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Multi-Valued Perturbations to a Couple of Differential Inclusions Governed by Maximal Monotone Operators

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Abstract. This paper concerns the existence of absolutely continuous solutions for a couple of evolution problems, governed by time and state dependent maximal monotone operators and sweeping process to subsmooth sets, with multi-valued perturbations.

1. Introduction

In [8], the authors have proven in a real Hilbert space H, the existence of absolutely continuous solutions for a couple of evolution differential inclusions governed by time and state dependent maximal monotone operators and a time and state dependent closed convex sweeping process, with single-valued perturbations of Carathéodory type.

As a continuation of this work, we study in this paper, the existence of absolutely continuous solutions to the following differential system

$$(S_1)\begin{cases} -\dot{u}(t) \in A(t,v(t))u(t) + F(t,u(t),v(t)), & \text{a.e. } t \in [0,T] \\ u(t) \in D(A(t,v(t))), & \forall t \in [0,T] \\ -\dot{v}(t) \in N_{C(t,u(t))}(v(t)) + G(t,u(t),v(t)), & \text{a.e. } t \in [0,T] \\ v(t) \in C(t,u(t)), & \text{a.e. } t \in [0,T] \\ u(0) = u_0 \in D(A(0,v_0)), v(0) = v_0 \in C(0,u_0), \end{cases}$$

where for all $(t,x) \in [0,T] \times H, A(t,x)$ is a time and state dependent maximal monotone operator, $\{C(t,x) : (t,x) \in I \times H\}$ is an equi-uniformly subsmooth family of closed sets, $N_{C(t,x)}(\cdot)$ the Fréchet normal cone to $C(t,x)$, and F, G are set-valued maps with nonempty convex and closed values. So that, our main theorem generalizes the result in [8] in two directions, since we deal with multi-valued perturbations and the class of subsmooth sets, which strictly contains that of convex sets.

Such problems find many applications in mechanics, hysteresis systems, traffic equilibria, social and economic modelings, optimal control; to cite but a few topics. There is a vast related bibliography in the

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literature, we can cite for instance [3–6, 10, 11, 16, 18, 21–23, 25, 29, 31–34] and their references, and for some contributions to sweeping process with subsmooth sets we refer to [2, 19, 20, 26].

Our paper is organized as follows: In the next section we recall the background materials that we need in our proof, and in section 3 we state and prove our main result.

2. Notation and Preliminaries

Throughout this paper, $I := [0, T]$ ($T > 0$) and H is a real separable Hilbert space, with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. We denote by $\overline{B}_H(x,r)$ (resp. $B_H(x,r)$) the closed (resp. open) ball of H of center $x \in H$ and radius $r > 0$, and by $\overline{B}_H$ its closed unit ball. On I we consider the Lebesgue sigma-algebra $\mathcal{L}(I)$, and on H the Borel sigma-algebra $\mathcal{B}(H)$.

For $p \in [0, +\infty[$, we denote by $L^p(I,H)$ the quotient Banach space of measurable maps $u : I \longrightarrow H$, such that $t \mapsto \|u(t)\|^p$ is Lebesgue-integrable, equipped with its standard norm $\|u\|_{L^p} = \left(\int_0^T \|u(t)\|^p\right)^{\frac{1}{p}}$, and by $C(I,H)$ we denote the Banach space of continuous mappings $u : I \longrightarrow H$ endowed with the sup-norm $\|\cdot\|_C$. Finally, $W^{1,q}(I,H)$ ($q = 1, 2$), denotes the space of mappings $u \in C(I,H)$ such that their first derivatives $\dot{u} \in L^q(I,H)$.

For $S \subset H$, we denote by $co(S)$ the convex hull of S and by $\overline{co}(S)$ its closed convex hull. Recall that if S is a nonempty subset of H, then

$$\overline{co}(S) = \left\{x \in S : \ \forall x' \in H, \langle x', x\rangle \le \delta^*(x', S)\right\}, \tag{1}$$

where $\delta^*(\cdot, S)$ is the support function of S, i.e., $\delta^*(z, S) := \sup\limits_{x\in S}\langle z, x\rangle$, $\forall z \in H$.

A subset S of H is said to be ball compact, if its intersection with any closed ball of H is compact.

We denote by $\mathcal{H}(S_1, S_2)$ the Hausdorff distance between closed subsets S_1 and S_2 of H, which is defined by $\mathcal{H}(S_1, S_2) := \max\left\{\sup\limits_{x\in S_2} d(x, S_1), \sup\limits_{x\in S_1} d(x, S_2)\right\}$, where $d(x, S) := \inf\{\|x - y\| : \ y \in S\}$ is the distance from $x \in H$ to $S \subset H$, sometimes denoted by $d_S(x)$. The projection set of x into S is the set $Proj(x, S) := \left\{y \in S : \ d(x, S) = \|x - y\|\right\}$, if $Proj(x, S)$ is a singleton, its unique element will be denoted by $proj(x, S)$, in the case $x = 0$, this element, which is the element of minimal norm of S, will be denoted by S^0, i.e., $S^0 := proj(0, S)$.

2.1. Cones and subdifferentials

For more information on these notions and properties we refer the reader to [13–15, 24, 27, 28].

Let f be a proper lower semicontinuous function from H into $\mathbb{R} \cup \{+\infty\}$ and let x be any point where f is finite. We recall that the Clarke subdifferential of f at x is defined by

$$\partial^C f(x) = \left\{\xi \in H : \ \langle \xi, h\rangle \le f^{\uparrow}(x, h) \ \ \forall h \in H\right\},$$

where $f^{\uparrow}(x,\cdot)$ is the generalized Rockafellar directional derivative given by

$$f^{\uparrow}(x, h) = \limsup_{\substack{x' \longrightarrow^f x \\ t\downarrow 0}} \inf_{h' \longrightarrow h} \frac{f(x' + th') - f(x')}{t}.$$

The notation $x' \longrightarrow^f x$ means that $x' \longrightarrow x$ and $f(x') \longrightarrow f(x)$.

