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Upper and Lower Solutions for Second-Order with M-Point Impulsive Boundary Value Problems

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Abstract. In this paper, we consider a second-order m-point impulsive boundary value problem. By applying the upper and lower solutions method and the Schauder's fixed point theorem, we obtain the existence of at least one positive solution. We also give an example to illustrate our main result.

1. Introduction

The theory of impulsive differential equations is a new and significant subsection of differential equation theory, which has an extensive physical ecology, biological systems, population dynamics, and engineering background. Moreover, impulsive differential equations advance a more realistic approach to modeling many tangible problems encountered in various fields such as control theory, electronics, mechanics, economics, electrical circuits and medicine. For the introduction of the fundamental theory of impulsive equations, see [1, 4, 6, 22, 24] and the papers [2, 21].

Some authors in the literature have obtained results about the solutions of second-order impulsive boundary value problems, for some see [5, 10–14, 16, 19, 20, 23, 27] in the references. Recently, some works, such as Liu and Yu [3], Meiqiang and Dongxiu [8], Zhao et al. [18] and Tian and Liu [26], deal with second-order m-point impulsive boundary value problems. Yet, many of the results are usually obtained using fixed point theorems on the cone. There are also other methods to prove the existence of solutions. The upper and lower solutions method, in particular, is a powerful method for proving the existence of results for boundary value problems, for some see [7, 9, 17, 18, 25]. There is no study on second-order with m-point impulsive boundary value problems using the method upper and lower solutions, except that in [18].

In [18], Zhao and Ge considered

$$\begin{cases} \left(\phi_p\left(u'(t)\right)\right)' + q(t)f(t,u(t),u'(t)) = 0, \ t \in J' = [0,1]\backslash\{t_1,t_2,...,t_n\} \\ \triangle u|_{t=t_k} = I_{1k}(u(t_k)), \ k = 1,2,...,n, \\ \triangle \phi_p(u')|_{t=t_k} = I_{2k}(u(t_k),u'(t_k)), \ k = 1,2,...,n, \\ u(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u(\eta_i), \ \ u'(1) = 0 \end{cases}$$

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by using the upper and lower solutions method, the authors have existence of solutions to the above boundary value pronblem.

Motivated by the mentioned above result, in this study, we consider the following second-order m-point impulsive boundary value problem (IBVP)

$$\begin{cases} z''(t) + f(t, z(t), z'(t)) = 0, \ t \in J = [0,1], t \neq t_k, k = 1,2,...,n, \\ \triangle z|_{t=t_k} = I_k(z(t_k)), \\ \triangle z'|_{t=t_k} = -\bar{I}_k(z(t_k), z'(t_k)), \\ z(0) = \sum_{j=1}^{m-2} \gamma_j z(\varsigma_j), \ \ az(1) + bz'(1) = \sum_{j=1}^{m-2} \delta_j z(\varsigma_j) \end{cases} \tag{1}$$

where $J = [0,1]$, $t \neq t_k$, $k = 1,2,...,n$ with $0 < t_1 < t_2 < ... < t_n < 1$. $\triangle z|_{t=t_k}$ and $\triangle z'|_{t=t_k}$ indicate the jump of $z(t)$ and $z'(t)$ at $t = t_k$, i.e.,

$$\triangle z|_{t=t_k} = z(t_k^+) - z(t_k^-), \ \ \triangle z'|_{t=t_k} = z'(t_k^+) - z'(t_k^-),$$

where $z(t_k^+), z'(t_k^+)$ and $z(t_k^-), z'(t_k^-)$ symbolize the right-hand limit and left-hand limit of $z(t)$ and $z'(t)$ at $t = t_k$, $k = 1,2,...,n$, respectively.

We assume that following conditions are provided throughout this paper.

$(K1)$ $a, b \in [0,\infty)$, $\gamma_j, \delta_j \in [0,\infty)$, $\varsigma_j \in (0,1)$, for $j \in \{1,...,m-2\}$,

$(K2)$ $f \in C(J \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$,

$(K3)$ $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a bounded function, $\bar{I}_k \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ such that $[b + a(1-t_k)]\bar{I}_k(z(t_k), z'(t_k)) > aI_k(z(t_k))$, $t < t_k$, $k = 1,2,...,n$.

Our aim in this article is to investigate the existence of postive solutions for the IBVP (1). For this, Schauder's Fixed Point Theorem and the upper and lower solutions method will be the main tool.

The main structure of this paper is as follows. In Section 2, we provide some definitions and preliminary lemmas are key tools for our main result. In Section 3, we give and prove our main result. In Section 4, we give an example to demonstrate our main result.

2. Preliminaries

In this section, we first introduce some background definitions in Banach spaces, and then present auxiliary lemmas which will be used later.
Let $J' = J \backslash \{t_1, t_2, ..., t_n\}$. $C(J, \mathbb{R}^+)$ indicate the Banach space of all continuous mapping $z : J \to \mathbb{R}^+$ with the norm $\|z\| = \sup_{t \in J} |z(t)|$, $PC(J, \mathbb{R}^+) = \{z : J \to \mathbb{R}^+ : z \in C(J'),\ z(t_k^+)$ and $z(t_k^-)$ exist and $z(t_k^-) = z(t_k),\ k = 1,2,...,n\}$ is also a Banach space with norm $\|z\|_{PC} = \sup_{t\in J}|z(t)|$, and $PC^1(J, \mathbb{R}^+) = \{z \in PC(J, \mathbb{R}^+) : z' \in PC(J'),\ z'(t_k^+)$ and $z'(t_k^-)$ exist and $z'(t_k^-) = z'(t_k),\ k = 1,2,...,n\}$ is a real Banach space with norm $\|z\|_{PC^1} = \max\{\|z\|_{PC}, \|z'\|_{PC}\}$ where $\|z\|_{PC} = \sup_{t\in J}|z(t)|$, $\|z'\|_{PC} = \sup_{t\in J}|z'(t)|$. A function $z \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ is referred a solution of the IBVP (1) provided that it yields (1).

