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Trace Class Operators via OPV-Frames

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Abstract. Trace class operators for quaternionic Hilbert spaces (QHS) were studied by Moretti and Oppio [18]. In this paper, we study trace class operators via operator valued frames (OPV-frames). We introduce OPV-frames in a right quaternionic Hilbert space $\mathcal{H}$ with range in a two sided quaternionic Hilbert space $\mathcal{K}$ and obtain various results including several characterizations of OPV-frames. Also, we obtain a necessary and sufficient condition for a bounded operator on a right QHS to be a trace class operator which generalizes a similar result by Attal [2]. Moreover, we construct a trace class operator on a two sided QHS. Finally, we study quaternionic quantum channels as completely positive trace preserving maps and obtain various Choi-Kraus type representations of quaternionic quantum channels using OPV-frames in quaternionic Hilbert spaces.

1. Introduction

Duffin and Schaeffer [12] introduced frames in a study of non-harmonic Fourier series. Later, Daubechies, Grossmann and Meyer [11] reintroduced frames which gather a lot of attention among the researchers. The main reason for frames to be popular among researchers in recent years is their applications in digital signal processing [3] and other areas having physical and engineering problems [8].

Frames are integrally connected to time-frequency analysis. It is difficult to find a particular category of frames that is suitable to most of the physical problems, as there is no comprehensive class of frames that suits to all types of problems. Keeping this in mind, researchers across the disciplines come together for finding tools for the theory of frames to tackle various physical problems including solutions of operator equations in Hilbert spaces with fixed dual pairing [4]. Duffin and Schaeffer [12] defined frames as follows:

Let H be a Hilbert space. A sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq H$ is a *frame* for H, if there exist numbers $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad x \in H. \tag{1}$$

The scalars A and B are called the *lower* and *upper frame bounds* of the frame, respectively. They are not unique. If $A = B$, then $\{x_n\}_{n\in\mathbb{N}}$ is called an *A-tight frame* and if $A = B = 1$, then $\{x_n\}_{n\in\mathbb{N}}$ is called a *Parseval*

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frame. The inequality in (1) is called the *frame inequality* of the frame. The operator $T : \ell^2 \to H$ defined as $T(\{c_k\}_{k\in\mathbb{N}}) = \sum_{k\in\mathbb{N}} c_k x_k$, $\{c_k\}_{k\in\mathbb{N}} \in \ell^2$, is called the *pre-frame operator* (or *synthesis operator*) and its adjoint operator $T^* : H \to \ell^2$ is called the *analysis operator* and is given by

$$T^*(x) = \{\langle x, x_k\rangle\}_{k\in\mathbb{N}},\ x \in H.$$

Composing T and T^*, we obtain the *frame operator* $S = TT^* : H \to H$ given by

$$S(x) = \sum_{k\in\mathbb{N}} \langle x, x_k\rangle x_k,\ x \in H.$$

The frame operator S is a positive, self-adjoint and invertible operator on H and gives the following *reconstruction formula*:

$$x = SS^{-1}x = \sum_{k\in\mathbb{N}} \langle S^{-1}x, x_k\rangle x_k \quad \left(= \sum_{k\in\mathbb{N}} \langle x, S^{-1}x_k\rangle x_k \right),\ \ x \in H.$$

For various details related to frames and applications, one may allude to [5, 8].

In recent years, many generalizations of frames in Hilbert spaces and quaternionic Hilbert spaces have been introduced and studied. In 2004, Casazza and Kutyniok [6] defined frames of subspaces which has many applications in sensor networks and packet encoding. Khokulan, Thirulogasanthar and Srisatkunarajah [17] introduced and studied frames for finite dimensional quaternionic Hilbert spaces. Sharma and Virender [25] studied some different types of dual frames of a given frame in a finite dimensional quaternionic Hilbert space and gave various types of reconstructions with the help of dual frames. Sharma and Goel [23] introduced and studied frames for seperable quaternionic Hilbert spaces. Muraleetharan and Thirulogasanthar [19] studied the invariance of the Fredholm index under small norm operator and compact operator perturbations and with the association of the Fredholm operators, developed the theory of essential S-spectrum. K-frames in quaternionic Hilbert spaces were studied in [13]. Very recently, Sharma, Jarrah and Kaushik [24] introduced frame of operators in quaternionic Hilbert spaces and proved that they generalizes various notions like Pseduo frames, bounded quasi-projectors and frame of subspaces (fusion frames) in separable quaternionic Hilbert spaces.

Overview. We organize the paper as follows: In Section 2, we state some known results and standard definitions which are necessary for understanding the main content of the paper. In Section 3, we define and study operator valued frames (OPV-frames) in a quaternionic Hilbert space and give a necessary and sufficient condition for the existence of an OPV-frame. Also, we obtain conditions under which an OPV-frame is a Riesz and orthonormal OPV-frame. Further, it is proved that an OPV-frame is a compression of a Riesz OPV-frame and a Parseval OPV-frame is a compression of an orthonormal OPV-frame. In Section 4, we study trace class operators and quaternionic quantum channels and obtain various results including the Choi-Kraus type representations of quaternionic quantum channels using OPV-frames in quaternionic Hilbert spaces.

2. Prerequisites

Throughout this paper, until specified, we will denote $\mathbb{H}$ to be the non-commutative field of quaternions, $\mathbb{N}$ the set of natural numbers, $\mathcal{H}$ a separable right quaternionic Hilbert space, $\mathcal{K}$ a separable two-sided quaternionic Hilbert space. By the term "right linear operator", we mean a "right $\mathbb{H}$-linear operator", $B(\mathcal{H}, \mathcal{K})$ denotes the set of all bounded (right $\mathbb{H}$-linear) operators from $\mathcal{H}$ to $\mathcal{K}$ and the space $\ell_2(\mathbb{H})$ denotes the set of all sequences $\{q_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ such that $\sum_{n\in\mathbb{N}} |q_n|^2 < +\infty$.

Let D be a division ring and F be the center of D. A unital D-bimodule X is called a *D-vector space* if X is a F-vector space(or vector space over F) under the restriction of the scalar multiplication to F.

Here, we deal with the case when $D = \mathbb{H}$, the set of all real quaternions. One may observe that, $\mathbb{H} \otimes \mathbb{H}$ together with the scalar multiplication, $\alpha(a \otimes b)\beta = \alpha a \otimes b\beta$ is a $\mathbb{H}$-vector space.

For basic definitions of *right quaternionic pre-Hilbert space (or right quaternionic inner product space)*, right quaternionic Hilbert space and other terminologies related to quaternionic Hilbert spaces one may refer to [20, 23].

In [1], Daniel Alpay and H. Turgay Kaptanoğlu observed that if $\mathcal{H}$ is a right quaternionic Hilbert space and $\mathcal{K}$ is a two-sided quaternionic Hilbert space, then $\mathcal{H} \otimes \mathcal{K}$ is also a right quaternionic Hilbert space with the inner product $\langle .|. \rangle : (\mathcal{H} \otimes \mathcal{K}) \times (\mathcal{H} \otimes \mathcal{K}) \to \mathbb{H}$ given by

$$\Big\langle h_1 \otimes k_1 | h_2 \otimes k_2 \Big\rangle_{\mathcal{H}\otimes\mathcal{K}} = \Big\langle \langle h_2 | h_1 \rangle_{\mathcal{H}} k_1 | k_2 \Big\rangle_{\mathcal{K}}, \ \ h_1, h_2 \in \mathcal{H} \text{ and } k_1, k_2 \in \mathcal{K}.$$

Further, if $\{h_i\}_{i\in\mathbb{N}}$ and $\{k_i\}_{i\in\mathbb{N}}$ are orthonormal sets in $\mathcal{H}$ and $\mathcal{K}$ respectively, then $\{h_i \otimes k_j\}_{i,j\in\mathbb{N}}$ is an orthonormal set in $\mathcal{H} \otimes \mathcal{K}$.

Definition 2.1. *[18] Let $\mathcal{H}$ be a right quaternionic Hilbert space. Then, $T \in \mathcal{B}(\mathcal{H})$ is said to be of trace-class if*

$$\sum_{u\in\mathcal{N}} \langle u | |T| u \rangle < \infty, \text{ for some orthonormal basis } \mathcal{N} \subset \mathcal{H}, \tag{2}$$

where $|T| = (T^*T)^{1/2}$. Let us denote $\mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ as the set of all trace-class operators on $\mathcal{H}$.