When f is locally Lipschitz, the Clarke subdifferential has also an other useful description

$$\partial^C f(x) = \left\{\xi \in H : \ \langle \xi, h\rangle \le f^0(x, h) \ \ \forall h \in H\right\},$$

where

$$f^0(x, h) = \limsup_{(x',t)\longrightarrow(x,0)} \frac{f(x' + th) - f(x')}{t}.$$

The Fréchet subdifferential of f at x is defined by:

$$\partial^F f(x) := \Big\{\zeta \in H : \forall \epsilon > 0, \exists \delta > 0, \forall \bar{x} \in B_H(x,\delta), \langle \zeta, \bar{x} - x\rangle \le f(\bar{x}) - f(x) + \epsilon\|\bar{x} - x\|\Big\}.$$

It is known that we always have the inclusion $\partial^F f(x) \subset \partial^C f(x)$ for all $x \in H$. By convention we set $\partial^C f(x) = \partial^F f(x) = \emptyset$, if $f(x)$ is not finite.

Let S be a nonempty and closed subset of H and $x \in S$. Let us recall that the Clarke normal cone (resp. the Fréchet normal cone) of S at x, $N_S^C(x)$ (resp. $N_S^F(x)$) is the subdifferential of the indicator function of S, i.e., $N_S^C(x) = \partial^C \delta_S(x)$ (resp. $N_S^F(x) = \partial^F \delta_S(x)$), where $\delta_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise, so that we have always the inclusion $N_S^F(x) \subset N_S^C(x)$ for all $x \in S$. The Fréchet normal cone is also related to the Fréchet subdifferential of the distance function since for all $x \in S$

$$\partial^F d_S(x) = N_S^F(x) \cap \overline{B}_H \tag{2}$$

and observe that if $y \in Proj(x,S)$, one has

$$x - y \in N_S^F(y) \text{ and so } x - y \in N_S^C(y). \tag{3}$$

We introduce in the following, the definition and some properties of subsmooth sets, and we refer the reader to [1] for more details. See also [30].

Definition 2.1. *We say that a nonempty and closed subset S of H is subsmooth at $x_0 \in S$, if for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $x_1, x_2 \in B_H(x_0,\delta) \cap S$ and all $\zeta_i \in N_S^C(x_i) \cap \overline{B}_H$ $(i = 1,2)$, we have*

$$\langle \zeta_1 - \zeta_2, x_1 - x_2\rangle \ge -\epsilon\|x_1 - x_2\|. \tag{4}$$

The set S is subsmooth, if it is subsmooth at each point of S. We further say that S is uniformly subsmooth, if for every $\epsilon > 0$, there exists $\delta > 0$, such that (4) holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and for all $\zeta_i \in N_S^C(x_i) \cap \overline{B}_H$ $(i = 1,2)$.

Proposition 2.2. *Let S be a closed subset of H and $x_0 \in S$. If S is subsmooth at x_0, then it is normally Fréchet regular at x_0, that is $N_S^F(x_0) = N_S^C(x_0)$. Furthermore, $\partial^C d_S(x_0) = \partial^F d_S(x_0)$.*

We also introduce the concept of equi-uniform subsmoothness for a family of closed sets.

Definition 2.3. *[1] Let $(S(q))_{q\in Q}$ be a family of closed sets of H with parameter $q \in Q$. This family is called equi-uniformly subsmooth, if for every $\epsilon > 0$, there exists $\delta > 0$, such that for each $q \in Q$, the inequality (4) holds for all $x_1, x_2 \in S(q)$ satisfying $\|x_1 - x_2\| < \delta$, and for all $\zeta_i \in N_{S(q)}^C(x_i) \cap \overline{B}_H$ $(i = 1,2)$.*

Hereafter, we will denote by $N_S(\cdot)$ and $\partial d_S(\cdot)$, the Clarke normal cone and subdifferential of the distance function to S, instead of $N_S^C(\cdot)$ and $d_S^C(\cdot)$.

We close this subsection by the following proposition, which is crucial for the statement of our main theorem. We refer the reader to [1] for the proof.

Proposition 2.4. *Let $\{K(t,x) : (t,x) \in I \times H\}$ be a family of nonempty closed sets of H, which is equi-uniformly subsmooth and let a real $\eta \ge 0$. Assume that there exist a real constant $L \ge 0$ and a continuous function $\vartheta : I \to \mathbb{R}$ such that, for any $x_1, x_2, y \in H$ and $t, s \in I$*

$$|d(y, K(t,x_1)) - d(y, K(s,x_2))| \le |\vartheta(t) - \vartheta(s)| + L\|x_1 - x_2\|.$$

Then, the following assertions hold.
(i) *For all $(t,x,y) \in gph(K)$ we have $\eta\partial d_{K(t,x)}(y) \subset \eta\overline{B}_H$.*
(ii) *For any sequence $(t_n)_n \subset I$ converging to t, any sequence $(x_n)_n$ converging to x, any sequence $(y_n)_n$ converging to $y \in K(t,x)$ with $y_n \in K(t_n,x_n)$, and any $\xi \in H$, we have*

$$\limsup_{n\to+\infty} \delta^*(\xi, \eta\partial d_{K(t_n,x_n)}(y_n)) \le \delta^*(\xi, \eta\partial d_{K(t,x)}(y)).$$

2.2. Maximal monotone operators

We recall in this subsection, see [7, 9, 35], definition and some properties of maximal monotone operators that we need after.

A set-valued operator A from H to H is a mapping from H into 2^H. Its domain and range are defined by

$$D(A) = \Big\{x \in H : \ Ax \neq \emptyset\Big\}, \qquad R(A) = \underset{x \in D(A)}{\cup} Ax,$$

while its graph is the following set

$$gph(A) = \Big\{(x, y) \in H \times H : \ x \in D(A), \ y \in Ax\Big\}.$$

The operator $A : D(A) \longrightarrow 2^H$ is monotone, if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ for all $(x_1, y_1), (x_2, y_2) \in gph(A)$. It is maximal monotone, if it is monotone and its graph could not be strictly contained in the graph of any other monotone operator, i.e., from Minty's Theorem, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, where I_H stands for the identity mapping of H.

If A is a maximal monotone operator, then for every $x \in D(A)$, Ax is nonempty, closed and convex. So that, the projection of the origin into Ax, $A^0(x)$, exists and is unique.