Definition 2.1. *The function $\underline{z} \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ is called a lower solution for the IBVP* (1) *if*

$$\begin{cases} \underline{z}''(t) + f(t, \underline{z}(t), \underline{z}'(t)) \geq 0, \ t \in J = [0,1], t \neq t_k, k = 1,2,...,n, \\ \triangle \underline{z}|_{t=t_k} = I_k(\underline{z}(t_k)), \\ \triangle \underline{z}'|_{t=t_k} = -\bar{I}_k(\underline{z}(t_k), \underline{z}'(t_k)), \\ \underline{z}(0) \leq \sum_{j=1}^{m-2} \gamma_j \underline{z}(\varsigma_j), \ \ a\underline{z}(1) + b\underline{z}'(1) \leq \sum_{j=1}^{m-2} \delta_j \underline{z}(\varsigma_j). \end{cases}$$

The function $\overline{z} \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ is called an upper solution for the IBVP (1) *if*

$$\begin{cases} \overline{z}''(t) + f(t, \overline{z}(t), \overline{z}'(t)) \leq 0, \ t \in J = [0,1], t \neq t_k, k = 1, 2, ..., n, \\ \triangle \overline{z}|_{t=t_k} = I_k(\overline{z}(t_k)), \\ \triangle \overline{z}'|_{t=t_k} = -\overline{I}_k(\overline{z}(t_k), \overline{z}'(t_k)), \\ \overline{z}(0) \geq \sum_{j=1}^{m-2} \gamma_j \overline{z}(\varsigma_j), \ \ a\overline{z}(1) + b\overline{z}'(1) \geq \sum_{j=1}^{m-2} \delta_j \overline{z}(\varsigma_j). \end{cases}$$

For $\underline{z}$, $\overline{z} \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$, we can write $\underline{z} \leq \overline{z}$ if $\underline{z}(t) \leq \overline{z}(t)$, $\forall t \in J$.

Definition 2.2. *Let $\underline{z}$, $\overline{z} \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ be such that $\underline{z} \leq \overline{z}$ on J. We express that f provides the Nagumo condition with respect to $\underline{z}, \overline{z}$ if for*

$$\varrho = \max_{1 \leq k \leq n} \left\{ \frac{\underline{z}(t_{k+1}) - \overline{z}(t_k^+)}{t_{k+1} - t_k}, \frac{\overline{z}(t_{k+1}) - \underline{z}(t_k^+)}{t_{k+1} - t_k} \right\},$$

there exists a constant σ such that

$$\sigma > \max\{\varrho, \|\underline{z}'\|_{PC}, \|\overline{z}'\|_{PC}\},$$

a continuous function $\varphi : [0, \infty) \to [0, \infty)$ and constants $D \geq 0$, $E \geq 0$ such that

$$|f(t, z, z')| \leq D|z'|\varphi(|z'|) + E, t \in J, \ \underline{z} \leq \overline{z}, \ z' \in \mathbb{R}^+$$

and

$$\int_{\varrho}^{\sigma} \frac{1}{\varphi(s)} ds > D[\max_{t \in J} \overline{z}(t) - \min_{t \in J} \underline{z}(t)] + E \max_{s \geq \varrho} \frac{1}{\varphi(s)}.$$

Also, suppose that the following conditions are provided:

($K4$) f provides the Nagumo condition with respect to $\underline{z}$, $\overline{z}$;

($K5$) $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\overline{I}_k \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\overline{I}_k(z, z')$ is nondecreasing in $z' \in [-\sigma, \sigma]$ for all $1 \leq k \leq n$;

($K6$) $\sum_{j=1}^{m-2} \delta_j < a, \ \sum_{j=1}^{m-2} \gamma_j < 1.$

We consider the modified problem

$$\begin{cases} z''(t) + f^*(t, z(t), \frac{d}{dt} m_{\underline{z}, \overline{z}}(t, z)) = 0, \ t \in J = [0,1], t \neq t_k, k = 1, 2, ..., n, \\ \triangle z|_{t=t_k} = I_k(m_{\underline{z}, \overline{z}}(t_k, z(t_k))), \\ \triangle z'|_{t=t_k} = -\overline{I}_k(m_{\underline{z}, \overline{z}}(t_k, z(t_k)), n(t_k, z'(t_k))), \\ z(0) = \sum_{j=1}^{m-2} \gamma_j z(\varsigma_j), \ \ az(1) + bz'(1) = \sum_{j=1}^{m-2} \delta_j z(\varsigma_j) \end{cases} \tag{2}$$

where

$$f^*(t, z, z') = f(t, m_{\underline{z}, \overline{z}}(t, z), n(t, z')) + \frac{m_{\underline{z}, \overline{z}}(t, z) - z}{1 + (z - m_{\underline{z}, \overline{z}}(t, z))^2}$$

and $m_{\underline{z}, \overline{z}}(t, z) = \max\{\underline{z}(t), \min\{z, \overline{z}(t)\}\}$, $n(t, z) = \max\{-\sigma, \min\{z, \sigma\}\}$.

Lemma 2.3. *[23] For each $z \in PC^1(J,\mathbb{R}^+) \cap C^2(J',\mathbb{R})$, the following two properties hold:*
(i) $\frac{d}{dt} m_{\underline{z},\bar{z}}(t,z)$ exists for a.e. $t \in J'$;
(ii) if $z, z_n \in PC^1(J,\mathbb{R}^+) \cap C^2(J',\mathbb{R})$ and $z_n \to z \in PC^1(J,\mathbb{R}^+) \cap C^2(J',\mathbb{R})$, then $\frac{d}{dt} m_{\underline{z},\bar{z}}(t,z_n(t)) \to \frac{d}{dt} m_{\underline{z},\bar{z}}(t,z(t))$ for a.e. $t \in J'$.