Theorem 2.2. *[18] The set $B_1(\mathcal{H})$ enjoys the following properties:*

(a) If $T \in B_1(\mathcal{H})$, then (2) is valid for every orthonormal basis $\mathcal{M} \subset \mathcal{H}$ and $\sum_{u\in\mathcal{M}} \langle u \big| |T| u \rangle$ does not depend on $\mathcal{M}$.

(b) $T \in B_1(\mathcal{H})$ if and only if

(i) T is a compact operator, and

(ii) $\|T\|_1 = \sum_{\lambda\in\sigma(|T|)} \lambda d_\lambda < \infty$, where $\sigma(|T|) \subset [0, +\infty)$ is the spherical point spectrum of the self adjoint compact operator $|T| = \sqrt{T^*T}$ and $d_\lambda = 1, 2 \cdots < \infty$ is the dimension of the λ-eigenspace of $|T|$.

(c) If $T \in B_1(\mathcal{H})$, then for every orthonormal basis $\mathcal{M} \subset \mathcal{H}$, $\|T\|_1 = \sum_{u\in\mathcal{M}} \langle u | |T| u \rangle$.

Theorem 2.3. *[14] (Polar Decomposition of an Operator) Let $T \in B(\mathcal{H})$. Then, there exists a unique operator $W \in B(\mathcal{H})$ such that*

(a) $T = W|T|$.

(b) $N(|T|) \subset N(W)$.

(c) $\|W(u)\| = \|u\|$, $u \in N(|T|)^\perp$.

A quaternionic quantum channel is defined as a completely positive trace-preserving map. Matthew A. Graydon in [15] has given a Choi-Kraus type representation of quaternionic quantum channels as follows:

Definition 2.4. *[15] A quaternionic quantum channel is a map $\phi : B(\mathcal{H}) \to B(\mathcal{K})$ whose action on any $T \in B(\mathcal{H})$ is defined in terms of some bounded operators $\{F_n\}_{n\in\mathbb{N}} \subset B(\mathcal{H}, \mathcal{K})$ such that*

$$\phi(T) = \sum_{n\in\mathbb{N}} F_n T F_n^*, \ \ T \in B(\mathcal{H}) \ \text{ and } \ \sum_{n\in\mathbb{N}} F_n^* F_n = \mathcal{I}_H.$$

3. Operator valued frames

The classical frame inequalities can be expressed in terms of operator inequalities involving sums of rank one operators. This case, called the multiplicity one case motivated Kaftal et al. [16] in 2009 to introduce the notion of operator-valued frames (OPV-frames).
In this section, we define and study OPV-frames for quaternionic Hilbert spaces. More precisely, we have

Definition 3.1. *Let $\mathcal{H}$ be a right quaternionic Hilbert space, $\mathcal{K}$ a two-sided quaternionic Hilbert space, and $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$. Then $\mathcal{F}$ is said to be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ if there exist positive constants $r_1 \leq r_2$ such that*

$$r_1\mathcal{I} \leq \sum_{n\in\mathbb{N}} F_n^* F_n \leq r_2\mathcal{I}. \tag{3}$$

The positive constants r_1 and r_2, are called lower and upper frame bounds for the OPV-frame $\mathcal{F}$, respectively and the inequality (3) is called the OPV-frame inequality. The family $\mathcal{F}$ is called an OPV Bessel sequence for $\mathcal{H}$ with range in $\mathcal{K}$ with Bessel bound r_2, if $\mathcal{F}$ satisfies the right hand side of the inequality (3). The family $\mathcal{F}$ is said to be an OPV tight frame for $\mathcal{H}$ with range in $\mathcal{K}$ if there exist positive constants r_1, r_2 satisfying inequality (3) with $r_1 = r_2$ and is called an OPV Parseval frame if it is tight and $r_1 = r_2 = 1$. Further, $\mathcal{F}$ is called exact if it ceases to be an OPV-frame in case any one of its elements is removed. The multiplicity of an OPV-frame $\mathcal{F}$ is defined as $\sup\{\text{rank } F_n\}_{n\in\mathbb{N}}$.

For each $h \in \mathcal{H}$, define $|h\rangle_{\mathcal{H}} : \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ as

$$|h\rangle_{\mathcal{H}}(k) = h \otimes k, \ k \in \mathcal{K}. \tag{4}$$

Also, define ${}_{\mathcal{H}}\langle h| : \mathcal{H} \otimes \mathcal{K} \to \mathcal{K}$ as

$${}_{\mathcal{H}}\langle h|(h' \otimes k') = \langle h'|h\rangle k', \ \ h' \otimes k' \in \mathcal{H} \otimes \mathcal{K}. \tag{5}$$

One may easily observe that, the operators defined in (4) and (5) are bounded and satisfies $\| \, |h\rangle_{\mathcal{H}} \, \| = \| \, {}_{\mathcal{H}}\langle h| \, \| = \|h\|$ and $|h\rangle_{\mathcal{H}}^* = {}_{\mathcal{H}}\langle h|$, $h \in \mathcal{H}$.
In particular, given a two-sided quaternionic Hilbert space $\mathcal{K}$, we can define the partial isometries with mutually orthogonal ranges $|e_n\rangle_{\ell_2(\mathbb{H})} : \mathcal{K} \to \ell_2(\mathbb{H}) \otimes \mathcal{K}$, $|e_n\rangle_{\ell_2(\mathbb{H})}(k) = e_n \otimes k$ for $k \in \mathcal{K}$, and the adjoint is given by ${}_{\ell_2(\mathbb{H})}\langle e_n| : \ell_2(\mathbb{H}) \otimes \mathcal{K} \to \mathcal{K}$, ${}_{\ell_2(\mathbb{H})}\langle e_n|(q \otimes k) = \langle q|e_n\rangle k$, where $q = \{q_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{H}), k \in \mathcal{K}$ and $\{e_n\}_{n\in\mathbb{N}}$ is the standard orthonormal basis of $\ell_2(\mathbb{H})$.

Next, we give a result in the form of a lemma, related to some basic properties of the above defined partial isometries, which will be used in subsequent results.

Lemma 3.2. *Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\ell_2(\mathbb{H})$. Then*

(a) $\sum\limits_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} \, {}_{\ell_2(\mathbb{H})}\langle e_n| = \mathcal{I}_{\ell_2(\mathbb{H})\otimes\mathcal{K}}$.

(b) ${}_{\ell_2(\mathbb{H})}\langle e_i| \, |e_j\rangle_{\ell_2(\mathbb{H})} = \delta_{ij}\mathcal{I}_{\mathcal{K}}$.

Proof. (a) Let $q \otimes k \in \ell_2(\mathbb{H}) \otimes \mathcal{K}$. Then, we compute

$$\begin{aligned} \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} \, {}_{\ell_2(\mathbb{H})}\langle e_n|(q \otimes k) &= \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})}(q_n k), \text{ where } q = \{q_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{H}) \\ &= \sum_{n\in\mathbb{N}} e_n \otimes (q_n k) \\ &= \sum_{n\in\mathbb{N}} e_n q_n \otimes k = q \otimes k. \end{aligned}$$

(b) Let $k \in \mathcal{K}$. Then, we have

$${}_{\ell_2(\mathbb{H})}\langle e_i| \, |e_j\rangle_{\ell_2(\mathbb{H})}(k) = {}_{\ell_2(\mathbb{H})}\langle e_i|(e_j \otimes k) = \delta_{ij} k. \qquad \square$$

In the following result, we give a necessary and sufficient condition for the existence of an OPV Bessel sequence in a right quaternionic Hilbert space.

Theorem 3.3. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$. Then $\mathcal{F}$ is an OPV Bessel sequence for $\mathcal{H}$ with range in $\mathcal{K}$ with Bessel bound r if and only if there exists a bounded right linear operator Q from $\ell_2(\mathbb{H})\otimes\mathcal{K}$ into $\mathcal{H}$ with $\|Q\| \leq \sqrt{r}$ such that*

$$Q(q\otimes k) = \sum_{n\in\mathbb{N}} F_n^*(q_n k) = \sum_{n\in\mathbb{N}} F_n^* \,{}_{\ell_2(\mathbb{H})}\langle e_n|(q\otimes k), \quad q\otimes k \in \ell_2(\mathbb{H})\otimes\mathcal{K},$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\ell_2(\mathbb{H})$ and $q = \{q_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{H})$.