For $\lambda > 0$, the resolvent and the Yosida approximation of A are the single valued operators defined on all of H by $J_\lambda^A = (I_H + \lambda A)^{-1}$ and $A_\lambda = \frac{1}{\lambda}\left(I_H - J_\lambda^A\right)$, respectively. Furthermore, we have

$$J_\lambda^A(x) \in D(A) \quad \forall x \in H. \tag{5}$$

$$\|A_\lambda(x)\| \leq \|A^0(x)\| \quad \forall x \in D(A). \tag{6}$$

$$\|A_\lambda(x) - A_\lambda(y)\| \leq \frac{1}{\lambda}\|x - y\| \quad \forall x, y \in H. \tag{7}$$

We close this subsection by the definition of Vladimirov's pseudo distance (see [34]), and some fundamental lemmas crucial for our proof. We refer the reader to [21] for details.

Let $A : D(A) \longrightarrow 2^H$ and $B : D(B) \longrightarrow 2^H$ be maximal monotone operators, then we denote by $dis(A, B)$, the Vladimirov's pseudo-distance between A and B defined by

$$dis(A, B) = \sup\Big\{\frac{\langle y - y', x' - x\rangle}{1 + \|y\| + \|y'\|} : \ (x, y) \in gph(A), \ (x', y') \in gph(B)\Big\}. \tag{8}$$

Lemma 2.5. *Let $A : D(A) \longrightarrow 2^H$ be a maximal monotone operator. If $x \in \overline{D(A)}$ and $y \in H$ are such that*

$$\langle A^0(z) - y, z - x\rangle \geq 0 \ \ \forall z \in D(A),$$

then $x \in D(A)$ and $y \in Ax$.

Lemma 2.6. *Let $A_n : D(A_n) \longrightarrow 2^H$ $(n \in \mathbb{N})$ and $A : D(A) \longrightarrow 2^H$ be maximal monotone operators such that $dis(A_n, A) \longrightarrow 0$. Suppose also that $x_n \in D(A_n)$ with $x_n \longrightarrow x$ and $y_n \in A_n x_n$ with $y_n \longrightarrow y$ weakly for some $x, y \in H$. Then $x \in D(A)$ and $y \in Ax$.*

Lemma 2.7. *Let $A_n : D(A_n) \longrightarrow 2^H$ $(n \in \mathbb{N})$ and $A : D(A) \longrightarrow 2^H$ be maximal monotone operators such that $dis(A_n, A) \longrightarrow 0$ and for some $c > 0$, all $n \in \mathbb{N}$ and $x \in D(A_n)$, $\|A_n^0(x)\| \leq c\,(1 + \|x\|)$. Then for every $z \in D(A)$ there exists a sequence (ζ_n) such that*

$$\zeta_n \in D(A_n), \ \ \zeta_n \longrightarrow z \ \ \text{and} \ \ A_n^0(\zeta_n) \longrightarrow A^0(z). \tag{9}$$

3. Main result

Let H be a real separable Hilbert space. Let for every $(t,x) \in I \times H$, $A(t,x) : D(A(t,x)) \subset H \longrightarrow 2^H$ be a maximal monotone operator, and let $C : I \times H \rightrightarrows H$ be a set-valued map with nonempty closed values and $F, G : I \times H \times H \rightrightarrows H$ be set-valued maps with nonempty, closed and convex values.
We will state our main result under the following assumptions.
$(\mathcal{H}_A^0)$ $D(A(t,x)) = D$ for all $(t,x) \in I \times H$, that is $A(t,x)$ has fixed domain.
$(\mathcal{H}_A^1)$ There exist a nonnegative real constant λ and a nonnegative and nondecreasing function $\beta \in W^{1,1}(I,\mathbb{R})$, such that

$$dis\,(A(t,x), A(s,y)) \leq |\beta(t) - \beta(s)| + \lambda\|x-y\| \ \ \forall t,s \in I, \ \forall x,y \in H.$$

$(\mathcal{H}_A^2)$ There exists a nonnegative real constant c such that

$$\|A^0(t,x)y\| \leq c(1 + \|x\| + \|y\|) \ \ \forall t \in I, \ \forall x \in H, \ \forall y \in D(A(t,x)).$$

$(\mathcal{H}_A^3)$ D is relatively ball compact.
$(\mathcal{H}_C^1)$ The family $\left\{C(t,x) : \ (t,x) \in I \times H\right\}$ is equi-uniformly subsmooth.
$(\mathcal{H}_C^2)$ There exist a nonnegative real constant α, satisfying $\alpha\lambda < 1$, and a nonnegative and nondecreasing function $\eta \in W^{1,1}(I,\mathbb{R})$, such that

$$\left|d_{C(t,x)}(z) - d_{C(s,y)}(z)\right| \leq |\eta(t) - \eta(s)| + \alpha\|x-y\| \ \ \forall t,s \in I, \ \forall x,y,z \in H.$$

$(\mathcal{H}_C^3)$ For any bounded subset $E \subset H$, the set $C(I \times E)$ is ball compact.
$(\mathcal{H}_F^1)$ (resp. $(\mathcal{H}_G^1)$) F (resp. G) is $\mathcal{L}(I) \otimes \mathcal{B}(H) \otimes \mathcal{B}(H)$-measurable.
$(\mathcal{H}_F^2)$ (resp. $(\mathcal{H}_G^2)$) For each $t \in I$, $F(t,\cdot,\cdot)$ (resp. $G(t,\cdot,\cdot)$) is scalarly upper semi-continuous.
$(\mathcal{H}_F^3)$ (resp. $(\mathcal{H}_G^3)$) There exists a non negative real constant M_F (resp. M_G) such that

$$d(0, F(t,x,y)) \leq M_F(1 + \|x\| + \|y\|) \ \ \forall (t,x,y) \in I \times H \times H$$

(resp.

$$d(0, G(t,x,y)) \leq M_G(1 + \|x\| + \|y\|) \ \ \forall (t,x,y) \in I \times H \times H).$$

Now we present our main theorem. We follow ideas of the proof of Theorem 3.1 in [8]. We stress the fact that in [8] the functions β and η were taken in $W^{1,2}(I,\mathbb{R})$, in this work we have weaken these hypotheses by taking β and η in $W^{1,1}(I,\mathbb{R})$, with an additional assumption, that the domain of $A(t,x)$ is fixed (hypothesis $(\mathcal{H}_A^0)$).