Set

$$\rho = a + b, \tag{3}$$

and

$$\triangle = \begin{vmatrix} -\sum_{j=1}^{m-2}\gamma_j\varsigma_j & \rho - \sum_{j=1}^{m-2}\gamma_j[b + a(1-\varsigma_j)] \\ \rho - \sum_{j=1}^{m-2}\delta_j\varsigma_j & -\sum_{j=1}^{m-2}\delta_j[b + a(1-\varsigma_j)] \end{vmatrix}. \tag{4}$$

Lemma 2.4. *Let (K1)-(K3) hold. Assume that*
(K7) $\triangle \neq 0$.
If $z \in PC^1(J,\mathbb{R}^+) \cap C^2(J',\mathbb{R})$ is a solution of the equation

$$z(t) = \int_0^1 G(t,s) f(s,z(s),z'(s))ds + \sum_{k=1}^{n} W_k(t,z) + tA(f) + (b + a(1-t))B(f), \tag{5}$$

where

$$W_k(t,z) = \frac{1}{\rho}\begin{cases} t[-aI_k(z(t_k)) + (b + a(1-t_k))\bar{I}_k(z(t_k),z'(t_k))], & t < t_k, \\ (b + a(1-t))[I_k(z(t_k)) + t_k\bar{I}_k(z(t_k),z'(t_k))], & t_k \leq t, \end{cases} \tag{6}$$

$$G(t,s) = \frac{1}{\rho}\begin{cases} s[a(1-t) + b], & s \leq t, \\ t[a(1-s) + b], & t \leq s, \end{cases} \tag{7}$$

$$A(f) = \frac{1}{\triangle}\begin{vmatrix} \sum_{j=1}^{m-2}\gamma_j K(\varsigma_j,z) & \rho - \sum_{j=1}^{m-2}\gamma_j[b + a(1-\varsigma_j)] \\ \sum_{j=1}^{m-2}\delta_j K(\varsigma_j,z) & -\sum_{j=1}^{m-2}\delta_j[b + a(1-\varsigma_j)] \end{vmatrix}, \tag{8}$$

$$B(f) = \frac{1}{\triangle}\begin{vmatrix} -\sum_{j=1}^{m-2}\gamma_j\varsigma_j & \sum_{j=1}^{m-2}\gamma_j K(\varsigma_j,z) \\ \rho - \sum_{j=1}^{m-2}\delta_j\varsigma_j & \sum_{j=1}^{m-2}\delta_j K(\varsigma_j,z) \end{vmatrix}, \tag{9}$$

and

$$K(\varsigma_j,z) = \int_0^1 G(\varsigma_j,s) f(s,z(s),z'(s))ds + \sum_{k=1}^{n} W_k(\varsigma_j,z), \tag{10}$$

then z is a solution of the IBVP (1).

Proof. Let z satisfies the integral equation (5), then we get

$$z(t)=\int_0^1 G(t,s)f(s,z(s),z'(s))ds+\sum_{k=1}^{n}W_k(t,z)+tA(f)+(b+a(1-t))B(f),$$

i.e.,

$$\begin{aligned} z(t) &= \frac{1}{\rho}\int_0^t s(b+a(1-t))f(s,z(s),z'(s))ds+\frac{1}{\rho}\int_t^1 t(b+a(1-s))f(s,z(s),z'(s))ds \\ &+\frac{1}{\rho}\sum_{0<t_k<t}(b+a(1-t))[I_k(z(t_k))+t_k\bar{I}_k(z(t_k),z'(t_k))] \\ &+\frac{1}{\rho}\sum_{t<t_k<1}t[-aI_k(z(t_k))+(b+a(1-t_k))\bar{I}_k(z(t_k),z'(t_k))] \\ &+tA(f)+(b+a(1-t))B(f), \end{aligned}$$

$$\begin{aligned} z'(t) &= \frac{1}{\rho}\int_0^t(-as)f(s,z(s),z'(s))ds+\frac{1}{\rho}\int_t^1(b+a(1-s))f(s,z(s),z'(s))ds \\ &+\frac{1}{\rho}\sum_{0<t_k<t}(-a)[I_k(z(t_k))+t_k\bar{I}_k(z(t_k),z'(t_k))] \\ &+\frac{1}{\rho}\sum_{t<t_k<1}[-aI_k(z(t_k))+(b+a(1-t_k))\bar{I}_k(z(t_k),z'(t_k))] \\ &+A(f)+(-a)B(f). \end{aligned}$$

So that

$$\begin{aligned} z''(t) &= \frac{1}{\rho}(-at-(b+a(1-t)))f(t,z(t),z'(t)) \\ &= -f(t,z(t),z'(t)), \\ & z''(t)+f(t,z(t),z'(t))=0. \end{aligned}$$

Since

$$z(0)=\rho B(f),$$

we have that

$$\begin{aligned} z(0) &= \rho B(f) \\ &= \sum_{j=1}^{m-2}\gamma_j\left[\int_0^1 G(\varsigma_j,s)f(s,z(s),z'(s))ds+\sum_{k=1}^{n}W_k(\varsigma_j,z)+\varsigma_jA(f)+(b+a(1-\varsigma_j))B(f)\right]. \end{aligned} \tag{11}$$

Since

$$z(1)=\frac{b}{\rho}\int_0^1 sf(s,z(s),z'(s))ds+\frac{b}{\rho}\sum_{k=1}^{n}t_k\bar{I}_k(z(t_k),z'(t_k))+\frac{b}{\rho}\sum_{k=1}^{n}I_k(z(t_k))+A(f)+bB(f)$$

and

$$z'(1) = \frac{-a}{\rho}\int_0^1 t_k f(s,z(s),z'(s))ds + \frac{-a}{\rho}\sum_{k=1}^{n} t_k \bar{I}_k(z(t_k),z'(t_k)) + \frac{-a}{\rho}\sum_{k=1}^{n} I_k(z(t_k)) + A(f) + (-a)B(f),$$

we get that

$$\begin{aligned} az(1)+bz'(1) &= \rho A(f) \\ &= \sum_{j=1}^{m-2}\delta_j\left[\int_0^1 G(\varsigma_j,s)f(s,z(s),z'(s))ds + \sum_{k=1}^{n} W_k(\varsigma_j,z) + \varsigma_j A(f) + (b+a(1-\varsigma_j))B(f)\right]. \end{aligned} \tag{12}$$

