial solution $y_0(x)=0$ enable us to conclude that $|y(x)-y_0(x)|\leq C\,\sigma(x)$ with $\sigma(x)=\frac{2}{x^2-\alpha^2-2b^2+2ab}$ a non-decreasing continuous function and $C=\theta$. This means that under the above conditions, the Bessel equation (1) with initial conditions (3) has the σ-semi-Hyers-Ulam stability. □

4. Stabilities for the modified Bessel equation

The present section is devoted to present sufficient conditions for the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the modified Bessel equation (2), where $x\in[a,b]$ and $\alpha\in\mathbb{R}$, for some fixed positive real numbers a and b.

Theorem 4.1. *Let $\alpha\in\mathbb{R}$ and $y\in C^2([a,b])$ with $0\leq y'(x)\leq ky(x)$ for all $x\in[a,b]$ and some $k>0$.*
If y is such that

$$\left|x^2y''(x)+xy'(x)-(x^2+\alpha^2)y(x)\right| \leq \theta, \qquad x\in[a,b], \tag{24}$$

where $\theta\geq 0$ and $\alpha^2-(k^2+1)b^2+2ab>0$, then there is a function $y_0\in C^2([a,b])$ and a constant $C>0$ such that

$$|y(x)-y_0(x)| \leq C\,\theta \tag{25}$$

for all $x\in[a,b]$.
This means that under the above conditions, the modified Bessel equation (2) with initial conditions (3) has the Hyers-Ulam stability.

Proof. For every $\theta\geq 0$, by the differential inequality (24), we have

$$-\theta \leq x^2y''(x)+xy'(x)-(x^2+\alpha^2)y(x) \leq \theta.$$

Multiplying the above inequality by $y'(x)$, integrating from a to t and using the initial conditions (3) we obtain that

$$\begin{aligned}
&-\theta\int_a^t y'(x)dx \le \int_a^t x^2y''(x)y'(x)dx + \int_a^t xy'(x)y'(x)dx - \int_a^t (x^2+\alpha^2)y(x)y'(x)dx \le \theta\int_a^t y'(x)dx\\
&\Leftrightarrow -\theta y(t) \le \frac{1}{2}t^2(y'(t))^2 - \frac{1}{2}(t^2+\alpha^2)(y(t))^2 + \int_a^t x(y(x))^2dx \le \theta y(t)\\
&\Leftrightarrow -2\theta y(t) \le t^2(y'(t))^2 - (t^2+\alpha^2)(y(t))^2 + 2\int_a^t x(y(x))^2dx \le 2\theta y(t).
\end{aligned}$$

Using the inequality on the left-hand side and taking into account that $y'(x) \le ky(x)$ for all $x \in [a,b]$ we have

$$\begin{aligned}
(t^2+\alpha^2)(y(t))^2 &\le t^2(y'(t))^2 + 2\theta y(t) + 2\int_a^t x(y(x))^2dx\\
&\le k^2t^2(y(t))^2 + 2\theta y(t) + 2b\int_a^t (y(x))^2dx.
\end{aligned}$$

When considering

$$M = \max_{t\in[a,b]} |y(t)|,$$

we obtain

$$(t^2+\alpha^2)M^2 \le k^2t^2M^2 + 2\theta M + 2b(b-a)M^2$$

and so

$$\begin{aligned}
(\alpha^2-(k^2+1)b^2+2ab)M^2 \le 2\theta M &\Leftrightarrow (\alpha^2-(k^2+1)b^2+2ab)M \le 2\theta\\
&\Leftrightarrow M \le \frac{2}{\alpha^2-(k^2+1)b^2+2ab}\theta
\end{aligned}$$

if $\alpha^2-(k^2+1)b^2+2ab > 0$. Thus,

$$|y(x)| \le \frac{2}{\alpha^2-(k^2+1)b^2+2ab}\theta$$

for all $x \in [a,b]$.

The trivial solution $y_0(x) = 0$ allows us to conclude that $|y(x)-y_0(x)| \le C\,\theta$ with $C = \frac{2}{\alpha^2-(k^2+1)b^2+2ab}$. This means that under the above conditions, the modified Bessel equation (2) with initial conditions (3) has the Hyers-Ulam stability. □

Theorem 4.2. *Let σ be a non-negative continuous function defined on $[a,b]$. In addition, suppose that there is $\beta \in \mathbb{R}$ such that*

$$\int_a^x \sigma(t)dt \le \beta\sigma(x) \tag{26}$$

for all $x \in [a,b]$.