Theorem 3.1. *Let for every $(t,x) \in I \times H$, $A(t,x) : D \longrightarrow 2^H$ be a maximal monotone operator satisfying $(\mathcal{H}_A^0)$, $(\mathcal{H}_A^1)$, $(\mathcal{H}_A^2)$ and $(\mathcal{H}_A^3)$.*
Let $C : I \times H \rightrightarrows H$ be a set-valued map with nonempty closed values satisfying $(\mathcal{H}_C^1)$, $(\mathcal{H}_C^2)$ and $(\mathcal{H}_C^3)$, and let $F : I \times H \times H \rightrightarrows H$ (resp. $G : I \times H \times H \rightrightarrows H$) be a set-valued map with nonempty, closed and convex values satisfying $(\mathcal{H}_F^1)$, $(\mathcal{H}_F^2)$ and $(\mathcal{H}_F^3)$ (resp. $(\mathcal{H}_G^1)$, $(\mathcal{H}_G^2)$ and $(\mathcal{H}_G^3)$). Then, for any $(u_0,v_0) \in D \times C(0,u_0)$, there exists an absolutely continuous solution $(u,v) : I \longrightarrow H \times H$ of the evolution problem (S_1). Furthermore, this solution satisfies, for almost every $t \in I$ and for some nonnegative real constants K and γ, the following estimates:

$$\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t) + \dot{\eta}(t)) \quad \text{and} \quad \|\dot{v}(t)\| \leq \gamma(1 + \dot{\beta}(t) + \dot{\eta}(t)).$$

Proof. For each $n \geq 1$, consider the partition $\{t_i^n : i = 0,1,..,n\}$ of the interval I, and for $i = 0,1,...,n-1$, set

$$\delta_{i+1}^n := |t_{i+1}^n - t_i^n|, \ \ \beta_{i+1}^n := |\beta(t_{i+1}^n) - \beta(t_i^n)|, \ \ \eta_{i+1}^n := |\eta(t_{i+1}^n) - \eta(t_i^n)|. \tag{10}$$

Without loss of generality, we suppose that $\beta(0) = \eta(0) = 0$.
Set $b(t) := t + \beta(t) + \eta(t)$ for all $t \in I$. Since β and η are absolutely continuous, the partition $\{t_i^n : i = 0,...,n\}$ can be chosen such that for all $i = 0,...,n-1$ and $n \geq 1$,

$$k_{i+1}^n := \delta_{i+1}^n + \beta_{i+1}^n + \eta_{i+1}^n \leq \frac{1}{n} b(T). \tag{11}$$

For each $(t,x,y) \in I \times H \times H$, let us denote by $f(t,x,y)$ (resp. $g(t,x,y)$) the element of minimal norm of the closed convex set $F(t,x,y)$ (resp. $G(t,x,y)$) of H, that is

$$f(t,x,y) = F^0(t,x,y) = proj(0, F(t,x,y))$$

$$g(t,x,y) = G^0(t,x,y) = proj(0, G(t,x,y)).$$

For each $(x,y) \in H \times H$, the map $t \mapsto f(t,x,y)$ (resp $t \mapsto g(t,x,y)$) is $\mathcal{L}(I)$-measurable thanks to $(\mathcal{H}_F^1)$ (resp. $(\mathcal{H}_G^1)$) and the separability of H; see Theorem III-41 in [12] for details. Furthermore, by hypotheses $(\mathcal{H}_F^3)$, $(\mathcal{H}_G^3)$,

$$\|f(t,x,y)\| \leq M_F(1 + \|x\| + \|y\|) \;\; \forall (t,x,y) \in I \times H \times H, \tag{12}$$

$$\|g(t,x,y)\| \leq M_G(1 + \|x\| + \|y\|) \;\; \forall (t,x,y) \in I \times H \times H. \tag{13}$$

Step 1. Construction of step mappings $(u_n(.))_n$ and $(v_n(.))_n$.
For each $n \geq 1$, define the mappings $u_n, v_n : I \longrightarrow H$ as follows: for $t \in [t_i^n, t_{i+1}^n[$, $0 \leq i \leq n-1$,

$$u_n(t) = u_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}\Big(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n, v_i^n)ds\Big) - \int_{t_i^n}^{t} f(s, u_i^n, v_i^n)ds \tag{14}$$

$$v_n(t) = v_i^n + \frac{b(t) - b(t_i^n)}{k_{i+1}^n}\Big(v_{i+1}^n - v_i^n + \int_{t_i^n}^{t_{i+1}^n} g(s, u_i^n, v_i^n)ds\Big) - \int_{t_i^n}^{t} g(s, u_i^n, v_i^n)ds \tag{15}$$

and $u_n(T) = u_n^n$, $v_n(T) = v_n^n$, where $v_0^n = v_0$, $u_0^n = u_0$ and for $i = 0, ..., n-1$,

$$v_{i+1}^n \in Proj\Big(v_i^n - \int_{t_i^n}^{t_{i+1}^n} g(s, u_i^n, v_i^n)ds,\, C(t_{i+1}^n, u_i^n)\Big), \tag{16}$$

$$u_{i+1}^n \;\;=\;\; J_{i+1}^n\Big(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n, v_i^n)ds\Big). \tag{17}$$

Here, $J_{i+1}^n := J_{\delta_{i+1}^n}^{A(t_{i+1}^n, v_{i+1}^n)} = \Big(I_H + \delta_{i+1}^n A(t_{i+1}^n, v_{i+1}^n)\Big)^{-1}$.

