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Some Existence Results on Implicit Fractional Differential Equations

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Abstract. In this paper, we study the existence of a solution for the nonlinear implicit fractional differential equation of the type

$$D^{\alpha}u(t) = f\left(t, u(t), D^{\alpha}u(t)\right),$$

with Riemann-Liouville fractional derivative via the different boundary conditions $u(0) = u(T)$, and the three point boundary conditions $u(0) = \beta_1 u(\eta)$ and $u(T) = \beta_2 u(\eta)$, where $T > 0, t \in I = [0, T], 0 < \alpha < 1, 0 < \eta < T$, $0 < \beta_1 < \beta_2 < 1$.

1. Introduction

The theory of fractional calculus has been available and applicable to various fields of study. The investigation of the theory of fractional differential and integral equations has started quite recently, see [1, 10, 15, 22] and the references therein. Due to their significant applications in various science and technology disciplines, differential equations of fractional order have received considerable attention from researchers over the past few decades. Fractional derivatives introduce excellent tools for describing the general properties of the various materials and processes. The benefits of fractional derivatives are evident in the modeling of mechanical and electrical properties of real materials, as well as in the description of the properties of gases , liquids, rocks and in many other fields, see [12, 13, 15, 23] and the references therein.

Fractional order differential equations are used to model real world problems more effectively than ordinary differential equations. The field deals with the study of the existence and uniqueness of solutions to initial / boundary value problems for fractional differential and integrodifferential equations has been studied very well and a lot of papers are available in the literature on it, see for example [2, 5–7, 9, 17, 20, 21, 24, 25] and the references therein.

Integral boundary conditions are encountered in population dynamics, blood flow models, chemical engineering, cellular systems, heat transmission, plasma physics, thermoelasticity, etc. They come up when values of the function on the boundary are connected to its values inside the domain, they have physical

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significations such as total mass, moments, etc. Sometimes it is better to impose integral conditions because they lead to more precise measures than those proposed by a local condition. An equation of type

$$f(t, x(t), x'(t), x''(t), ..., x^{n-1}(t), x^n(t)) = 0,$$

is called the n^{th} order implicit ordinary differential equation. Such equations have different applications in economics and managerial sciences. Motivated by the significance of implicit differential equations, many authors extend this form in fractional order form and investigated the existence results with various initial and boundary conditions, see [3, 4, 8, 11, 14, 18, 19] and the references therein.

The aim of the present paper is to prove the existence of a solution for the nonlinear implicit fractional differential equation with Riemann-Liouville fractional derivative via the different boundary conditions. The main tools employed in our analysis are based on the theory of fractional calculus and fixed point theorems.

2. Preliminaries

Before proceeding to the statement of our main results, we setforth definitions, preliminaries and hypotheses that will be used in our subsequent discussion.

Definition 2.1. *([10],[15]): The Caputo fractional derivative of order α, for a continuous function f is defined by*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n-1}} ds, \ \ (n-1 < \alpha < n, n = [\alpha] + 1) \tag{1}$$

where $[\alpha]$ denotes the integer part of α, provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. *([10],[15]): The Riemann-Liouville fractional derivative of order α $(n-1 < \alpha < n)$ is defined as*

$$D^{\alpha}_{0+} u(t) = \left(\frac{d}{dt}\right)^n \left(I^{n-\alpha}_{0+} u(t)\right) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds, \ t > 0. \tag{2}$$

We need the following definitions of $\alpha - \psi$-contractive mapping and α-admissible mapping which have been recently introduced in **[16]**.

Definition 2.3. *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi$-contractive mapping whenever there exist two functions $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that $\alpha(x, y)d(Tx, Ty) \le \psi(d(x, y))$ for all $x, y \in X$. Also, we say that T is $\alpha-$admissible whenever $\alpha(x, y) \ge 1$ implies that $\alpha(Tx, Ty) \ge 1$.*

Here, Ψ is a family of non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$. We need the following result which has been proved in [16].