Proof. Let $\{F_n\}_{n\in\mathbb{N}}$ be an OPV Bessel sequence for $\mathcal{H}$ with range in $\mathcal{K}$ with Bessel bound r. Define $U : \mathcal{H} \to \ell_2(\mathbb{H})\otimes\mathcal{K}$ as

$$U(h) = \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} F_n(h) = \sum_{n\in\mathbb{N}} e_n \otimes F_n(h), \; h\in\mathcal{H}.$$

Then, using hypothesis we have

$$\begin{aligned}\|U(h)\|^2 &= \Big\langle \sum_{n\in\mathbb{N}} e_n\otimes F_n(h) \Big| \sum_{j\in\mathbb{N}} e_j\otimes F_j(h)\Big\rangle_{\ell_2(\mathbb{H})\otimes\mathcal{K}} \\ &= \sum_{n,j\in\mathbb{N}} \Big\langle \langle e_j|e_n\rangle_{\ell_2(\mathbb{H})} F_n(h) \Big| F_j(h)\Big\rangle_{\mathcal{K}} \\ &= \sum_{n\in\mathbb{N}} \|F_n(h)\|^2 \leq r\|h\|^2, \; h\in\mathcal{H}.\end{aligned}$$

Therefore, the operator U is well defined and bounded such that $\|U\| \leq \sqrt{r}$. Take $Q = U^*$. For each $q\otimes k \in \ell_2(\mathbb{H})\otimes\mathcal{K}$, we have

$$Q(q\otimes k) = \sum_{n\in\mathbb{N}} F_n^* \,{}_{\ell_2(\mathbb{H})}\langle e_n|(q\otimes k) = \sum_{n\in\mathbb{N}} F_n^*(q_n k), \; q = \{q_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{H}).$$

Since U is bounded, so is Q and $\|Q\| \leq \sqrt{r}$.

Conversely, let $Q : \ell_2(\mathbb{H})\otimes\mathcal{K} \to \mathcal{H}$ be a well- defined bounded right linear operator with $\|Q\| \leq \sqrt{r}$. As $Q^*(h) = \sum_{n\in\mathbb{N}} e_n\otimes F_n(h)$, we obtain

$$\|Q^*(h)\|^2 = \sum_{n\in\mathbb{N}} \|F_n(h)\|^2, \; h\in\mathcal{H}.$$

This gives

$$\sum_{n\in\mathbb{N}} \|F_n(h)\|^2 \leq r\|h\|^2, \; h\in\mathcal{H}.$$

Thus, $\mathcal{F}$ is an OPV Bessel sequence for $\mathcal{H}$ with range in $\mathcal{K}$. □

Given a Bessel sequence $\{v_n\}_{n\in\mathbb{N}}$ with Bessel bound r in $\mathcal{H}$. For each $n\in\mathbb{N}$, define an operator $F_n \in \mathcal{B}(\mathcal{H},\mathbb{H})$ as

$$F_n(x) = \langle v_n|x\rangle, \; x\in\mathcal{H}.$$

Then, the family $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ is an OPV Bessel sequence for $\mathcal{H}$ with range in $\mathbb{H}$ with Bessel bound r. Conversely, by Quaternionic representation Riesz theorem[9, 14], every OPV Bessel sequence $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H},\mathbb{H})$ can be identified with the vectors $\{v_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$. Then, the sequence $\{v_n\}_{n\in\mathbb{N}}$ is a Bessel sequence in $\mathcal{H}$ with the same Bessel bound. Thus, a Bessel sequence in a right quaternionic Hilbert space $\mathcal{H}$ corresponds to an OPV Bessel sequence for $\mathcal{H}$ with range in $\mathbb{H}$.

Further, if we consider an OPV-frame $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ with range in $\mathcal{K}$ of multiplicity one, then each operator F_n can be identified with a vector of $\mathcal{H}$. Therefore, every OPV-frame of multiplicity one gives rise to a frame for $\mathcal{H}$ with the same bounds. Indeed, let $\{F_n\}_{n\in\mathbb{N}}$ be an OPV-frame of multiplicity one. Then, without loss of generality, using Quaternionic representation Riesz theorem, for each $n \in \mathbb{N}$, $F_n(x) = e_n\langle v_n|x\rangle$, for some unit vector $e_n \in \mathcal{K}$, $x \in \mathcal{H}$, and $v_n \in \mathcal{H}$. So, we get

$$\sum_{n\in\mathbb{N}} \|F_n(x)\|^2 = \sum_{n\in\mathbb{N}} |\langle v_n|x\rangle|^2, \ x \in \mathcal{H}.$$

Next, we give a characterization of an OPV-frame in a right quaternionic Hilbert space.

Theorem 3.4. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then, $\mathcal{F}$ is an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ if and only if there exists a bounded, right linear operator Q from $\ell_2(\mathbb{H}) \otimes \mathcal{K}$ onto $\mathcal{H}$ such that*

$$Q(q \otimes k) = \sum_{n\in\mathbb{N}} F_n^*(q_n k) = \sum_{n\in\mathbb{N}} F_n^* \ {}_{\ell_2(\mathbb{H})}\langle e_n|(q \otimes k), \quad q \otimes k \in \ell_2(\mathbb{H}) \otimes \mathcal{K}, \tag{6}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\ell_2(\mathbb{H})$ and $q = \{q_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{H})$.

Proof. Let $\mathcal{F}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. Since, $\mathcal{F}$ is an OPV Bessel sequence, there exists a bounded, right linear operator Q satisfying (6) and the adjoint $U : \mathcal{H} \to \ell_2(\mathbb{H}) \otimes \mathcal{K}$ of Q is given by

$$U(h) = \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} F_n(h) = \sum_{n\in\mathbb{N}} e_n \otimes F_n(h), \ h \in \mathcal{H}.$$

This gives

$$QU(h) = \sum_{n\in\mathbb{N}} F_n^* F_n(h), \ h \in \mathcal{H}.$$

Therefore, QU is invertible. Hence Q is a surjective operator.

Conversely, since Q is a surjective operator, Q^* is one-one and hence there exists a positive constant r such that

$$r\|h\|^2 \leq \|Q^*(h)\|^2 = \sum_{n\in\mathbb{N}} \|F_n(h)\|^2, \ h \in \mathcal{H}.$$

This gives

$$\sum_{n\in\mathbb{N}} \|F_n(h)\|^2 = \sum_{n\in\mathbb{N}} \langle F_n(h)|F_n(h)\rangle = \left\langle h \middle| \sum_{n\in\mathbb{N}} F_n^* F_n(h) \right\rangle, \ h \in \mathcal{H}.$$

Also by Theorem 3.3, $\mathcal{F}$ is a Bessel sequence. Hence $\mathcal{F}$ is an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. □

Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\ell_2(\mathbb{H})$. Then, the *analysis operator* $\mathcal{T} : \mathcal{H} \to \ell_2(\mathbb{H}) \otimes \mathcal{K}$ of the OPV-frame $\mathcal{F}$ is given by

$$\mathcal{T}(h) \quad = \quad \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} F_n(h) = \sum_{n\in\mathbb{N}} e_n \otimes F_n(h), \ h \in \mathcal{H},$$

and the *synthesis operator* $\mathcal{T}^* : \ell_2(\mathbb{H}) \otimes \mathcal{K} \to \mathcal{H}$ of $\mathcal{F}$ is given by

$$\mathcal{T}^* = \sum_{n\in\mathbb{N}} F_n^* \ {}_{\ell_2(\mathbb{H})}\langle e_n|.$$

By composing $\mathcal{T}^*$ with its adjoint $\mathcal{T}$, we obtain the frame operator $\mathcal{S}$ given by

$$\mathcal{S} = \mathcal{T}^*\mathcal{T} = \sum_{n\in\mathbb{N}} F_n^* F_n.$$

Observe that an OPV-frame $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ is a Parseval OPV- frame if and only if $\mathcal{T}$ is an isometry. This is because

$$\|\mathcal{T}(h)\|^2 = \sum_{n\in\mathbb{N}} \|F_n(h)\|^2 = \left\langle h \middle| \sum_{n\in\mathbb{N}} F_n^* F_n(h) \right\rangle = \langle h|\mathcal{S}(h)\rangle, \ h \in \mathcal{H}.$$

Remark 3.5. *The analysis operator $\mathcal{T} : \mathcal{H} \to \ell_2(\mathbb{H}) \otimes \mathcal{K}$ of an OPV-frame $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ is a bounded injective operator.*

Next, we observe that the information carried by an OPV-frame can be fully encoded with the help of its analysis operator. In other words, an OPV-frame can be fully reconstructed from its analysis operator.