Observe that relation (16) is well defined since the sets $C(t,x)$ are ball-compact, and clearly, for $i = 0, ..., n-1$,

$$v_{i+1}^n \in C(t_{i+1}^n, u_i^n), \tag{18}$$

and from (3) and (16) we have

$$-(v_{i+1}^n - v_i^n) \in N_{C(t_{i+1}^n, u_i^n)}(v_{i+1}^n) + \int_{t_i^n}^{t_{i+1}^n} g(s, u_i^n, v_i^n)ds. \tag{19}$$

On the other hand, using relation (5), we have from (17),

$$u_{i+1}^n \in D\Big(A(t_{i+1}^n, v_{i+1}^n)\Big) = D, \tag{20}$$

and

$$-\frac{u_{i+1}^n - u_i^n}{\delta_{i+1}^n} \in A\left(t_{i+1}^n, v_{i+1}^n\right) u_{i+1}^n + \frac{1}{\delta_{i+1}^n}\int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n, v_i^n)ds. \tag{21}$$

Obviously, the mappings u_n and v_n are absolutely continuous, $u_n(t_i^n) = u_i^n$, $v_n(t_i^n) = v_i^n$, and for $t \in]t_i^n, t_{i+1}^n[$,

$$\dot{u}_n(t) = \frac{1}{\delta_{i+1}^n}\Big(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n, v_i^n)ds\Big) - f(t, u_i^n, v_i^n), \tag{22}$$

$$\dot{v}_n(t) = \frac{\dot{b}(t)}{k_{i+1}^n}\Big(v_{i+1}^n - v_i^n + \int_{t_i^n}^{t_{i+1}^n} g(s, u_i^n, v_i^n)ds\Big) - g(t, u_i^n, v_i^n). \tag{23}$$

From (21) and (22) we have

$$-\dot{u}_n(t) - f(t, u_i^n, v_i^n) \in A\left(t_{i+1}^n, v_{i+1}^n\right) u_{i+1}^n, \tag{24}$$

and from (19) and (23)

$$-\dot{v}_n(t) \in N_{C(t_{i+1}^n, u_i^n)}(v_{i+1}^n) + g(t, u_i^n, v_i^n). \tag{25}$$

To be able to continue the proof, we need the following estimates. We refer to the proof of Theorem 3.1 in [8].

Lemma 3.2. *Under the hypotheses $(\mathcal{H}_A^1)$, $(\mathcal{H}_A^2)$, $(\mathcal{H}_C^2)$, $(\mathcal{H}_F^3)$ and $(\mathcal{H}_G^3)$, there exist nonnegative real constants K, γ such that, for $n \geq 1$ and $i = 0, ..., n$,*

$$\|u_i^n\| \leq K \quad \textit{and} \quad \|v_i^n\| \leq \gamma. \tag{26}$$

Now, observe that from (22) we have for all $t \in]t_i^n, t_{i+1}^n[$,

$$\begin{aligned}
\|\dot{u}_n(t)\| &\leq \Big\|\frac{1}{\delta_{i+1}^n}\Big(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n, v_i^n)ds\Big)\Big\| + \|f(t, u_i^n, v_i^n)\| \\
&= \Big\|\frac{1}{\delta_{i+1}^n}\Big(J_{i+1}^n\Big(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n, v_i^n)ds\Big) - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n, v_i^n)ds\Big)\Big\| + \|f(t, u_i^n, v_i^n)\| \\
&= \Big\|A_{\delta_{i+1}^n}(t_{i+1}^n, v_{i+1}^n)\Big(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n, v_i^n)ds\Big)\Big\| + \|f(t, u_i^n, v_i^n)\| \\
&\leq \Big\|A_{\delta_{i+1}^n}(t_{i+1}^n, v_{i+1}^n)\Big(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n, v_i^n)ds\Big) - A_{\delta_{i+1}^n}(t_{i+1}^n, v_{i+1}^n)(u_i^n)\Big\| \\
&+ \|A_{\delta_{i+1}^n}(t_{i+1}^n, v_{i+1}^n)(u_i^n)\| + \|f(t, u_i^n, v_i^n)\|,
\end{aligned}$$

here, $A_{\delta_{i+1}^n}(t_{i+1}^n, v_{i+1}^n)$ is the Yosida approximation of $A(t_{i+1}^n, v_{i+1}^n)$, i.e., $A_{\delta_{i+1}^n}(t_{i+1}^n, v_{i+1}^n)(x) = \frac{1}{\delta_{i+1}^n}(x - J_{i+1}^n(x))$. Since $u_i^n \in D(A(t_i^n, v_i^n)) = D(A(t_{i+1}^n, v_{i+1}^n)) = D$, using (6) and (7) together with $(\mathcal{H}_F^3)$, $(\mathcal{H}_A^2)$ and (26), we get

$$\begin{aligned}
\|\dot{u}_n(t)\| &\leq \frac{1}{\delta_{i+1}^n}\int_{t_i^n}^{t_{i+1}^n} \|f(s, u_i^n, v_i^n)\|ds + \|A^0(t_{i+1}^n, v_{i+1}^n)(u_i^n)\| + \|f(t, u_i^n, v_i^n)\| \\
&\leq 2M_F(1 + \|u_i^n\| + \|v_i^n\|) + c(1 + \|u_i^n\| + \|v_{i+1}^n\|) \\
&\leq (2M_F + c)(1 + K + \gamma) =: R,
\end{aligned}$$

so that, there is a negligible subset N' of I such that for all $n \geq 1$,

$$\|\dot{u}_n(t)\| \leq R \quad \forall t \in I \setminus N'. \tag{27}$$

On the other hand, by $(\mathcal{H}_G^3)$, (23) and (26), for all $t \in]t_i^n, t_{i+1}^n[$,

$$\begin{aligned}
\|\dot{v}_n(t)\| &\leq \frac{\dot{b}(t)}{k_{i+1}^n}\|v_{i+1}^n - v_i^n\| + (\dot{b}(t) + 1)M_G(1 + K + \gamma) \\
&\leq \gamma\dot{b}(t) + (\dot{b}(t) + 1)M_G(1 + K + \gamma) =: \psi(t),
\end{aligned}$$

that is, there is a negligible subset N'' of I such that for all $n \geq 1$,

$$\|\dot{v}_n(t)\| \leq \psi(t) \quad \forall t \in I \setminus N'', \tag{28}$$

and this shows that $(\dot{v}_n)$ is integrably bounded since $\psi \in L^1(I, \mathbb{R})$.