Theorem 2.4. *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi$-contractive mapping and $\alpha-$ admissible self-map on X such that $\alpha(x_0, Tx_0) \ge 1$ for some $x_0 \in X$. If x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x$ for some $x \in X$, then $\alpha(x_n, x) \ge 1 \forall n$. Then T has a fixed point.*

3. Existence Results

In this section, we prove the main results of the article. We divide this section into three important parts.

3.1. Case I :

In this case, we discuss the nonlinear implicit fractional differential equation

$$ {}^{c}D^{\alpha}x(t) = f(t, x(t), {}^{c}D^{\alpha}x(t)), (0 < t < 1, 1 < \alpha \leq 2) \tag{3}$$

via the integral boundary condition

$$x(0) = 0, x(1) = \int_0^{\eta} x(s)ds, (0 < \eta < 1),$$

where ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α and $f : [0,1] \times X \times X \to X$ is a continuous function. Note that $(X, \|\cdot\|)$ is a Banach space and $C = C([0,1], X)$ denotes the Banach space of continuous functions from $[0,1]$ to X endowed with uniform topology.

Theorem 3.1. *Suppose that*

1. *there exists $p \in C(I)$ and $M \in (0,1)$ such that $P < \frac{1}{2}(1-M)$ where $P = max\{p(t)|t \in [0,1]\}$ and here exists a function $\xi : R^2 \to R$ and $\psi \in \Psi$ such that*
$$\begin{aligned}|f(t, a_1, a_2) - f(t, b_1, b_2)| &\leq p(t)|a_1 - b_1| + M|a_2 - b_2| \\ &\leq \frac{\Gamma(\alpha+1)}{10}\psi(|a_1 - b_1|) + \frac{\Gamma(\alpha+1)}{5}\psi(|a_2 - b_2|),\end{aligned}$$
for all $t \in [0,1]$, $a_1, a_2, b_1, b_2 \in R$ and $\xi(a_1, b_1) \geq 0$, $\xi(a_2, b_2) \geq 0$;
2. *there exists $x_0 \in C(I)$ such that $\xi(x_0(t), Fx_0(t)) \geq 0$ for all $t \in I$, where the operator $F : C(I) \to C(I)$ is defined by*
$$\begin{aligned}Fx(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^{c}D^{\alpha}x(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, x(s), {}^{c}D^{\alpha}x(s))ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^{\eta}\left[\int_0^s (s-m)^{\alpha-1} f(m, x(m), {}^{c}D^{\alpha}x(m))dm\right]ds;\end{aligned} \tag{4}$$
3. *For each $t \in I$ and $x, y \in C(I)$, $\xi(x(t), y(t)) \geq 0 \Rightarrow \xi(Fx(t), Fy(t)) \geq 0$; and*
4. *if $\{x_n\}$ is a sequence in $C(I)$ such that $x_n \to x$ in $C(I)$ and $\xi(x_n, x_{n+1}) \geq 0$ for all n, then $\xi(x_n, x) \geq 0$ for all n.*

Then the problem (3) has at least one solution on $[0,1]$.

Proof. We know that $x \in C(I)$ is a solution of the problem (3) iff $x \in C(I)$ is a solution of the integral equation

$$\begin{aligned}x(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^{c}D^{\alpha}x(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, x(s), {}^{c}D^{\alpha}x(s))ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^{\eta}\left[\int_0^s (s-m)^{\alpha-1} f(m, x(m), {}^{c}D^{\alpha}x(m))dm\right]ds, \ (t \in [0,1])\end{aligned}$$

Define an operator $F : C(I) \to C(I)$ defined by

$$\begin{aligned}Fx(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^{c}D^{\alpha}x(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, x(s), {}^{c}D^{\alpha}x(s))ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^{\eta}\left[\int_0^s (s-m)^{\alpha-1} f(m, x(m), {}^{c}D^{\alpha}x(m))dm\right]ds, \ (t \in [0,1]);\end{aligned}$$