Theorem 3.6. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an OPV- frame for $\mathcal{H}$ with range in $\mathcal{K}$ and $\mathcal{T}, \mathcal{T}^*$ be the analysis and synthesis operators of $\mathcal{F}$, respectively. Then, for each $n \in \mathbb{N}$, $F_n = {}_{\ell_2(\mathbb{H})}\langle e_n|\mathcal{T}$ and $F_n^* = \mathcal{T}^*|e_n\rangle_{\ell_2(\mathbb{H})}$, where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\ell_2(\mathbb{H})$.*

Proof. For $h \in \mathcal{H}$, we compute ${}_{\ell_2(\mathbb{H})}\langle e_n|\mathcal{T}(h) = \sum_{j\in\mathbb{N}} \langle e_j|e_n\rangle F_j(h) = F_n(h)$. Similarly, for $k \in \mathcal{K}$, we have $\mathcal{T}^*|e_n\rangle_{\ell_2(\mathbb{H})}(k) = \sum_{j\in\mathbb{N}} F_j^*(\langle e_n|e_j\rangle k) = F_n^*(k)$. □

In the following result, we give some properties of the frame operator for an OPV-frame in a right quaternionic Hilbert space.

Theorem 3.7. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ having lower and upper frame bounds r_1 and r_2, respectively with the frame operator $\mathcal{S}$. Then $\mathcal{S}$ is a right-linear, positive, self- adjoint, bounded and bijective operator.*

Proof. Clearly, $\mathcal{S} = \mathcal{T}^*\mathcal{T}$ is self adjoint bounded operator. Also, in view of the frame inequality, we have $r_1\mathcal{I} \leq \mathcal{S} \leq r_2\mathcal{I}$ and so $\mathcal{S}$ is positive. Let $\mathcal{S}(h) = 0$. Then, $\|h\|^2 = \langle h|h\rangle \leq \langle \mathcal{S}(h)|h\rangle = 0$. This implies $h = 0$. Further, as $\mathcal{T}^*$ is surjective, for any $h \in \mathcal{H}$, there exists $q \otimes k \in \ell_2(\mathbb{H}) \otimes \mathcal{K}$ such that $\mathcal{T}^*(q \otimes k) = h$. Thus $q \otimes k \in N(\mathcal{T}^*)^\perp = R(\mathcal{T})$. Hence $\mathcal{S}$ is a bijective operator. □

Definition 3.8. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. Then, $\mathcal{F}$ is said to be*

(i) a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ if $\mathcal{T}(\mathcal{H}) = \ell_2(\mathbb{H}) \otimes \mathcal{K}$.

(ii) an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ if it is a Parseval Riesz OPV-frame.

Observations (I) Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then, $\mathcal{F}$ is a Riesz OPV frame for $\mathcal{H}$ with range in $\mathcal{K}$ if and only if there exists a bounded, bijective and right linear operator $Q : \ell_2(\mathbb{H}) \otimes \mathcal{K} \longrightarrow \mathcal{H}$ such that

$$Q(q \otimes k) = \sum_{n\in\mathbb{N}} F_n^*(q_n k) = \sum_{n\in\mathbb{N}} F_n^* \; {}_{\ell_2(\mathbb{H})}\langle e_n|(q \otimes k), \quad q \otimes k \in \ell_2(\mathbb{H}) \otimes \mathcal{K},$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\ell_2(\mathbb{H})$ and $q = \{q_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{H})$.
(II) Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $\mathcal{F}$ is an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ if and only if there exists a unitary operator $Q : \ell_2(\mathbb{H}) \otimes \mathcal{K} \longrightarrow \mathcal{H}$ such that

$$Q(q \otimes k) = \sum_{n\in\mathbb{N}} F_n^*(q_n k) = \sum_{n\in\mathbb{N}} F_n^* \; {}_{\ell_2(\mathbb{H})}\langle e_n|(q \otimes k), \quad q \otimes k \in \ell_2(\mathbb{H}) \otimes \mathcal{K},$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\ell_2(\mathbb{H})$ and $q = \{q_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{H})$.

(III) Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and $\mathcal{S}$ be its frame operator. Write $\mathcal{P} = \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*$. Then, $\mathcal{P}$ is a projection from $\ell_2(\mathbb{H}) \otimes \mathcal{K}$ onto $\mathcal{T}(\mathcal{H})$. Further, if $\mathcal{F}$ is a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$, then $\mathcal{P} = \mathcal{I}_{\ell_2(\mathbb{H})\otimes\mathcal{K}}$.

In the following result, we characterize Riesz OPV-frames and orthonormal OPV-frames in a right quaternionic Hilbert space.

Theorem 3.9. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an OPV- frame for $\mathcal{H}$ with range in $\mathcal{K}$ with frame operator $\mathcal{S}$. Then*

(a) $\mathcal{F}$ is Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ if and only if for each $i, j \in \mathbb{N}$, $F_i\mathcal{S}^{-1}F_j^* = \delta_{ij}I_{\mathcal{K}}$.

(b) $\mathcal{F}$ *is orthonormal OPV-frame for* $\mathcal{H}$ *with range in* $\mathcal{K}$ *if and only if* $F_iF_j^* = \delta_{ij}I_{\mathcal{K}}$ *and* $\{F_n\}_{n\in\mathbb{N}}$ *is Parseval OPV-frame.*

Proof. (a) Let $\mathcal{F}$ be a Riesz OPV-frame. Then, for each $k \in \mathcal{K}$ and $i,\ j \in \mathbb{N}$, we compute

$$F_i\mathcal{S}^{-1}F_j^*(k) = {}_{\ell_2(\mathbb{H})}\langle e_i|\mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*\ |e_j\rangle_{\ell_2(\mathbb{H})}(k) = {}_{\ell_2(\mathbb{H})}\langle e_i|\mathcal{I}_{\ell_2(\mathbb{H})\otimes\mathcal{K}}\ |e_j\rangle_{\ell_2(\mathbb{H})}(k) = \delta_{ij}k.$$

Conversely, for $q \otimes k \in \ell_2(\mathbb{H}) \otimes \mathcal{K}$, we have

$$\begin{aligned}
\mathcal{P}(q\otimes k) &= \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*(q\otimes k)\\
&= \mathcal{T}\mathcal{S}^{-1}\sum_{n\in\mathbb{N}}F_n^*(q_nk)\\
&= \sum_{n\in\mathbb{N}}\mathcal{T}(\mathcal{S}^{-1}F_n^*(q_nk))\\
&= \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}}e_j\otimes F_j(\mathcal{S}^{-1}F_n^*(q_nk))\\
&= \sum_{n\in\mathbb{N}}e_n\otimes(q_nk) = q\otimes k.
\end{aligned}$$

Therefore, $\mathcal{P} = \mathcal{I}_{\ell_2(\mathbb{H})\otimes\mathcal{K}}$.

(b) Straight forward. □

In the next result, we give a characterization of OPV-frame and orthonormal OPV-frame with the help of frame and orthonormal basis in $\mathcal{H}$, respectively.

Theorem 3.10. *Let* $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ *be an OPV-frame for* $\mathcal{H}$ *with range in* $\mathcal{K}$ *and* $\{k_j\}_{j\in\mathbb{N}}$ *be an orthonormal basis of* $\mathcal{K}$*. Then*

(a) $\mathcal{F}$ *is an OPV-frame if and only if* $\{F_n^*(k_j)\}_{j,n\in\mathbb{N}}$ *is a frame for* $\mathcal{H}$*.*

(b) $\mathcal{F}$ *is an orthonormal OPV-frame if and only if* $\{F_n^*(k_j)\}_{j,n\in\mathbb{N}}$ *is an orthonormal basis of* $\mathcal{H}$*.*

Proof. (a) Let $h \in \mathcal{H}$. Then, for each $n \in \mathbb{N}$, we have

$$F_n(h) = \sum_{i\in\mathbb{N}}k_i\langle k_i|F_n(h)\rangle = \sum_{i\in\mathbb{N}}k_i\langle F_n^*(k_i)|h\rangle.$$

Also, we compute

$$\begin{aligned}
\|F_n(h)\|^2 &= \left\langle\sum_{i\in\mathbb{N}}k_i\langle F_n^*(k_i)|h\rangle\middle|\sum_{j\in\mathbb{N}}k_j\langle F_n^*(k_j)|h\rangle\right\rangle_K\\
&= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\langle h|F_n^*(k_i)\rangle_{\mathcal{H}}\langle k_i|k_j\rangle_{\mathcal{K}}\langle F_n^*(k_j)|h\rangle_{\mathcal{H}}\\
&= \sum_{j\in\mathbb{N}}\langle h|F_n^*(k_j)\rangle_{\mathcal{H}}\langle F_n^*(k_j)|h\rangle_{\mathcal{H}}\\
&= \sum_{j\in\mathbb{N}}|\langle h|F_n^*(k_j)\rangle|^2.
\end{aligned}$$

This gives

$$\sum_{n\in\mathbb{N}}\|F_n(h)\|^2 = \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}}|\langle h|F_n^*(k_j)\rangle|^2.$$

Hence $\mathcal{F}$ is an OPV-frame if and only if $\{F_n^*(k_j)\}_{j,n\in\mathbb{N}}$ is a frame for $\mathcal{H}$.