From (11) and (12), we have that

$$\begin{cases} -\left[\sum_{j=1}^{m-2}\gamma_j\varsigma_j\right]A(f) + \left[\rho - \sum_{j=1}^{m-2}\gamma_j(b+a(1-\varsigma_j))\right]B(f) = \sum_{j=1}^{m-2}\gamma_j K(\varsigma_j,z), \\ \left[\rho - \sum_{j=1}^{m-2}\delta_j\varsigma_j\right]A(f) + \left[-\sum_{i=1}^{m-2}\delta_j(b+a(1-\varsigma_j))\right]B(f) = \sum_{j=1}^{m-2}\delta_j K(\varsigma_j,z) \end{cases}$$

which yields that $A(f)$ and $B(f)$ satisfy (8) and (9), respectively. □

Lemma 2.5. *Let (K1)-(K3) hold. Then for all $t,s\in J$, we have*

$$G(t,s)\le G(s,s).$$

Lemma 2.6. *Let (K1)-(K3) hold. Assume that*

(K8) $\triangle<0,\;\; \rho-\sum_{j=1}^{m-2}\delta_j\varsigma_j>0,\;\; \rho-\sum_{j=1}^{m-2}\gamma_j[b+a(1-\varsigma_j)]>0$

holds. Then for $z\in PC^1(J,\mathbb{R}^+)\cap C^2(J',\mathbb{R})$, the solution z of the IBVP (1) *satisfies*

$$z(t)\ge 0 \;\; for \;\; t\in J.$$

Proof. It can be easily seen from the facts of $G(t,s)\ge 0$ on $J\times J$ and $A(f)\ge 0,\; B(f)\ge 0$. □

Lemma 2.7. *If z is a solution of the IBVP* (2)*, $\underline{z}(t)$ and $\overline{z}(t)$ are lower and upper solutions of the IBVP* (1)*, respectively, $\underline{z}\le\overline{z}$, and*

$$I_k(\underline{z}(t_k))\le I_k(z)\le I_k(\overline{z}(t_k)),\;\; k=1,...,n, \tag{13}$$

for

$$\underline{z}(t_k)\le z\le\overline{z}(t_k),\;\; k=1,...,n,$$

then

$$\underline{z}(t)\le z(t)\le\overline{z}(t),\;\; t\in J.$$

Proof. Indicate $\vartheta(t)=z(t)-\overline{z}(t)$, we will just prove that $z(t)\le\overline{z}(t)$ for all $t\in J$. With the similar technique it can be shown that $z(t)\ge\underline{z}(t)$ for all $t\in J$ is provided. Let $z(t)>\overline{z}(t), t\in J$ holds. Then, $\sup_{t\in J}(z(t)-\overline{z}(t))>0$, there are three cases.

Case 1. Assume that $\max_{0\le t\le 1}\vartheta(t)=\sup_{t\in J}(z(t)-\overline{z}(t))=\vartheta(0)$, or $\max_{0\le t\le 1}\vartheta(t)=\vartheta(1)$, we only see that $\max_{0\le t\le 1}\vartheta(t)=\vartheta(1)$. It is clear that $\vartheta(1)>0$. By virtue of definition 2.1 and (K6), we get

$$\begin{aligned}\vartheta(1)= \quad z(1)-\overline{z}(1) \quad &\le \frac{1}{a}\sum_{j=1}^{m-2}\delta_j z(\varsigma_j)-\frac{b}{a}z'(1)-\frac{1}{a}\sum_{j=1}^{m-2}\delta_j\overline{z}(\varsigma_j)+\frac{b}{a}\overline{z}'(1)\\ &=\frac{1}{a}\sum_{j=1}^{m-2}\delta_j[z(\varsigma_j)-\overline{z}(\varsigma_j)]-\frac{b}{a}[z'(1)-\overline{z}'(1)]\\ &=\frac{1}{a}\sum_{j=1}^{m-2}\delta_j\vartheta(\varsigma_j)-\frac{b}{a}\vartheta'(1)\\ &\le \max_{t\in J}\vartheta(t)\frac{1}{a}\sum_{j=1}^{m-2}\delta_j\\ &<\max_{t\in J}\vartheta(t).\end{aligned}$$

This is a contradiction. So, our assumption is wrong.

Case 2. Assume that there exist $k\in\{0,1,...,n\}$ and $\zeta\in(t_k,t_{k+1})$ such that $\sup_{t\in(t_k,t_{k+1}]}\vartheta(t)=\vartheta(\zeta)=z(\zeta)-\overline{z}(\zeta)>0$. Then $\vartheta'(\zeta)=0$ and $\vartheta''(t)\le 0$. On the other hand,

$$\begin{aligned}\vartheta''(\zeta)= \quad z''(\zeta)-\overline{z}''(\zeta) \quad &\ge -f^*(\zeta,z(\zeta),\frac{d}{d\zeta}m_{\underline{z},\overline{z}}(\zeta,z))+f(\zeta,\overline{z}(\zeta),\overline{z}'(\zeta))\\ &=-f(\zeta,m(\zeta,z(\zeta)),\frac{d}{d\zeta}m_{\underline{z},\overline{z}}(\zeta,z))-\frac{m_{\underline{z},\overline{z}}(\zeta,z(\zeta))-z(\zeta)}{1+(z(\zeta)-m_{\underline{z},\overline{z}}(\zeta,z(\zeta)))^2}\\ &\quad +f(\zeta,\overline{z}(\zeta),\overline{z}'(\zeta))\\ &=-f(\zeta,\overline{z}(\zeta),\overline{z}'(\zeta))-\frac{m_{\underline{z},\overline{z}}(\zeta,z(\zeta))-z(\zeta)}{1+(z(\zeta)-m_{\underline{z},\overline{z}}(\zeta,z(\zeta)))^2}\\ &\quad +f(\zeta,\overline{z}(\zeta),\overline{z}'(\zeta))\\ &=\frac{z(\zeta)-m_{\underline{z},\overline{z}}(\zeta,z(\zeta))}{1+(z(\zeta)-m_{\underline{z},\overline{z}}(\zeta,z(\zeta)))^2}\\ &=\frac{\vartheta(\zeta)}{1+\vartheta^2(\zeta)}>0.\end{aligned}$$

This is a contradiction. So, our assumption is wrong. Consequently, the function ϑ cannot have any positive maximum in the interval (t_k,t_{k+1}) for $k=1,2,...,n$.
Case 3. In accordance with Case 2, if $\sup_{t\in J}\vartheta(t)>0$, then $\sup_{t\in J}\vartheta(t)=\vartheta(t_k^+)$ or $\sup_{t\in J}\vartheta(t)=\vartheta(t_k^-)=\vartheta(t_k)$, we just prove that $\sup_{t\in J}\vartheta(t)=\vartheta(t_k^-),\ k=1,2,...,n$. Assume that $\sup_{t\in J}\vartheta(t)=\vartheta(t_1^-),\ \vartheta'(t_1)=\vartheta'(t_1^-)\ge 0$.