Now, consider the step functions $\theta_n, \varphi_n : I \longrightarrow I$ defined by $\theta_n(t) = t_{i+1}^n$ and $\varphi_n(t) = t_i^n$ if $t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = \varphi_n(0) = 0$. Clearly,

$$|\theta_n(t) - t| \to 0 \ \text{ and } \ |\varphi_n(t) - t| \to 0 \ \text{ as } n \to +\infty. \tag{29}$$

For all $t \in I$, set $f_n(t) = f(t, u_n(\varphi_n(t)), v_n(\varphi_n(t)))$ and $g_n(t) = g(t, u_n(\varphi_n(t)), v_n(\varphi_n(t)))$.
From (18), (20), (24) and (25), it follows that for each $n \geq 1$, there exists a negligible subset N_n of I such that

$$-\dot{u}_n(t) \in A\left(\theta_n(t), v_n(\theta_n(t))\right) u_n(\theta_n(t)) + f_n(t) \ \ \forall t \in I \setminus N_n. \tag{30}$$

$$u_n(\theta_n(t)) \in D\left(A(\theta_n(t), v_n(\theta_n(t)))\right) = D \ \ \forall t \in I. \tag{31}$$

$$f_n(t) \in F(t, u_n(\varphi_n(t)), v_n(\varphi_n(t))) \ \ \forall t \in I. \tag{32}$$

$$-\dot{v}_n(t) \in N_{C\left(\theta_n(t), u_n(\varphi_n(t))\right)}(v_n(\theta_n(t))) + g_n(t) \ \ \forall t \in I \setminus N_n. \tag{33}$$

$$v_n(\theta_n(t)) \in C\left(\theta_n(t), u_n(\varphi_n(t))\right) \ \ \forall t \in I. \tag{34}$$

$$g_n(t) \in G(t, u_n(\varphi_n(t)), v_n(\varphi_n(t))) \ \ \forall t \in I. \tag{35}$$

Step 2. Convergence of the sequences.
For every $t, s \in I$ ($s \leq t$), we have by (27),

$$\|u_n(t) - u_n(s)\| \leq \int_s^t \|\dot{u}_n(\tau)\| \, d\tau \leq R|t - s| \quad \forall n \geq 1. \tag{36}$$

This shows that the sequence (u_n) is equicontinuous. On the other hand, from (26) and (31), we have for every $t \in I$,

$$(u_n(\theta_n(t)))_n \subset D \cap K\overline{\mathbf{B}}_H,$$

but, by $(\mathcal{H}_A^3)$, we know that the set in the right hand side of this inclusion is relatively compact. Then, the sequence $(u_n(\theta_n(t)))_n$ is relatively compact. Since from (36) and (29) we have for all $t \in I$, $\|u_n(\theta_n(t)) - u_n(t)\| \longrightarrow 0$ as $n \longrightarrow \infty$, we conclude that, for all $t \in I$, $\{u_n(t), n \geq 1\}$ is also relatively compact. By Arzelà-Ascoli Theorem, up to a subsequence, (u_n) converges uniformly and strongly to some mapping $u \in C(I, H)$. Moreover, we have for all $t \in I$,

$$\|u_n(\theta_n(t)) - u(t)\| \quad \leq \quad \|u_n(t) - u(t)\| + \|u_n(t) - u_n(\theta_n(t))\|.$$

Then,

$$\|u_n(\theta_n(t)) - u(t)\| \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty, \tag{37}$$

and similarly

$$\|u_n(\varphi_n(t)) - u(t)\| \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty. \tag{38}$$

As regards the convergence of (v_n), for every $t, s \in I$ ($s \leq t$), we have by (28),

$$\|v_n(t) - v_n(s)\| \leq \int_s^t \|\dot{v}_n(\tau)\| \, d\tau \leq \int_s^t \psi(\tau) d\tau \quad \forall n \geq 1. \tag{39}$$

This shows that the sequence (v_n) is equicontinuous. On the other hand, from (26) and (34), we have for all $t \in I$,

$$(v_n(\theta_n(t)))_n \subset C(I \times K\overline{\mathbf{B}}_H) \cap \gamma\overline{\mathbf{B}}_H.$$

Whence, by $(\mathcal{H}_C^3)$, the sequence $(v_n(\theta_n(t))_n$ is relatively compact, and since from (39) and (29), for all $t \in I$, $\|v_n(\theta_n(t)) - v_n(t)\| \longrightarrow 0$ as $n \longrightarrow \infty$, we conclude that for all $t \in I$, $\{v_n(t), n \geq 1\}$ is relatively compact in H. Then, $(v_n)_n$ is relatively compact in $\mathcal{C}(I, H)$, by extracting a subsequence, that we do not relabel, we conclude that it converges uniformly and strongly to some mapping $v \in \mathcal{C}(I, H)$. Moreover, for all $t \in I$, we have

$$\|v_n(\theta_n(t)) - v(t)\| \leq \|v_n(t) - v(t)\| + \|v_n(\theta_n(t)) - v_n(t)\| \underset{n \longrightarrow \infty}{\longrightarrow} 0, \tag{40}$$

and similarly

$$\|v_n(\varphi_n(t)) - v(t)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{41}$$

Now, from (27) and (28), we know that the sequences $(\dot{u}_n)$ and $(\dot{v}_n)$ are integrably bounded so, by extracting subsequences, not relabeled, we may assume that they converge weakly in $L^1(I, H)$ to $\dot{u}$ and $\dot{v}$, respectively (see [17]).

Step 3. Existence of a solution.
We start by proving that $u(t) \in D(A(t, v(t))) = D$ and $v(t) \in C(t, u(t))$ for all $t \in I$. Indeed, if we put for any fixed $t \in I$ and each $n \geq 1$, $A_n := A(\theta_n(t), v_n(\theta_n(t)))$, $A := A(t, v(t))$, $x_n := u_n(\theta_n(t)) \longrightarrow u(t)$ and $y_n := A_n^0(x_n)$, we get by $(\mathcal{H}_A^1)$, (10), (11) and (40),

$$dis\Big(A_n, A\Big) \leq |\beta(\theta_n(t)) - \beta(t)| + \lambda \|v_n(\theta_n(t)) - v(t)\| \underset{n \to +\infty}{\longrightarrow} 0, \tag{42}$$

and by (31), $x_n \in D(A_n)$, and by $(\mathcal{H}_A^2)$, (26), the sequence (y_n) is bounded, hence, up to a subsequence, it is weakly convergent in H. So that, by virtu of Lemma 2.6, we conclude that $u(t) \in D$ for all $t \in I$.
On the other hand, by $(\mathcal{H}_C^2)$, (11), (34), (38) and (40), for all $t \in I$,