(b) As $\mathcal{F}$ is an orthonormal OPV-frame, $F_iF_j^* = \delta_{ij}\mathcal{I}_{\mathcal{K}}$ and $\sum_{n\in\mathbb{N}} F_n^*F_n = \mathcal{I}_{\mathcal{H}}$. Also, for each $n \in \mathbb{N}$, $F_n(h) = \sum_{j\in\mathbb{N}} k_j\langle F_n^*(k_j)|h\rangle$. This gives

$$h = \sum_{n\in\mathbb{N}} F_n^*F_n(h) = \sum_{n,j\in\mathbb{N}} F_n^*(k_j)\langle F_n^*(k_j)|h\rangle,\ h \in \mathcal{H}.$$

Also, we have

$$\|F_n^*(k_j)\|^2 = \langle k_j|F_nF_n^*(k_j)\rangle = \|k_j\|^2 = 1.$$

and $\langle F_{n_1}^*(k_{j_1})|F_{n_2}^*(k_{j_2})\rangle = 0$.

Conversely, let $\{F_n^*(k_j)\}_{j,n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then

$$h = \sum_{n,j\in\mathbb{N}} F_n^*(k_j)\langle F_n^*(k_j)|h\rangle = \sum_{n\in\mathbb{N}} F_n^*F_n(h),\ h \in \mathcal{H}.$$

Therefore $\{F_n\}_{n\in\mathbb{N}}$ is a Parseval OPV-frame. Also, for each $k \in \mathcal{K}$, we compute

$$F_nF_n^*(k) = \sum_{j\in\mathbb{N}} k_j\langle k_j|F_nF_n^*(k)\rangle = \sum_{j\in\mathbb{N}} k_j\left\langle F_n^*(k_j)\middle|F_n^*\Big(\sum_{i\in\mathbb{N}} k_i\langle k_i|k\rangle\Big)\right\rangle = \sum_{i\in\mathbb{N}} k_i\langle k_i|k\rangle = k.$$

and

$$F_nF_m^*(k) = \sum_{j\in\mathbb{N}} k_j\langle F_n^*(k_j)|F_m^*(k)\rangle = \sum_{i,j\in\mathbb{N}} k_j\langle F_n^*(k_j)|F_m^*(k_i)\rangle\langle k_i|k\rangle = 0.$$

Hence, by Theorem 3.9, $\mathcal{F}$ is an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. □

In the following result, we show that an OPV-frame is a compression of a Riesz OPV-frame in a right quaternionic Hilbert space.

Theorem 3.11. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an OPV- frame for $\mathcal{H}$ with range in $\mathcal{K}$ with lower and upper frame bounds r_1 and r_2 respectively, and frame operator $\mathcal{S}$. Then, there exists a right quaternionic Hilbert space $\mathcal{G}$ and a sequence of bounded right linear operators $\{E_n : \mathcal{G} \to \mathcal{K}\}_{n\in\mathbb{N}}$ such that $\{E_n\}_{n\in\mathbb{N}}$ is a Riesz OPV-frame for $\mathcal{G}$ with range in $\mathcal{K}$ and $E_n|_{\mathcal{H}} = F_n$, $n \in \mathbb{N}$.*

Proof. Let $Q = \mathcal{I}_{\ell_2(\mathbb{H})\otimes\mathcal{K}} - \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*$ and $\mathcal{G} = \mathcal{H} \oplus \ker\mathcal{T}^*$, where $\mathcal{T}$ is the analysis operator for $\mathcal{F}$. For each $n \in \mathbb{N}$, define $E_n : \mathcal{G} \to \mathcal{K}$ as

$$E_n(h \oplus y) = F_n(h) + {}_{\ell_2(\mathbb{H})}\langle e_n|Q(y),\ h \in \mathcal{H},\ y \in \ker\mathcal{T}^*,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\ell_2(\mathbb{H})$. Also, define $E : \mathcal{G} \to \ell_2(\mathbb{H}) \otimes \mathcal{K}$ as

$$E(h \oplus y) = \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})}E_n(h \oplus y) = \sum_{n\in\mathbb{N}} e_n \otimes E_n(h \oplus y),\ h \oplus y \in \mathcal{G}.$$

Since $y \in \ker \mathcal{T}^*$ we have $Q(y) = y$. Therefore, we compute

$$\begin{aligned}
\|E(h \oplus y)\|^2 &= \sum_{n\in\mathbb{N}} \|E_n(h \oplus y)\|^2 \\
&= \sum_{n\in\mathbb{N}} \|F_n(h) + {}_{\ell_2(\mathbb{H})}\langle e_n|Q(y)\|^2 \\
&= \sum_{n\in\mathbb{N}} \|F_n(h) + {}_{\ell_2(\mathbb{H})}\langle e_n|(y)\|^2 \\
&= \sum_{n\in\mathbb{N}} \|F_n(h)\|^2 + \sum_{n\in\mathbb{N}} \| {}_{\ell_2(\mathbb{H})}\langle e_n|(y)\|^2 + \sum_{n\in\mathbb{N}} \langle F_n(h)| {}_{\ell_2(\mathbb{H})}\langle e_n|(y)\rangle + \sum_{n\in\mathbb{N}} \langle {}_{\ell_2(\mathbb{H})}\langle e_n|(y)|F_n(h)\rangle \\
&= \sum_{n\in\mathbb{N}} \|F_n(h)\|^2 + \left\langle \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})}\, {}_{\ell_2(\mathbb{H})}\langle e_n|(y) \middle| y \right\rangle + \left\langle \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} F_n(h) \middle| y \right\rangle + \left\langle y \middle| \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} F_n(h) \right\rangle \\
&= \sum_{n\in\mathbb{N}} \|F_n(h)\|^2 + \|y\|^2 + \langle \mathcal{T}(h)|y\rangle + \langle y|\mathcal{T}(h)\rangle \\
&= \sum_{n\in\mathbb{N}} \|F_n(h)\|^2 + \|y\|^2.
\end{aligned}$$

Thus, E is a well defined bounded operator. Now, for each $n \in \mathbb{N}$, $E_n^* = F_n^* + Q|e_n\rangle_{\ell_2(\mathbb{H})}$ and $E^* = \sum_{n\in\mathbb{N}} E_n^*\, {}_{\ell_2(\mathbb{H})}\langle e_n|$. Further, for $q \otimes k \in \ell_2(\mathbb{H}) \otimes \mathcal{K}$, the adjoint E^* of E is given by

$$\begin{aligned}
E^*(q \otimes k) &= \sum_{n\in\mathbb{N}} E_n^*\, {}_{\ell_2(\mathbb{H})}\langle e_n|(q \otimes k) \\
&= \sum_{n\in\mathbb{N}} \left(F_n^* + Q|e_n\rangle_{\ell_2(\mathbb{H})}\right) {}_{\ell_2(\mathbb{H})}\langle e_n|(q \otimes k) \\
&= \sum_{n\in\mathbb{N}} F_n^*\, {}_{\ell_2(\mathbb{H})}\langle e_n|(q \otimes k) \oplus Q \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})}\, {}_{\ell_2(\mathbb{H})}\langle e_n|(q \otimes k) \\
&= \mathcal{T}^*(q \otimes k) \oplus Q(q \otimes k).
\end{aligned}$$

Let $q \otimes k \in \ker E^*$. Then $\mathcal{T}^*(q \otimes k) = 0$ and $q \otimes k = \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*(q \otimes k)$. Since $\ker \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^* = \ker \mathcal{T}^*$, $q \otimes k = \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*(q \otimes k) = 0$. This shows that E^* is injective. Now, to show that E^* is surjective, let $h \oplus y \in \mathcal{H} \oplus \ker \mathcal{T}^*$. Since $\mathcal{T}^*$ is onto, there exists $\zeta \in \ell_2(\mathbb{H}) \otimes \mathcal{K}$ such that $h = \mathcal{T}^*(\zeta)$. Let $\mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*(\zeta) + y = q \otimes k$, for $q \otimes k \in \ell_2(\mathbb{H}) \otimes \mathcal{K}$. This gives $y = (\mathcal{I}_{\ell_2(\mathbb{H})\otimes\mathcal{K}} - \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*)(q \otimes k) = Q(q \otimes k)$.