Then the problem (3) is eqivalent to finding $x^{\star} \in C(I)$, which is a fixed point of F. Let $x, y \in C(I)$ be such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. By using (1), we have

$$\begin{aligned}
|Fx(t)-Fy(t)| &= \bigg|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),{}^cD^\alpha x(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,x(s),{}^cD^\alpha x(s))ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta \Big[\int_0^s (s-m)^{\alpha-1} f(m,x(m),{}^cD^\alpha x(m))dm\Big]ds \\
&- \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s),{}^cD^\alpha y(s))ds + \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,y(s),{}^cD^\alpha y(s))ds \\
&- \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta \Big[\int_0^s (s-m)^{\alpha-1} f(m,y(m),{}^cD^\alpha y(m))dm\Big]ds\bigg| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}|f(s,x(s),{}^cD^\alpha x(s)) - f(s,y(s),{}^cD^\alpha y(s))|ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 |1-s|^{\alpha-1}|f(s,x(s),{}^cD^\alpha x(s)) - f(s,y(s),{}^cD^\alpha y(s))|ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta \Big[\int_0^s (s-m)^{\alpha-1}\Big|f(m,x(m),{}^cD^\alpha x(m)) - f(m,y(m),{}^cD^\alpha y(m))dm\Big|ds\Big]
\end{aligned}$$

Consider

$$\begin{aligned}
|{}^cD^\alpha x(t) - {}^cD^\alpha y(t)| &= |{}^cD^\alpha (x(t)-y(t))| = |f(s,x(s),{}^cD^\alpha x(s)) - f(s,y(s),{}^cD^\alpha y(s))| \\
&\le \{p(t)|x(t)-y(t)| + M|{}^cD^\alpha x(t) - {}^cD^\alpha y(t)|\} \\
&\le \frac{p(t)}{1-M}|x(t)-y(t)|
\end{aligned}$$

Therefore,

$$|{}^cD^\alpha x(t) - {}^cD^\alpha y(t)| \le \frac{p(t)}{1-M}|x(t)-y(t)|$$

Since $\psi \in \Psi$ is non-decreasing, we have

$$\begin{aligned}
&|{}^cD^\alpha x(t) - {}^cD^\alpha y(t)| \le \frac{p(t)}{1-M}|x(t)-y(t)| \\
&\Rightarrow \psi\,(|{}^cD^\alpha x(t) - {}^cD^\alpha y(t)|) \le \psi\left(\frac{p(t)}{1-M}|x(t)-y(t)|\right) \\
&\Rightarrow \psi\Big(|{}^cD^\alpha x(t) - {}^cD^\alpha y(t)|\Big) \le \frac{P}{1-M}\psi(|x(t)-y(t)|)
\end{aligned}$$

Therefore,

$$|f(t,x(t),{}^cD^\alpha x(t)) - f(t,y(t),{}^cD^\alpha y(t))| \le \frac{\Gamma(\alpha+1)}{10}\psi(|x(t)-y(t)|) + \frac{\Gamma(\alpha+1)}{5}\frac{P}{1-M}\psi(|x(t)-y(t)|)$$

Hence, using above inequality, we get

$$\begin{aligned}
|Fx(t)-Fy(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t |t-s|^{\alpha-1}\Big\{\frac{\Gamma(\alpha+1)}{10}\psi(|x(s)-y(s)|) + \frac{\Gamma(\alpha+1)}{5}\frac{P}{1-M}\psi(|x(s)-y(s)|)\Big\}ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 |1-s|^{\alpha-1}\Big\{\frac{\Gamma(\alpha+1)}{10}\psi(|x(s)-y(s)|) + \frac{\Gamma(\alpha+1)}{5}\frac{P}{1-M}\psi(|x(s)-y(s)|)\Big\}ds \\
&+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta \int_0^s \Big[|s-m|^{\alpha-1}\Big\{\frac{\Gamma(\alpha+1)}{10}\psi(|x(m)-y(m)|)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\Gamma(\alpha+1)}{5}\frac{P}{1-M}\psi(|x(m)-y(m)|)\Big\}dm\Big]ds \\
&\leq \frac{\Gamma(\alpha+1)}{10\Gamma(\alpha)}\psi(\|x-y\|_\infty)\sup_{t\in I}\Big\{\int_0^t |t-s|^{\alpha+1}ds + \frac{2t}{(2-\eta^2)}\int_0^1 |t-s|^{\alpha-1}ds \\
&+ \frac{2t}{(2-\eta^2)}\int_0^\eta\int_0^s |s-m|^{\alpha-1}dm\ \ ds\Big\} \\
&+ \frac{P}{(1-M)\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{5}\psi(\|x-y\|_\infty) \\
&\times \sup_{t\in I}\Big\{\int_0^t |t-s|^{\alpha-1}ds + \frac{2t}{(2-\eta^2)}\int_0^1 |t-s|^{\alpha-1}ds + \frac{2t}{(2-\eta^2)}\int_0^\eta\int_0^s |s-m|^{\alpha-1}dm\ \ ds\Big\} \\
&\leq \Big[\frac{1}{2} + \frac{P}{1-M}\Big]\psi(\|x-y\|_\infty)
\end{aligned}
$$