Let $\alpha \in \mathbb{R}$ and $y \in C^2([a,b])$ with $0 \le y'(x) \le ky(x)$ for all $x \in [a,b]$ and some $k > 0$.

If $y \in C^2([a,b])$ is such that

$$\left|x^2y''(x) + xy'(x) - (x^2+\alpha^2)y(x)\right| \le \sigma(x), \qquad x \in [a,b], \tag{27}$$

where $\alpha^2-(k^2+1)b^2+2ab > 0$, then there is a function $y_0 \in C^2([a,b])$ and a constant $C > 0$ such that

$$|y(x)-y_0(x)| \le C\,\sigma(x) \tag{28}$$

for all $x \in [a,b]$.

This means that under the above conditions, the modified Bessel equation (2) with initial conditions (3) has the Hyers-Ulam-Rassias stability.

Proof. By the differential inequality (27), we have

$$-\sigma(x) \le x^2y''(x) + xy'(x) - (x^2+\alpha^2)y(x) \le \sigma(x).$$

Multiplying the above inequality by $y'(x)$, integrating from a to t and using the initial conditions (3) we obtain that

$$\begin{aligned}
&-\int_a^t \sigma(x)y'(x)dx \le \int_a^t x^2y''(x)y'(x)dx + \int_a^t xy'(x)y'(x)dx - \int_a^t (x^2+\alpha^2)y(x)y'(x)dx \le \int_a^t \sigma(x)y'(x)dx\\
&\Leftrightarrow -\int_a^t \sigma(x)y'(x)dx \le \frac{1}{2}t^2(y'(t))^2 - \frac{1}{2}(t^2+\alpha^2)(y(t))^2 + \int_a^t x(y(x))^2dx \le \int_a^t \sigma(x)y'(x)dx\\
&\Leftrightarrow -2\int_a^t \sigma(x)y'(x)dx \le t^2(y'(t))^2 - (t^2+\alpha^2)(y(t))^2 + 2\int_a^t x(y(x))^2dx \le 2\int_a^t \sigma(x)y'(x)dx.
\end{aligned}$$

Taking profit from the inequality on the left-hand side and taking into account that $y'(x) \le ky(x)$ for all $x \in [a,b]$ we get that

$$\begin{aligned}
(t^2+\alpha^2)(y(t))^2 &\le t^2(y'(t))^2 + 2\int_a^t x(y(x))^2dx + 2\int_a^t \sigma(x)y'(x)dx\\
&\le k^2t^2(y(t))^2 + 2b\int_a^t (y(x))^2dx + 2k\int_a^t \sigma(x)y(x)dx.
\end{aligned}$$

Setting

$$M = \max_{t\in[a,b]} |y(t)|,$$

it follows

$$\begin{aligned}
(t^2+\alpha^2)M^2 &\le k^2t^2M^2 + 2b(b-a)M^2 + 2kM\int_a^t \sigma(x)dx\\
&\le k^2t^2M^2 + 2b(b-a)M^2 + 2kM\beta\sigma(t)
\end{aligned}$$

and so

$$(\alpha^2 - (k^2+1)b^2 + 2ab)M^2 \le 2k\beta\sigma(t)M \quad \Leftrightarrow \quad M \le \frac{2k\beta}{\alpha^2-(k^2+1)b^2+2ab}\sigma(t)$$

if $\alpha^2 - (k^2+1)b^2 + 2ab > 0$, for all $t \in [a,b]$. Thus,

$$|y(x)| \le \frac{2k\beta}{\alpha^2-(k^2+1)b^2+2ab}\sigma(x)$$

for all $x \in [a,b]$.

The trivial solution $y_0(x) = 0$ enable us to verify that $|y(x) - y_0(x)| \le C\,\sigma(x)$ with $C = \frac{2k\beta}{\alpha^2-(k^2+1)b^2+2ab}$. This means that under the above conditions, the modified Bessel equation (2) with initial conditions (3) has the Hyers-Ulam-Rassias stability. □

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