Also, we compute

$$\mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*(q \otimes k) = \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*\mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*(\zeta) + \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*(y) = \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*(\zeta).$$

This gives $\mathcal{T}^*(\zeta) = \mathcal{T}^*(q \otimes k)$. Therefore

$$h = \mathcal{T}^*(\zeta) = \mathcal{T}^*(q \otimes k) \text{ and } h \oplus y = \mathcal{T}^*(q \otimes k) \oplus Q(q \otimes k) = E^*(q \otimes k).$$

This verifies that E^* is surjective. Hence, $\{E_n\}_{n\in\mathbb{N}}$ is a Riesz OPV-frame for $\mathcal{G}$ with range in $\mathcal{K}$. Further, $E_n|_{\mathcal{H}} = F_n$, $n \in \mathbb{N}$. □

Next, we show that a Parseval OPV-frame can also be expressed as a compression of an orthonormal OPV-frame in a right quaternionic Hilbert space.

Theorem 3.12. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be a Parseval OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. Then, there exists a right quaternionic Hilbert space $\mathcal{G}$ and a sequence of bounded right linear operators $\{E_n : \mathcal{G} \to \mathcal{K}\}_{n\in\mathbb{N}}$ such that $\{E_n\}_{n\in\mathbb{N}}$ is an orthonormal OPV-frame for $\mathcal{G}$ with range in $\mathcal{K}$ and $E_n|_{\mathcal{H}} = F_n$, $n \in \mathbb{N}$.*

Proof. Let $Q = \mathcal{I}_{\ell_2(\mathbb{H})\otimes\mathcal{K}} - \mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*$ and $\mathcal{G} = \mathcal{H} \oplus \ker\mathcal{T}^*$, where $\mathcal{T}$ is the analysis operator of $\mathcal{F}$. For each $n \in \mathbb{N}$, define $E_n : \mathcal{G} \to \mathcal{K}$ by

$$E_n(h \oplus y) = F_n(h) + {}_{\ell_2(\mathbb{H})}\langle e_n|Q(y),\ h \in \mathcal{H},\ y \in \ker\mathcal{T}^*.$$

Let $h \oplus y \in \mathcal{H} \oplus \ker\mathcal{T}^*$. Define $E : \mathcal{G} \to \ell_2(\mathbb{H}) \otimes \mathcal{K}$ as

$$E(h \oplus y) = \sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} E_n(h \oplus y) = \sum_{n\in\mathbb{N}} e_n \otimes E_n(h \oplus y).$$

Proceeding as in Theorem 3.11, we have

$$\|E(h \oplus y)\|^2 = \sum_{n\in\mathbb{N}} \|F_n(h)\|^2 + \|y\|^2 = \|h\|^2 + \|y\|^2 = \|h \oplus y\|^2,\ h \oplus y \in \mathcal{H} \oplus \ker\mathcal{T}^*.$$

Therefore E is an isometry. So, $\{E_n\}_{n\in\mathbb{N}}$ is a Parseval OPV-frame for $\mathcal{G}$ with range in $\mathcal{K}$. Also, it is a Riesz OPV-frame by Theorem 3.11 and hence $\{E_n\}_{n\in\mathbb{N}}$ is an orthonormal OPV-frame for $\mathcal{G}$ with range in $\mathcal{K}$. □

4. Trace class operators and their applications in quaternionic Hilbert spaces

In [7], Choi proved that in a Hilbert space H, a linear operator $\mathfrak{f}$ on $\mathcal{B}(H)$ is completely positive and trace preserving if and only if $\mathfrak{f}(F) = \sum_{n\in\mathbb{N}} F_n^* F F_n$, where $\{F_n\}_{n\in\mathbb{N}}$ is a collection of operators in $B(H)$ such that $\sum_{n\in\mathbb{N}} F_n^* F_n = \mathcal{I}$. In [21], Poumai, Kaushik, and Djordjević obtained Choi-Kraus representation in the context of OPV-frames in Hilbert spaces. In this section, we will prove these result in the context of quaternionic Hilbert spaces.

In [2], Stéphane Attal gave a necessary and sufficient condition for an operator to be of trace class in a Hilbert space. The following is an extension of this result for a right quaternionic Hilbert space.

Theorem 4.1. *A bounded operator T on a right quaternionic Hilbert space $\mathcal{H}$ is trace-class if and only if $\sum_{n\in\mathbb{N}} |\langle g_n|Th_n\rangle| < \infty$, for all orthonormal families $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$. Further, there exist orthonormal families $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $\|T\|_1 = \sum_{n\in\mathbb{N}} |\langle g_n|Th_n\rangle|$.*

Proof. Let $T \in B_1(\mathcal{H})$. Then, T is compact and there exist orthonormal sequences $\{\phi_n\}_{n\in\mathbb{N}}$, $\{\xi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ and a sequence of positive real numbers $\{\alpha_n\}_{n\in\mathbb{N}}$ (see [10, 22]) such that

$$Tx = \sum_{n\in\mathbb{N}} \xi_n \alpha_n \langle \phi_n|x\rangle,\ x \in \mathcal{H}.$$

Also, we have

$$\begin{aligned}
\sum_{n\in\mathbb{N}} |\langle g_n|Th_n\rangle| &= \sum_{n\in\mathbb{N}} \left|\left\langle g_n \middle| \sum_{k\in\mathbb{N}} \xi_k \alpha_k \langle \phi_k|h_n\rangle \right\rangle\right| \\
&\le \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \alpha_k |\langle g_n|\xi_k\rangle| |\langle \phi_k|h_n\rangle| \\
&\le \sum_{k\in\mathbb{N}} \alpha_k \left(\sum_{n\in\mathbb{N}} |\langle g_n|\xi_k\rangle|^2\right)^{1/2} \left(\sum_{n\in\mathbb{N}} |\langle \phi_k|h_n\rangle|^2\right)^{1/2} \\
&= \sum_{k\in\mathbb{N}} \alpha_k \|\xi_k\| \|\phi_k\| \\
&= \sum_{k\in\mathbb{N}} \alpha_k < \infty.
\end{aligned}$$

If we take $g_n = \xi_n$ and $h_n = \phi_n$, for all $n \in \mathbb{N}$, then

$$\sum_{n\in\mathbb{N}} |\langle g_n|T(h_n)\rangle| = \sum_{k\in\mathbb{N}} \alpha_k = \|T\|_1.$$

Conversely, let $T \in B(\mathcal{H})$ be such that $\sum\limits_{n\in\mathbb{N}} |\langle g_n|Th_n\rangle| < \infty$, for all orthonormal families $\{g_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$. Let $T = U|T|$ be the polar decomposition of T (see Theorem 2.3). Choose an orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$ of $\overline{Ran(|T|)}$ and put $g_n = Uh_n,\ n \in \mathbb{N}$. Since U is an isometry on $\overline{Ran(|T|)}$, $U^*(g_n) = h_n,\ n \in \mathbb{N}$. Therefore, we obtain

$$\sum_{n\in\mathbb{N}} |\langle g_n|Th_n\rangle| = \sum_{n\in\mathbb{N}} |\langle h_n||T|h_n\rangle| = \sum_{n\in\mathbb{N}} \langle h_n||T|h_n\rangle.$$

Further, one may observe that $\sum\limits_{n\in\mathbb{N}} \langle h_n||T|h_n\rangle < \infty$. Extending the family $\{h_n\}_{n\in\mathbb{N}}$ into an orthonormal basis $\{\tilde{h_n}\}$ of $\mathcal{H}$ by completing with orthonormal vectors in $\overline{Ran(|T|)}^{\perp} = N(|T|)$, we get

$$\sum_{n\in\mathbb{N}} \langle \tilde{h_n}||T|\tilde{h_n}\rangle = \sum_{n\in\mathbb{N}} \langle h_n||T|h_n\rangle < \infty.$$

Hence, T is a trace-class operator on $\mathcal{H}$. □

In the next result, we obtain a trace class operator on a two-sided quaternionic Hilbert space $\mathcal{K}$ using a trace class operator on $\mathcal{H} \otimes \mathcal{K}$.