From (2) and (13), we can get

$$\begin{aligned} z(t_1^+) - z(t_1) &= z(t_1^+) - z(t_1^-) = I_1(m_{\underline{z},\overline{z}}(t_1, z(t_1))) \\ &= I_1(\overline{z}(t_1)) = \overline{z}(t_1^+) - \overline{z}(t_1^-) = \overline{z}(t_1^+) - \overline{z}(t_1), \end{aligned}$$

$$\begin{aligned} z'(t_1^+) - z'(t_1) &= z'(t_1^+) - z'(t_1^-) = \overline{I}_1(m_{\underline{z},\overline{z}}(t_1, z(t_1)), n(t_1, z'(t_1))) \\ &\geq \overline{I}_1(\overline{z}(t_1), \overline{z}'(t_1)) = \overline{z}'(t_1^+) - \overline{z}'(t_1^-). \end{aligned}$$

Hence

$$\begin{aligned} \vartheta(t_1^+) &= z(t_1^+) - \overline{z}(t_1^+) = z(t_1) - \overline{z}(t_1) > 0, \\ \vartheta'(t_1^+) &= z'(t_1^+) - \overline{z}'(t_1^+) = z'(t_1) - \overline{z}'(t_1) \geq 0. \end{aligned}$$

Assume that $\vartheta'(t_1^+) = 0$ and ϑ is nonincreasing on some interval $(t_1, t_1 + \kappa) \subset (t_1, t_2)$, where $\kappa > 0$ is small enough such that $\vartheta(t) > 0$ on $t \in (t_1, t_1 + \kappa)$. For $t \in (t_1, t_1 + \kappa)$,

$$\begin{aligned} z''(t) - \overline{z}''(t) &\geq -f(t, \overline{z}(t), \frac{d}{dt} m_{\underline{z},\overline{z}}(t, z)) - \frac{m_{\underline{z},\overline{z}}(t, z) - z}{1 + (z - m_{\underline{z},\overline{z}}(t, z))^2} \\ &\quad + f(t, \overline{z}(t), \overline{z}'(t)) \\ &= \frac{z - m_{\underline{z},\overline{z}}(t, z)}{1 + (z - m_{\underline{z},\overline{z}}(t, z))^2} > 0. \end{aligned}$$

This contradicts the assumption of monotonicity of ϑ. Therefore, we have

$$\begin{aligned} 0 < \vartheta(t_1^+) &< \vartheta(t_2) = \vartheta(t_2^-), \\ \vartheta'(t_2) &= \vartheta'(t_2^-) \geq 0. \end{aligned}$$

We practice the previous method and conclude by induction that

$$\vartheta(t_k) > 0, \quad \vartheta'(t_k) \geq 0, \quad k = 1, 2, ..., n+1.$$

This is a contradiction. Because $\vartheta(1)$ is not the maximum point. With the same analysis, we can obtain that $\vartheta(t_k) > 0$, $k = 2, 3, ..., n$, cannot hold. □

Lemma 2.8. *$-\sigma \leq z'(t) \leq \sigma$ on J, where $z(t)$ is the solution of the IBVP* (2).

Proof. We just show $z'(t) \leq \sigma$. Assume that there exists $\rho \in (t_k, t_{k+1})$ with $z'(\rho) = \dfrac{z(t_{k+1}) - z(t_k)}{t_{k+1} - t_k}$, $k = 1, 2, ..., n$, and as a result,

$$-\sigma < -\varrho \leq \frac{\underline{z}(t_{k+1}) - \overline{z}(t_k)}{t_{k+1} - t_k} \leq z'(\rho) \leq \frac{\overline{z}(t_{k+1}) - \underline{z}(t_k)}{t_{k+1} - t_k} \leq \varrho < \sigma.$$

Therefore, there exist $\eta_1, \eta_2 \in (t_k, t_{k+1})$ such that $z'(\eta_1) = \varrho$, $z'(\eta_2) = \sigma$ and either

$$\varrho \leq z'(t) \leq \sigma, \quad t \in (\eta_1, \eta_2),$$

or

$$\varrho \leq z'(t) \leq \sigma, \quad t \in (\eta_2, \eta_1).$$

we will only handle the first situation, as the other situation can be handled similarly. The following statement is due to the assumption

$$\begin{aligned}|z''(t)| &= |f(t,z(t),z'(t))| \\ &\le D|z'(t)|\varphi(|z'(t)|)+E, \quad for \;\; t\in(\eta_1,\eta_2).\end{aligned}$$

This implies that

$$\begin{aligned}\int_{\eta_1}^{\eta_2}\frac{|z''(t)|}{\varphi(|z'(t)|)}dt &\le D\int_{\eta_1}^{\eta_2}|z'(t)|dt+E\int_{\eta_1}^{\eta_2}\frac{dt}{\varphi(|z'(t)|)} \\ &= D(z(\eta_2)-z(\eta_1))+E\int_{\eta_1}^{\eta_2}\frac{dt}{\varphi(|z'(t)|)},\end{aligned}$$

which yields

$$\begin{aligned}\int_{\varrho}^{\sigma}\frac{ds}{\varphi(s)} = \int_{z'(\eta_1)}^{z'(\eta_2)}\frac{ds}{\varphi(s)} &\le D(z(\eta_2)-z(\eta_1))+E(\eta_2-\eta_1)\max_{s\ge\varrho}\frac{1}{\varphi(s)} \\ &\le D(\max_{t\in J}\overline{z}(t)-\max_{t\in J}\underline{z}(t))+E\max_{s\ge\varrho}\frac{1}{\varphi(s)}.\end{aligned}$$

This contradicts the choice of σ. Thus, it is obtained that $z'(t)\le\sigma$ is. $\square$

3. Main Result

In this section, we will obtain the existence of at least one positive solution for the IBVP (1). The theorem, which is fundamental and important for the proof of our main result, is the fixed point theorem below.