Hence, for each $x, y \in C(I)$ with $\xi(x(t), y(t)) \geq 0$ for all $t \in I$, we have

$$\|Fx - Fy\|_\infty \leq \Big[\frac{1}{2} + \frac{P}{1-M}\Big]\psi(\|x-y\|_\infty).$$

Next, define the function
$\alpha : C(I) \times C(I) \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} \left\{\frac{1}{2} + \frac{P}{1-M}\right\}^{-1}, & \text{if } \xi(x(t), y(t)) \geq 0 \text{ for all } t \in I, \\ 0, & \text{otherwise.} \end{cases}$$

As $P < \frac{1}{2}(1-M)$, so we must have $\alpha(x,y) \geq 1$. Therefore, $\alpha(x,y)d(Fx,Fy) \leq \psi(d(x,y))$ for all $x, y \in C(I)$. This implies that F is $\alpha - \psi$-contractive mapping. By using the condition (3), we get

$$\alpha(x,y) \geq 1 \Rightarrow \psi(x(t), y(t)) \geq 0 \Rightarrow \psi(Fx(t), Fy(t)) \geq 0 \Rightarrow \alpha(Fx, Fy) \geq 1$$

for all $x, y \in C(I)$. Therefore, using (4) and Theorem 2.4, we get the existence of $x^\star \in C(I)$ such that $F(x^\star) = x^\star$. Hence, $x^\star$ is a solution to (3). □

3.2. Case-II:

Now, we discuss nonlinear implicit fractional differential equation

$$D^\alpha x(t) + f(t, x(t), D^\alpha x(t)) = 0, (0 \leq t \leq 1, \alpha > 1) \tag{5}$$

via the two-point boundary conditions $x(0) = x(1) = 0$, where $f : I \times R^2 \to R$ is a continuous function and $I = [0, 1]$. Note that the Green's function associated with the problem (5) is given by

$$G(t,s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1 \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq t \leq s \leq 1 \end{cases}$$

Theorem 3.2. *Suppose that*

1. *there exists a function $\xi : R^2 \to R$, $p \in C(I)$, $M \in (0,1)$ with $P < \frac{1}{2}(1-M)$ and $\psi \in \Psi$ such that*

$$|f(t, a_1, a_2) - f(t, b_1, b_2)| \leq p(t)|a_1 - b_1| + M|a_2 - b_2| \leq \frac{1}{2}\psi(|a_1 - b_1|) + \psi(|a_2 - b_2|)$$

for all $t \in I$ and $a, b \in R$ with $\xi(a_1, b_1) \geq 0$, $\xi(a_1, b_1) \geq 0$.