$$\begin{aligned} & d\left(v(t), C(t, u(t))\right) \leq \|v_n(\theta_n(t)) - v(t)\| + d\left(v_n(\theta_n(t)), C(t, u(t))\right) \\ \leq\ & \|v_n(\theta_n(t)) - v(t)\| + |\eta(\theta_n(t)) - \eta(t)| + \alpha \|u_n(\varphi_n(t)) - u(t)\| \underset{n \to +\infty}{\longrightarrow} 0, \end{aligned}$$

which means that $v(t) \in C(t, u(t))$ since $C(t, u(t))$ is closed.
Next, since for all $t \in I$ and each $n \geq 1$,

$$\begin{aligned} \|f_n(t)\| &\leq M_F(1 + \|u_n(\varphi_n(t))\| + \|v_n(\varphi_n(t))\|) \\ &\leq M_F(1 + K + \gamma) \end{aligned}$$

and

$$\|g_n(t)\| \leq M_G(1 + K + \gamma), \tag{43}$$

we may suppose (taking subsequences if necessary) that (f_n) (resp. (g_n)) converges weakly in $L^1(I, H)$ to some mapping f (resp. g).

Now, since the sequence $(\dot{u}_n + f_n)$ (resp. (f_n)) converges weakly in $L^1(I, H)$ to $\dot{u} + f$ (resp. to f), by Mazur's Theorem, there is a sequence (z_j) (resp. $(\overline{z}_j)$) such that for each $j \in \mathbb{N}$, $z_j \in co\{\dot{u}_k + f_k, k \geq j\}$ (resp. $\overline{z}_j \in co\{f_k, k \geq j\}$ and (z_j) (resp. $(\overline{z}_j)$) converges strongly in $L^1(I, H)$ to $\dot{u} + f$ (resp. to f). Hence, there exists a subset I_0 (resp. I_1) of I with null Lebesgue-measure and a subsequence (j_p) (resp. $(j_{\bar{p}})$) of $\mathbb{N}$ such that for all $t \in I \setminus I_0$ (resp. $t \in I \setminus I_1$) $(z_{j_p}(t))$ (resp. $(\overline{z}_{j_{\bar{p}}}(t))$) converges to $\dot{u}(t) + f(t)$ (resp. to $f(t)$). So that, for $t \in I \setminus I_0$ (resp. $t \in I \setminus I_1$)

$$\dot{u}(t) + f(t) \in \bigcap_{p \in \mathbb{N}} \overline{co}\left\{\dot{u}_k(t) + f_k(t), k \geq j_p\right\},$$

(resp.

$$f(t) \in \bigcap_{\bar{p} \in \mathbb{N}} \overline{co}\left\{f_k(t), k \geq j_{\bar{p}}\right\},)$$

which means that for $t \in I \setminus I_0$ (resp. $t \in I \setminus I_1$) and for any $\zeta \in H$

$$\Big\langle \zeta, \dot{u}(t) + f(t) \Big\rangle \leq \limsup_{n\to\infty} \Big\langle \zeta, \dot{u}_n(t) + f_n(t) \Big\rangle. \tag{44}$$

(resp.

$$\Big\langle \zeta, f(t) \Big\rangle \leq \limsup_{n\to\infty} \Big\langle \zeta, f_n(t) \Big\rangle.) \tag{45}$$

Repeating the same arguments on the sequence $(\dot{v}_n + g_n)$ (resp. (g_n)), we infer the existence of a subset I'_0 (resp. I'_1) of I with null Lebesgue-measure such that for $t \in I \setminus I'_0$ (resp. $t \in I \setminus I'_1$) and for any $\zeta \in H$

$$\langle \zeta, \dot{v}(t) + g(t) \rangle \leq \limsup_{n\to\infty} \langle \zeta, \dot{v}_n(t) + g_n(t) \rangle \tag{46}$$

(resp.

$$\langle \zeta, g(t) \rangle \leq \limsup_{n\to\infty} \langle \zeta, g_n(t) \rangle.) \tag{47}$$

We finish par showing that u and v satisfy the system (S_1).
Using $(\mathcal{H}_A^2)$, we may apply Lemma 2.7, to the maximal monotone operators A_n and A, which verify (42), to ensure, for all $\xi \in D(A) = D$, the existence of a sequence (ξ_n) such that

$$\xi_n \in D\,(A_n) = D, \quad \xi_n \longrightarrow \xi \quad \text{and} \quad A_n^0(\xi_n) \longrightarrow A^0(\xi). \tag{48}$$

Since for every $(t, x) \in I \times H$, $A(t, x)$ is monotone, by (30), we have for $t \in I \setminus N_n$,

$$\Big\langle \dot{u}_n(t) + f_n(t), u_n(\theta_n(t)) - \xi_n \Big\rangle \leq \Big\langle A_n^0(\xi_n), \xi_n - u_n(\theta_n(t)) \Big\rangle. \tag{49}$$

Whence, from (27), (49) and $(\mathcal{H}_F^2)$ together with (26), we obtain for all $t \in I \setminus (\bigcup_{n\geq 1} N_n \bigcup N' \bigcup I_0)$,

$$\begin{aligned}
&\Big\langle \dot{u}_n(t) + f_n(t), u(t) - \xi \Big\rangle = \Big\langle \dot{u}_n(t) + f_n(t), u_n(\theta_n(t)) - \xi_n \Big\rangle \\
+ \quad &\Big\langle \dot{u}_n(t) + f_n(t), (u(t) - u_n(\theta_n(t))) - (\xi - \xi_n) \Big\rangle \\
\leq \quad &\Big\langle A_n^0(\xi_n), \xi_n - u_n(\theta_n(t)) \Big\rangle + (R + M_F(1 + K + \gamma))\Big(\|u_n(\theta_n(t)) - u(t)\| + \|\xi_n - \xi\|\Big).
\end{aligned}$$

Relations (37) and (48) then give

$$\limsup_{n\longrightarrow\infty} \Big\langle \dot{u}_n(t) + f_n(t), u(t) - \xi \Big\rangle \leq \Big\langle A^0(\xi), \xi - u(t) \Big\rangle,$$

which implies, by (44),

$$\Big\langle \dot{u}(t) + f(t), u(t) - \xi \Big\rangle \leq \Big\langle A^0(\xi), \xi - u(t) \Big\rangle.$$