Lemma 3.1. *[6] (Schauder's fixed point theorem) Let K be a convex subset of a normed linear space E. Each continuous, compact map $L: K\to K$ has a fixed point.*

Theorem 3.2. *Assume that conditions (K1)-(K8) hold. Then the IBVP* (1) *has at least one positive solution $z\in PC^1(J,\mathbb{R}^+)\cap C^2(J',\mathbb{R})$ such that*

$$\underline{z}(t)\le\overline{z}(t), \quad -\sigma\le z'(t)\le\sigma, \quad t\in J.$$

Proof. Solving the IBVP (2) is equivalent to finding $z\in PC^1(J,\mathbb{R}^+)\cap C^2(J',\mathbb{R})$ which satisfies

$$\begin{aligned}z(t) &= \int_0^1 G(t,s)f^*(s,z(s),\frac{d}{ds}m_{\underline{z},\overline{z}}(s,z))ds+\sum_{k=1}^{n}W_k^*(t,z) \\ &\quad +tA(f^*(s,z(s),\frac{d}{ds}m_{\underline{z},\overline{z}}(s,z)))+(b+a(1-t))B(f^*(s,z(s),\frac{d}{ds}m_{\underline{z},\overline{z}}(s,z))),\end{aligned}$$

where $I_k^*(z(t_k))=I_k(m_{\underline{z},\overline{z}}(t_k,z(t_k)))$, $\overline{I}_k^*(z(t_k),z'(t_k))=\overline{I}_k(m_{\underline{z},\overline{z}}(t_k,z(t_k)),n(t_k,z(t_k)))$ and

$$W_k^*(t,z)=\frac{1}{\rho}\begin{cases}t[-aI_k^*(z(t_k))+(b+a(1-t_k))\overline{I}_k^*(z(t_k),z'(t_k))], & t<t_k,\\ (b+a(1-t))[I_k^*(z(t_k))+t_k\overline{I}_k^*(z(t_k),z'(t_k))], & t_k<t.\end{cases}$$

Now, define the following operator $T : PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \to PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ by

$$\begin{aligned}(Tz)(t) &= \int_0^1 G(t,s) f^*(s, z(s), \frac{d}{ds} m_{\underline{z},\bar{z}}(s,z))ds + \sum_{k=1}^{n} W_k^*(t,z) \\ &\quad + tA(f^*(s, z(s), \frac{d}{ds} m_{\underline{z},\bar{z}}(s,z))) + (b + a(1-t))B(f^*(s, z(s), \frac{d}{ds} m_{\underline{z},\bar{z}}(s,z))).\end{aligned}$$

Let's show the completely continuous of operator $T : PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \to PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ in 2 steps. Define the following for convenience:

$$H(\varsigma_j) = \int_0^1 G(\varsigma_j, s)ds + 2n,$$

$$\overline{A} = \frac{1}{\triangle} \begin{vmatrix} \sum_{j=1}^{m-2} \gamma_j H(\varsigma_j) & \rho - \sum_{j=1}^{m-2} \gamma_j [b + a(1-\varsigma_j)] \\ \sum_{j=1}^{m-2} \delta_j H(\varsigma_j) & -\sum_{j=1}^{m-2} \delta_j [b + a(1-\varsigma_j)] \end{vmatrix}, \quad \overline{B} = \frac{1}{\triangle} \begin{vmatrix} -\sum_{j=1}^{m-2} \gamma_j \varsigma_j & \sum_{j=1}^{m-2} \gamma_j H(\varsigma_j) \\ \rho - \sum_{j=1}^{m-2} \delta_j \varsigma_j & \sum_{j=1}^{m-2} \delta_j H(\varsigma_j) \end{vmatrix}.$$

Step 1: In this step show that the operator $T : PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \to PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ is continuous. Assume that z_n be a sequence in $PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ such that $z_n \to z_0 \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ as $n \to \infty$. Thus, we can write $\|z_n - z_0\|_{PC^1} \to 0 \ (n \to \infty)$. From Lemma 2.3, we obtain that $\|\frac{d}{dt} m_{\underline{z},\bar{z}}(t, z_n) - \frac{d}{dt} m_{\underline{z},\bar{z}}(t, z_0)\|_{PC^1} \to 0 \ (n \to \infty)$. Since $\|z_n - z_0\|_{PC^1} \to 0 \ (n \to \infty)$ and $\|\frac{d}{dt} m_{\underline{z},\bar{z}}(t, z_n) - \frac{d}{dt} m_{\underline{z},\bar{z}}(t, z_0)\|_{PC^1} \to 0 \ (n \to \infty)$, from the definition of the norm in $PC^1(J, \mathbb{R}^+)$, we can write $\|z_n - z_0\|_{PC} \to 0 \ (n \to \infty)$ and $\|\frac{d}{dt} m_{\underline{z},\bar{z}}(t, z_n) - \frac{d}{dt} m_{\underline{z},\bar{z}}(t, z_0)\|_{PC} \to 0 \ (n \to \infty)$.
Also, from the continuity of the $f, I_k, \bar{I}_k$ functions, we conclude that

$$|f^*(s, z_n(s), \frac{d}{ds} m_{\underline{z},\bar{z}}(s, z_n)) - f^*(s, z_0(s), \frac{d}{ds} m_{\underline{z},\bar{z}}(s, z_0))| \to 0 \ (n \to \infty),$$

$$|I_k^*(z_n(t_k)) - I_k^*(z_0(t_k))| \to 0 \ (n \to \infty),$$

$$|\bar{I}_k^*(z_n(t_k), \frac{d}{dt_k} m_{\underline{z},\bar{z}}(t_k, z_n(t_k))) - \bar{I}_k^*(z_0(t_k), \frac{d}{dt_k} m_{\underline{z},\bar{z}}(t_k, z_0(t_k)))| \to 0 \ (n \to \infty).$$