Theorem 4.2. *If T is a trace-class operator on $\mathcal{H} \otimes \mathcal{K}$, then the operator ${}_{\mathcal{H}}\langle h|T|h\rangle_{\mathcal{H}}$ is a trace-class operator on $\mathcal{K}$, for all $h \in \mathcal{H}$. Moreover, for any orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$, the series $Tr_{\mathcal{H}}(T) = \sum_{n\in\mathbb{N}} {}_{\mathcal{H}}\langle h_n|T|h_n\rangle_{\mathcal{H}}$ is $\|.\|_1$ convergent.*

Proof. Let $\{k_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ be orthonormal families in $\mathcal{K}$. Without loss of generality, let $h \in \mathcal{H}$ be such that $\|h\| = 1$. Then

$$\sum_{n\in\mathbb{N}} |\langle k_n|\ {}_{\mathcal{H}}\langle h|T|h\rangle_{\mathcal{H}} u_n\rangle| = \sum_{n\in\mathbb{N}} |\langle h \otimes k_n|T(h \otimes u_n)\rangle|.$$

As T is a trace-class operator on $\mathcal{H} \otimes \mathcal{K}$ and $\{h \otimes k_n\}_{n\in\mathbb{N}}, \{h \otimes u_n\}_{n\in\mathbb{N}}$ are orthonormal families in $\mathcal{H} \otimes \mathcal{K}$, we have

$$\sum_{n\in\mathbb{N}} |\langle k_n|\ {}_{\mathcal{H}}\langle h|T|h\rangle_{\mathcal{H}} u_n\rangle| < \infty.$$

Thus, ${}_{\mathcal{H}}\langle h|T|h\rangle_{\mathcal{H}}$ is a trace-class operator on $\mathcal{K}$.

Moreover, as for each $n \in \mathbb{N}$, ${}_{\mathcal{H}}\langle h_n|T|h_n\rangle_{\mathcal{H}}$ is a trace-class operator on $\mathcal{K}$. Therefore, by Theorem 4.1, there exist orthonormal families $\{f_m^n\}_{m\in\mathbb{N}}$ and $\{g_m^n\}_{m\in\mathbb{N}}$ in $\mathcal{K}$ such that

$$\|{}_{\mathcal{H}}\langle h_n|T|h_n\rangle_{\mathcal{H}}\|_1 = \sum_{m\in\mathbb{N}} |\langle f_m^n|\ {}_{\mathcal{H}}\langle h_n|T|h_n\rangle_{\mathcal{H}}\ g_m^n\rangle|.$$

This gives

$$\sum_{n\in\mathbb{N}} \|{}_{\mathcal{H}}\langle h_n|T|h_n\rangle_{\mathcal{H}}\|_1 = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} |\langle f_m^n|\ {}_{\mathcal{H}}\langle h_n|T|h_n\rangle_{\mathcal{H}}\ g_m^n\rangle| = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} |\langle h_n \otimes f_m^n\ |T(h_n \otimes g_m^n)\rangle|.$$

Since $\{h_n \otimes f_m^n\}_{m,n\in\mathbb{N}}$ and $\{h_n \otimes g_m^n\}_{m,n\in\mathbb{N}}$ are orthonormal in $\mathcal{H} \otimes \mathcal{K}$, by Theorem 4.1, we obtain

$$\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} |\langle h_n \otimes f_m^n\ |T(h_n \otimes g_m^n)\rangle| < \infty.$$

Therefore, the series $\sum_{n\in\mathbb{N}} {}_{\mathcal{H}}\langle h_n|T|h_n\rangle_{\mathcal{H}}$ is $\|.\|_1$ convergent and hence the operator

$$Tr_{\mathcal{H}}(T) = \sum_{n\in\mathbb{N}} {}_{\mathcal{H}}\langle h_n|T|h_n\rangle_{\mathcal{H}}$$

is a well defined, trace-class operator on $\mathcal{K}$. □

Definition 4.3. *Let $\{h_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of a right quaternionic Hilbert space $\mathcal{H}$. Then, the partial trace $Tr_{\mathcal{H}}$ is a map from $\mathcal{B}_1(\mathcal{H}\otimes\mathcal{K})$ into $\mathcal{B}_1(\mathcal{K})$ defined by*

$$Tr_{\mathcal{H}}(L) = \sum_{n\in\mathbb{N}} {}_{\mathcal{H}}\langle h_n|L|h_n\rangle_{\mathcal{H}}.$$

Now, we prove three results in the form of lemmas which will be used in the subsequent results.

Lemma 4.4. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$, and $\{e_n\otimes k_n\}_{n\in\mathbb{N}}$ be an orthonormal system in $\ell_2(\mathbb{H})\otimes\mathcal{K}$. Then, $\{\mathcal{T}^*(e_n\otimes k_n)\}_{n\in\mathbb{N}}$ is an orthonormal system in $\mathcal{H}$.*

Proof. Since $\mathcal{F}$ is an orthonormal OPV-frame, by Observation (II), $\mathcal{T}\mathcal{T}^* = \mathcal{I}$. Therefore

$$\|\mathcal{T}^*(e_i\otimes k_i)\|^2 = \langle\mathcal{T}^*(e_i\otimes k_i)|T^*(e_i\otimes k_i)\rangle = \langle e_i\otimes k_i|e_i\otimes k_i\rangle = 1.$$

and

$$\langle\mathcal{T}^*(e_i\otimes k_i)|\mathcal{T}^*(e_j\otimes k_j)\rangle = 0. \quad \square$$

Lemma 4.5. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ with frame operator $\mathcal{S}$ and $\{e_n\otimes k_n\}_{n\in\mathbb{N}}$ be an orthonormal system in $\ell_2(\mathbb{H})\otimes\mathcal{K}$. Then $\{\mathcal{S}^{-1/2}\mathcal{T}^*(e_n\otimes k_n)\}_{n\in\mathbb{N}}$ is an orthonormal system in $\mathcal{H}$.*

Proof. It follows bearing in mind that

$$\|\mathcal{S}^{-1/2}\mathcal{T}^*(e_i\otimes k_i)\|^2 = \left\langle\mathcal{S}^{-1/2}\mathcal{T}^*(e_i\otimes k_i)|\mathcal{S}^{-1/2}\mathcal{T}^*(e_i\otimes k_i)\right\rangle = \langle(e_i\otimes k_i)|\mathcal{T}\mathcal{S}^{-1}\mathcal{T}^*(e_i\otimes k_i)\rangle = \|e_i\otimes k_i\|^2 = 1$$

and

$$\langle\mathcal{S}^{-1/2}\mathcal{T}^*(e_i\otimes k_i)|\mathcal{S}^{-1/2}\mathcal{T}^*(e_j\otimes k_j)\rangle = 0. \quad \square$$

In the following result, we construct a trace class operator on $\ell_2(\mathbb{H})\otimes\mathcal{K}$ using a trace class operator on $\mathcal{H}$.

Lemma 4.6. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and T be a trace class operator on $\mathcal{H}$. Then $\mathcal{T}T\mathcal{T}^*$ is a trace class operator on $\ell_2(\mathbb{H})\otimes\mathcal{K}$, where $\mathcal{T}$ is the analysis operator of the OPV-frame $\mathcal{F}$.*

Proof. Let $\{e_i\otimes k_i\}_{i\in\mathbb{N}}$ and $\{v_i\otimes u_i\}_{i\in\mathbb{N}}$ be orthonormal systems in $\ell_2(\mathbb{H})\otimes\mathcal{K}$. Then

$$\sum_{i\in\mathbb{N}} |\langle e_i\otimes k_i|\mathcal{T}T\mathcal{T}^*(v_i\otimes u_i)\rangle| = \sum_{i\in\mathbb{N}} |\langle\mathcal{T}^*(e_i\otimes k_i)|T\mathcal{T}^*(v_i\otimes u_i)\rangle|.$$

Note that $\{\mathcal{T}^*(e_i\otimes k_i)\}_{i\in\mathbb{N}}$ and $\{\mathcal{T}^*(v_i\otimes u_i)\}_{i\in\mathbb{N}}$ are orthonormal systems in $\mathcal{H}$ and $T\in\mathcal{B}_1(\mathcal{H})$. Then, one can obtain

$$\sum_{i\in\mathbb{N}} |\langle(e_i\otimes k_i)|\mathcal{T}T\mathcal{T}^*(v_i\otimes u_i)\rangle| < \infty.$$

Hence, $\mathcal{T}T\mathcal{T}^*$ is a trace class operator on $\ell_2(\mathbb{H})\otimes\mathcal{K}$. □

Next, we give a Choi-Kraus type representation using orthonormal OPV-frames in a right quaternionic Hilbert space.