Since for all $t \in I$, $u(t) \in D\,(A(t, v(t))) = D$, by Lemma 2.5, we conclude that

$$-\dot{u}(t) - f(t) \in A\,(t, v(t))\, u(t) \;\; a.e.\, t \in I.$$

Now, from (28) and (43), we have for almost every $t \in I$,
$(\dot{v}_n(t) - g_n(t)) \subset \bar{\psi}(t)\overline{B}_H$, where $\bar{\psi}(t) := \psi(t) + M_G(1 + K + \gamma)$, then from (33), (2) and Proposition 2.2, we have for almost every $t \in I$,

$$\begin{aligned}
-\dot{v}_n(t) - g_n(t) \quad &\in \quad N_{C(\theta_n(t), u_n(\varphi_n(t)))}(v_n(\theta_n(t))) \cap \bar{\psi}(t)\overline{B}_H \\
&= \quad \bar{\psi}(t)\partial d_{C(\theta_n(t), u_n(\varphi_n(t)))}(v_n(\theta_n(t))).
\end{aligned} \tag{50}$$

Fix any $t \in I \setminus \Big((\bigcup_{n\geq 1} N_n) \cup N'' \cup I'_0 \Big)$ and any $\zeta \in H$, this last relation and (46) give us, by the use of (*ii*) of Proposition 2.4 taking into account relations (29), (38) and (40)

$$\begin{aligned} \langle \zeta, \dot{v}(t) + g(t) \rangle &\leq \limsup_{n\to\infty} \delta^*\Big(\zeta, -\bar{\psi}(t)\partial d_{C(\theta_n(t), u_n(\varphi_n(t)))}(v_n(\theta_n(t)))\Big) \\ &\leq \delta^*\Big(\zeta, -\bar{\psi}(t)\partial d_{C(t,u(t))}(v(t))\Big). \end{aligned}$$

Since $-\bar{\psi}(t)\partial d_{C(t,u(t))}(v(t))$ is a convex closed set, we deduce from (1) that $-\dot{v}(t) - g(t) \in \bar{\psi}(t)\partial d_{C(t,u(t))}(v(t)) \subset N_{C(t,u(t))}(v(t))$.

It remains to check that for almost every $t \in I$, $f(t) \in F(t, u(t), v(t))$ (resp. $g(t) \in G(t, u(t), v(t))$). From (32) and (45) we have by the use of hypothesis $(\mathcal{H}_F^2)$,

$$\begin{aligned} \langle \zeta, f(t) \rangle &\leq \limsup_{n\to\infty} \delta^*\Big(\zeta, F(t, u_n(\theta_n(t)), v_n(\theta_n(t)))\Big) \\ &\leq \delta^*\Big(\zeta, F(t, u(t), v(t))\Big). \end{aligned}$$

Since F has closed and convex values, we conclude, by the use of relation (1), that $f(t) \in F(t, u(t), v(t))$ a.e. $t \in I$. Using similar arguments, we also get $g(t) \in G(t, u(t), v(t))$ a.e. $t \in I$. Consequently, (u, v) is a solution of (S_1). Furthermore, from (36) and (39) this solution satisfies

$$\|u(t) - u(s)\| \leq K|b(t) - b(s)|, \;\; \|v(t) - v(s)\| \leq \gamma|b(t) - b(s)| \;\; \forall t, s \in I,$$

that is, u and v are absolutely continuous. This completes the proof. □

Remark 3.3. *The statement of our Theorem 3.1 is also valid with no fixed domain $D(A(t, x))$ of the operator $A(t, x)$, if we take $\beta, \eta \in W^{1,2}(I, H)$. The proof will be similar to the proof of Theorem 3.1 in [8].*

We close the paper by the following corollaries.

Corollary 3.4. *Let $\mathbb{A}$ be a nonempty, convex and ball compact subset of H. Let C, F and G as in Theorem 3.1. Then, for any $(u_0, v_0) \in \mathbb{A} \times C(0, u_0)$, there exists a pair (u, v) of $W^{1,1}(I, H)$-mappings satisfying the evolution problem*

$$\begin{cases} -\dot{u}(t) \in N_{\mathbb{A}}(u(t)) + F(t, u(t), v(t)) \quad a.e.\, t \in I \\ -\dot{v}(t) \in N_{C(t,u(t))}(v(t)) + G(t, u(t), v(t)) \quad a.e.\, t \in I \\ u(t) \in \mathbb{A} \quad \forall t \in I \\ v(t) \in C(t, u(t)) \quad \forall t \in I \\ (u(0), v(0)) = (u_0, v_0). \end{cases}$$

Proof. It is known that if for all $(t, x) \in I \times H$, $B(t, x)$ is a closed convex subset of H, then the subdifferential of its indicator function, $\partial\delta_{B(t,x)} = N_{B(t,x)}$, is a maximal monotone operator, furthermore,

$$dis(N_{B(t,x)}, N_{B(s,y)}) = \mathcal{H}(B(t, x), B(s, y)) \quad \forall (t, x), (s, y) \in I \times H,$$

and $D(N_{B(t,x)}) = B(t, x)$. So that, our corollary is a direct consequence of Theorem 3.1 by taking for all $(t, x, y) \in I \times H \times H$, $A(t, x)y = \partial\delta_{\mathbb{A}}(y) = N_{\mathbb{A}}(y)$. □

Corollary 3.5. *Assume that for every $(t, x) \in I \times H$, $A(t, x) : D(A(t, x)) \subset H \longrightarrow 2^H$ is a maximal monotone operator satisfying $(\mathcal{H}_A^0)$, $(\mathcal{H}_A^1)$, $(\mathcal{H}_A^2)$ and $(\mathcal{H}_A^3)$. Let $h : H \longrightarrow H$ be a linear continuous and compact operator such that $\lambda\|h\| < 1$. Then, for any $u_0 \in D(A(0, h(u_0)))$, there is a $W^{1,1}(I, H)$-mapping u satisfying the evolution problem*

$$\begin{cases} -\dot{u}(t) \in A(t, h(u(t)))u(t) \quad a.e.\, t \in I \\ u(0) = u_0. \end{cases}$$

Proof. Apply Theorem 3.1 by taking for all $(t, x, y) \in I \times H \times H$, $F(t, x, y) = G(t, x, y) = \{0\}$ and $C(t, x) = \{h(x)\}$. □

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