Therefore, considering Lemma 2.5 and with the help of *Lebesgue Dominated Convergence Theorem*, we can obtain

$$\begin{aligned}\|Tz_n - Tz_0\|_{PC} \leq & \int_0^1 G(s,s)|f^*(s,z_n(s),\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z_n)) - f^*(s,z_0(s),\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z_0))|ds \\ & +2n\max\Big\{|I_k^*(z_n(t_k)) - I_k^*(z_0(t_k))|, \\ & |\bar{I}_k^*(z_n(t_k),\frac{d}{dt_k}m_{\underline{z},\bar{z}}(t_k,z_n(t_k))) - \bar{I}_k^*(z_0(t_k),\frac{d}{dt_k}m_{\underline{z},\bar{z}}(t_k,z_0(t_k)))|\Big\} \\ & +(\overline{A}+\rho\overline{B})\max\Big\{|I_k^*(z_n(t_k)) - I_k^*(z_0(t_k))|, \\ & |\bar{I}_k^*(z_n(t_k),\frac{d}{dt_k}m_{\underline{z},\bar{z}}(t_k,z_n(t_k))) - \bar{I}_k^*(z_0(t_k),\frac{d}{dt_k}m_{\underline{z},\bar{z}}(t_k,z_0(t_k)))|, \\ & |f^*(s,z_n(s),\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z_n)) - f^*(s,z_0(s),\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z_0))|\Big\} \to 0 \ \ (n\to\infty),\end{aligned}$$

$$\begin{aligned}\|T'z_n - T'z_0\|_{PC} \leq & \int_0^1 G_t(t,s)|_{t=s}|f^*(s,z_n(s),\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z_n)) - f^*(s,z_0(s),\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z_0))|ds \\ & +2n\max\Big\{|I_k^*(z_n(t_k)) - I_k^*(z_0(t_k))|, \\ & |\bar{I}_k^*(z_n(t_k),\frac{d}{dt_k}m_{\underline{z},\bar{z}}(t_k,z_n(t_k))) - \bar{I}_k^*(z_0(t_k),\frac{d}{dt_k}m_{\underline{z},\bar{z}}(t_k,z_0(t_k)))|\Big\} \\ & +(\overline{A}+\rho\overline{B})\max\Big\{|I_k^*(z_n(t_k)) - I_k^*(z_0(t_k))|, \\ & |\bar{I}_k^*(z_n(t_k),\frac{d}{dt_k}m_{\underline{z},\bar{z}}(t_k,z_n(t_k))) - \bar{I}_k^*(z_0(t_k),\frac{d}{dt_k}m_{\underline{z},\bar{z}}(t_k,z_0(t_k)))|, \\ & |f^*(s,z_n(s),\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z_n)) - f^*(s,z_0(s),\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z_0))|\Big\} \to 0 \ \ (n\to\infty).\end{aligned}$$

This yields that, $\|Tz_n - Tz_0\|_{PC^1} \to 0$ as $n\to\infty$. Hence, T is continuous.
Step 2: In order to prove the relatively compactness of operator $T : PC^1(J,\mathbb{R}^+)\cap C^2(J',\mathbb{R}) \to PC^1(J,\mathbb{R}^+)\cap C^2(J',\mathbb{R})$. Let B be any bounded subset of $PC^1(J,\mathbb{R}^+)\cap C^2(J',\mathbb{R})$. Then there exists $M_0 > 0$ such that $\|z\|_{PC^1} \leq M_0$ for all $z \in B$. Also, from $f \in C(J\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+,\mathbb{R}^+)$, $\bar{I}_k \in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, we can write

$$M_1 = \max_{\substack{s\in[0,1] \\ z,\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z)\in[0,M_0]}} f^*(s,z(s),\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z)) < \infty$$

and

$$M_2 = \max_{1\leq k\leq n}\left\{\max_{z\in[0,M_0]} I_k^*(z), \max_{\substack{z\in[0,M_0] \\ \frac{d}{ds}m_{\underline{z},\bar{z}}(s,z)\in[0,M_0]}} \bar{I}_k^*(z,\frac{d}{ds}m_{\underline{z},\bar{z}}(s,z))\right\} < \infty.$$

Then using conditions (K2), (K3) and Lemma 2.5, for $z \in B$, we have

$$\|Tz\|_{PC} \leq \left[\int_0^1 G(s,s)ds + 2n + \overline{A} + \rho\overline{B}\right]\max\{M_1, M_2\} < \infty$$

and

$$\|T'z\|_{PC} \leq \left[\int_0^1 G_t(t,s)|_{t=s}ds + 2n + \overline{A} + \rho\overline{B}\right]\max\{M_1, M_2\} < \infty.$$

This yields that $\|Tz\|_{PC^1} < \infty$. Hence, we obtain $T(B)$ is uniformly bounded.

Next, we show that T is equicontinuous on J. For all $t \in J$, we can get

$$\begin{aligned}(Tz)'(t) &\leq \left[\int_0^1 \frac{1}{\rho}(b + a(1-s))ds + n + \overline{A}\right]\max\{M_1, M_2\} \\ &:= M_3.\end{aligned}$$