Theorem 4.7. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. Then, there exists an operator $\phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{K})$ such that*

$$\phi(T) = \sum_{n\in\mathbb{N}} F_n T F_n^*, \ T \in \mathcal{B}_1(\mathcal{H}).$$

Proof. As T is a trace-class operator on $\mathcal{H}$, by Lemma 4.6, $\mathcal{T}T\mathcal{T}^*$ is a trace-class operator on $\ell_2(\mathbb{H}) \otimes \mathcal{K}$. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\ell_2(\mathbb{H})$. Proceeding as in Theorem 4.2, $\sum_{n\in\mathbb{N}} {}_{\ell_2(\mathbb{H})}\langle e_n|\mathcal{T}T\mathcal{T}^*|e_n\rangle_{\ell_2(\mathbb{H})}$ is $\|.\|_1$ convergent. Therefore,

$$Tr_{\ell_2(\mathbb{H})}(\mathcal{T}T\mathcal{T}^*) = \sum_{n\in\mathbb{N}} {}_{\ell_2(\mathbb{H})}\langle e_n|\mathcal{T}T\mathcal{T}^*|e_n\rangle_{\ell_2(\mathbb{H})} = \sum_{n\in\mathbb{N}} F_n T F_n^*.$$

Now define $\phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{K})$ as

$$\phi(T) = Tr_{\ell_2(\mathbb{H})}(\mathcal{T}T\mathcal{T}^*) = \sum_{n\in\mathbb{N}} F_n T F_n^*, \quad T \in \mathcal{B}_1(\mathcal{H}). \quad \square$$

Next, we give another type of Choi-Kraus type representation using Riesz OPV-frames.

Theorem 4.8. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. Then, there exists an operator $\phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{K})$ such that*

$$\phi(T) = \sum_{n\in\mathbb{N}} F_n \mathcal{S}^{-1/2} T \mathcal{S}^{-1/2} F_n^*,$$

where $\mathcal{S}$ is the frame operator for the Riesz OPV-frame $\mathcal{F}$. Further, we have

$$\phi(T) = Tr_{\ell_2(\mathbb{H})}(\mathcal{T}\mathcal{S}^{-1/2} T \mathcal{S}^{-1/2}\mathcal{T}^*).$$

Proof. Let $\{e_n \otimes k_n\}_{n\in\mathbb{N}}$ and $\{v_n \otimes u_n\}_{n\in\mathbb{N}}$ be orthonormal systems in $\ell_2(\mathbb{H}) \otimes \mathcal{K}$. Then

$$\sum_{n\in\mathbb{N}} |\langle e_n \otimes k_n|\mathcal{T}\mathcal{S}^{-1/2} T \mathcal{S}^{-1/2}\mathcal{T}^*(v_n \otimes u_n)\rangle| = \sum_{n\in\mathbb{N}} |\langle \mathcal{S}^{-1/2}\mathcal{T}^*(e_n \otimes k_n)|T\mathcal{S}^{-1/2}\mathcal{T}^*(v_n \otimes u_n)\rangle|.$$

Also, since $\{\mathcal{S}^{-1/2}\mathcal{T}^*(e_n \otimes k_n)\}_{n\in\mathbb{N}}$ and $\{\mathcal{S}^{-1/2}\mathcal{T}^*(v_n \otimes u_n)\}_{n\in\mathbb{N}}$ are orthonormal systems in $\mathcal{H}$ and $T \in \mathcal{B}_1(\mathcal{H})$, we have

$$\sum_{n\in\mathbb{N}} |\langle \mathcal{S}^{-1/2}\mathcal{T}^*(e_n \otimes k_n)|T\mathcal{S}^{-1/2}\mathcal{T}^*(v_n \otimes u_n)\rangle| < \infty.$$

This gives

$$\sum_{n\in\mathbb{N}} |\langle e_n \otimes k_n|\mathcal{T}\mathcal{S}^{-1/2} T \mathcal{S}^{-1/2}\mathcal{T}^*(v_n \otimes u_n)\rangle| < \infty.$$

Thus, $\mathcal{T}\mathcal{S}^{-1/2} T \mathcal{S}^{-1/2}\mathcal{T}^*$ is a trace-class operator on $\ell_2(\mathbb{H}) \otimes \mathcal{K}$. Further, proceeding as in Theorem 4.7, there exists an operator $\phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{K})$ such that

$$\phi(T) = Tr_{\ell_2(\mathbb{H})}(\mathcal{T}\mathcal{S}^{-1/2} T \mathcal{S}^{-1/2}\mathcal{T}^*) = \sum_{n\in\mathbb{N}} {}_{\ell_2(\mathbb{H})}\langle e_n|\mathcal{T}\mathcal{S}^{-1/2} T \mathcal{S}^{-1/2}\mathcal{T}^*|e_n\rangle_{\ell_2(\mathbb{H})} = \sum_{n\in\mathbb{N}} F_n \mathcal{S}^{-1/2} T \mathcal{S}^{-1/2} F_n^*. \quad \square$$

The following is another Choi-Kraus type representation using Parseval OPV-frames.

Theorem 4.9. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be a Parseval OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. Then, there exists a sequence of operators $\{E_n : \mathcal{H} \oplus \ker\mathcal{T}^* \to \mathcal{K}\}_{n\in\mathbb{N}}$ and an operator $\Phi : \mathcal{B}_1(\mathcal{H} \oplus \ker\mathcal{T}^*) \to \mathcal{B}_1(\mathcal{K})$ such that*

$$\Phi(T) = \sum_{n\in\mathbb{N}} E_n T E_n^*, \ T \in \mathcal{B}_1(\mathcal{H} \oplus \ker\mathcal{T}^*) \ \text{and} \ \sum_{n\in\mathbb{N}} E_n^* E_n = \mathcal{I}_{\mathcal{H}\oplus\ker\mathcal{T}^*}.$$

Proof. It can be worked out on the lines of the proof of Theorem 3.12 and Theorem 4.7. $\square$

Finally, we show that for any quaternionic quantum channel $\phi : B(\mathcal{H}) \to B(\mathcal{K})$, there exists a right quaternionic Hilbert space $\mathcal{Z}$ and an isometry $V : \mathcal{H} \to \mathcal{Z} \otimes \mathcal{K}$ such that for any $T \in B(\mathcal{H})$, $\phi(T) = Tr_{\mathcal{Z}}(VTV^*)$.

Theorem 4.10. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a Parseval OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. If there exists an operator $\Phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{K})$ such that*

$$\Phi(T) = \sum_{n\in\mathbb{N}} F_n T F_n^*, \quad T \in \mathcal{B}_1(\mathcal{H}),$$

then there exists an isometry $\mathcal{T} : \mathcal{H} \to \ell_2(\mathbb{H}) \otimes \mathcal{K}$ such that $\Phi(T) = Tr_{\ell_2(\mathbb{H})}(\mathcal{T} T \mathcal{T}^*)$ and $\mathcal{T}^*\mathcal{T} = \mathcal{I}_{\mathcal{H}}$.

Proof. Let $\mathcal{T}$ be the analysis operator for the Parseval OPV-frame $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$. Then, $\mathcal{T}$ is an isometry. Let $T \in \mathcal{B}_1(\mathcal{H})$. Then

$$\mathcal{T} T \mathcal{T}^* = \left(\sum_{n\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} F_n\right)(T)\left(\sum_{j\in\mathbb{N}} F_j^* \,{}_{\ell_2(\mathbb{H})}\langle e_j|\right) = \sum_{n,j\in\mathbb{N}} |e_n\rangle_{\ell_2(\mathbb{H})} F_n T F_j^* \,{}_{\ell_2(\mathbb{H})}\langle e_j|.$$

Also, we have

$$Tr_{\ell_2(\mathbb{H})}(\mathcal{T} T \mathcal{T}^*) = \sum_{i,n,j\in\mathbb{N}} {}_{\ell_2(\mathbb{H})}\langle e_i||e_n\rangle_{\ell_2(\mathbb{H})} F_n T F_j^* \,{}_{\ell_2(\mathbb{H})}\langle e_j||e_i\rangle_{\ell_2(\mathbb{H})} = \sum_{n\in\mathbb{N}} F_n T F_n^*.$$

Hence $\Phi(T) = Tr_{\ell_2(\mathbb{H})}(\mathcal{T} T \mathcal{T}^*)$ and $\mathcal{T}^*\mathcal{T} = \mathcal{I}_{\mathcal{H}}$. □

5. Conclusion

OPV-frames can be used in the study of quaternionic quantum mechanics (QQM). Quaternionic quantum channels are the transformation channels that can transform the initial associated state of any physical system. These channels act as a path or a pipe used to transmit quantum information. Keeping this in mind,we made an attempt to find Choi- Kraus type representations in quaternionic Hilbert spaces. To achieve this goal, we introduced and studied OPV-frames in quaternionic Hilbert spaces and obtained various results including a characterization for their existence. We also proved that an OPV-frame (Parseval OPV-frame) is a compression of a Riesz OPV-frame (orthonormal OPV-frame).

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