For all $t_1, t_2 \in J,\ t_1 < t_2$, if we integrate the last inequality from t_1 to t_2, then we have $(Tz)(t_2) - (Tz)(t_1) \leq M_3(t_2 - t_1)$. This means that

$$(Tz)(t_2) - (Tz)(t_1) \to 0, \ \ t_1 \to t_2. \tag{14}$$

On the other hand, we know that

$$\begin{aligned}(Tz)'(t) &= \frac{1}{\rho}\int_0^t (-as)f^*(s, z(s), \frac{d}{ds}m_{\underline{z},\overline{z}}(s,z))ds \\ &+\frac{1}{\rho}\int_t^1 (b + a(1-s))f^*(s, z(s), \frac{d}{ds}m_{\underline{z},\overline{z}}(s,z))ds \\ &+\frac{1}{\rho}\sum_{0<t_k<t}(-a)[I_k^*(z(t_k)) + t_k\overline{I}_k^*(z(t_k), z'(t_k))] \\ &+\frac{1}{\rho}\sum_{t<t_k<1}[-aI_k^*(z(t_k)) + (b + a(1-t_k))\overline{I}_k^*(z(t_k), z'(t_k))] \\ &+A(f^*(s, z(s), \frac{d}{ds}m_{\underline{z},\overline{z}}(s,z))) + (-a)B(f^*(s, z(s), \frac{d}{ds}m_{\underline{z},\overline{z}}(s,z))).\end{aligned}$$

From the above equation, we can obtain that

$$(Tz)'(t_2) - (Tz)'(t_1) \to 0, \ \ t_1 \to t_2. \tag{15}$$

Accordingly, from the equations (14) and (15), $T(B)$ is equicontinuous on J. Consequently, from steps 1-2 by *Arzela-Ascoli Theorem*, $T : PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \to PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ is completely continuous operator. With the help of the Schauder's Fixed Point Theorem, we get that T has a fixed point $z \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$. Thus, it is obtained that z is a solution of the IBVP (2). From Lemma 2.7 and Lemma 2.8, we know that $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$, $-\sigma \leq z'(t) \leq \sigma$, thereby IBVP (2) turns into the IBVP (1). Consequently, $z(t)$ is a solution of the IBVP (1). □

4. An Example

In this section, we give an example to demonstrate how our main result can be used in practice.

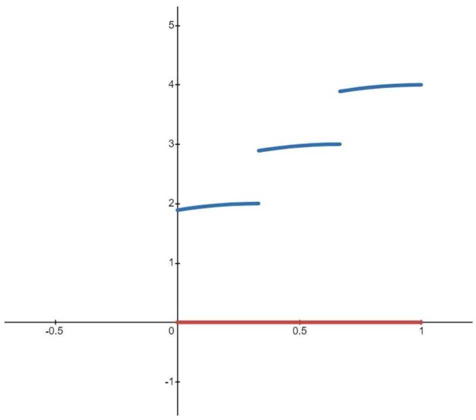
Example 4.1. *Consider the following the IBVP:*

$$\begin{cases} z''(t) + f(t, z(t), z'(t)) = 0, \ t \in J', \\ \triangle z|_{t=t_k} = z(t_k), \\ \triangle z'|_{t=t_k} = 4z(t_k) + z'(t_k), \\ z(0) = \dfrac{1}{5} z\left(\dfrac{1}{3}\right) + \dfrac{1}{6} z\left(\dfrac{2}{3}\right), \ 4z(1) + z'(1) = z(\dfrac{1}{3}) + \dfrac{1}{2} z(\dfrac{2}{3}), \end{cases} \tag{16}$$

where $f(t, z(t), z'(t)) = z'(t)$. *It is clear that*

$$\underline{z}(t) = 0, \quad \overline{z}(t) = \begin{cases} -(t - \dfrac{1}{3})^2 + 2, & t \in \left[0, \dfrac{1}{3}\right], \\ -(t - \dfrac{2}{3})^2 + 3, & t \in \left(\dfrac{1}{3}, \dfrac{2}{3}\right], \\ -(t-1)^2 + 4, & t \in \left(\dfrac{2}{3}, 1\right], \end{cases}$$

are lower and upper solutions of the IBVP (16), *respectively.*



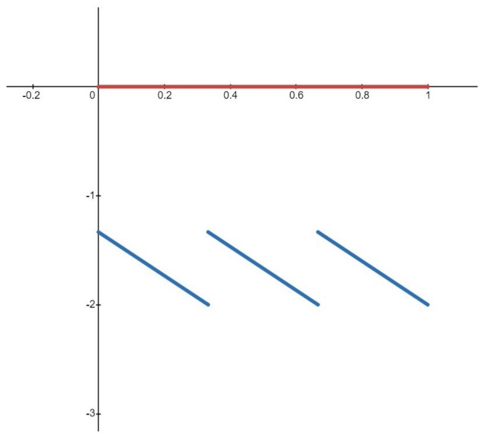


Figure 1: —: $\underline{z}(t)$ —: $\overline{z}(t)$ Figure 2: —: Lower —: Upper

The figures of $\underline{z}(t)$ *and* $\overline{z}(t)$ *are given in Figure 1. It is clearly seen from the Figure 1 that* $\underline{z}(t) \leq \overline{z}(t)$. *Substituting* $\underline{z}(t)$ *and* $\overline{z}(t)$ *in the equation* $z''(t) + f(t, z(t), z'(t)) = 0, \ t \in J'$, *it can be seen from Figure 2 that* $t \in J'$, $\overline{z}''(t) + f(t, \overline{z}(t), \overline{z}'(t)) \leq 0$ *inequality and* $\underline{z}''(t) + f(t, \underline{z}(t), \underline{z}'(t)) \geq 0$. *inequality are obtained. It can easily be obtained that other inequalities are satisfied. In accordance with Definition 2.1, it is obtained that* $\overline{z}(t)$ *and* $\underline{z}(t)$ *are upper and lower solutions of the IBVP* (16), *recpectively.*
Let $\varphi(s) = \sqrt{s+1}$; *then* $|f(t, z, z')| = |z'| \leq D|z'|(\sqrt{|z'|+1}) + E$, $t \in J$, $\underline{z} \leq z \leq \overline{z}$, $z' \in \mathbb{R}^+$ $D, E \geq 0$. *Noting that for any* $a > 0$,
$\displaystyle\int_a^\infty \frac{ds}{\varphi(s)} = \infty$ *and* $\displaystyle\max_{s \geq a} \frac{1}{\varphi(s)} \leq 1$, *there exists* $\sigma > 0$ *such that* f *satisfies the Nagumo condition with respect to* $\underline{z}$ *and* $\overline{z}$. *Hence, the IBVP* (16) *has at least one positive solution* $z(t) \in [\underline{z}, \overline{z}]$.

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