2. *there exists $x_0 \in C(I)$ such that $\xi\Big(x_0, \int_0^1 G(t,s)f(s, x_0(s), D^\alpha x_0(s))ds\Big) \geq 0$ for all $t \in I$.*

3. *for each* $t \in I$ *and* $x, y \in C(I)$, $\xi(x(t), y(t)) \geq 0$ *implies*

$$\psi\Big(\int_0^1 G(t,s)f(s,x(s),D^\alpha x(s))ds, \int_0^1 G(t,s)f(s,y(s),D^\alpha y(s))ds\Big) \geq 0;$$

and

4. *if* $\{x_n\}$ *is a sequence in* $C(I)$ *such that* $x_n \to x$ *in* $C(I)$ *and* $\xi(x_n, x_{n+1}) \geq 0$ *for all* n*, then* $\xi(x_n, x) \geq 0$ *for all* n*.*

Then the problem (5) has at least one solution.

Proof. We know that $x \in C(I)$ is a solution of (5) iff it is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s)f(s,x(s),D^\alpha x(s))ds, \text{for all } t \in I.$$

Define an operator $F : C(I) \to C(I)$ by

$$F(x(t)) = \int_0^1 G(t,s)f(s,x(s),D^\alpha x(s))ds.$$

So finding a solution for (5), it is enough to show that F has a fixed point. Now, let $x, y \in C(I)$ be such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. By using (1), we get

$$\begin{aligned}
|Fx(t) - Fy(t)| &= \Big|\int_0^1 G(t,s)f(s,x(s),D^\alpha x(s))ds - \int_0^1 G(t,s)f(s,y(s),D^\alpha y(s))ds\Big| \\
&\leq \int_0^1 |G(t,s)| \ \Big|f(s,x(s),D^\alpha x(s)) - f(s,y(s),D^\alpha y(s))\Big| ds \\
&\leq \int_0^1 |G(t,s)|\Big\{p(t)|x(s) - y(s)| + M|D^\alpha x(s) - D^\alpha y(s)|\Big\}ds \\
&\leq \int_0^1 |G(t,s)|\Big\{\frac{1}{2}\psi(|x(s) - y(s)|) + \frac{P}{1-M}\psi(|x(s) - y(s)|)\Big\}ds \\
&\leq \psi(\|x(s) - y(s)\|_\infty)\sup_{t\in I}\int_0^1 |G(t,s)|\Big[\frac{1}{2} + \frac{P}{1-M}\Big]ds \\
&\leq \psi(\|x(s) - y(s)\|_\infty)\Big[\frac{1}{2} + \frac{P}{1-M}\Big]
\end{aligned}$$

This implies that $x, y \in C(I)$ with $\xi(x(t), y(t)) \geq 0$ for all $t \in I$, we have

$$\|Fx - Fy\|_\infty \leq \Big[\frac{1}{2} + \frac{P}{1-M}\Big]\psi(\|x - y\|_\infty).$$

Next, define the function
$\alpha : C(I) \times C(I) \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} \Big\{\frac{1}{2} + \frac{P}{1-M}\Big\}^{-1}, & \text{if } \xi(x(t), y(t)) \geq 0 \text{ for all } t \in I, \\ 0, & \text{otherwise.} \end{cases}$$

As $P < \frac{1}{2}(1 - M)$, so we must have $\alpha(x, y) \geq 1$. Therefore, $\alpha(x, y)d(Fx, Fy) \leq \psi(d(x, y))$ for all $x, y \in C(I)$. This implies that F is $\alpha - \psi$-contractive mapping. By using the condition (3), we get

$$\alpha(x,y) \geq 1 \Rightarrow \psi(x(t), y(t)) \geq 0 \Rightarrow \psi(Fx(t), Fy(t)) \geq 0 \Rightarrow \alpha(Fx, Fy) \geq 1$$

for all $x, y \in C(I)$. Therefore, using (4) and Theorem 2.4, we get the existence of $x^\star \in C(I)$ such that $F(x^\star) = x^\star$. Hence, $x^\star$ is a solution to (5). □

3.3. Case-III:

Now, we study the nonlinear implicit fractional differential equation

$$D^{\alpha}x(t) + D^{\beta}x(t) = f(t, x(t), D^{\alpha}x(t))), \quad (0 \le t \le 1, 0 < \beta < \alpha < 1) \tag{6}$$

via the two point boundary conditions $x(0) = x(1) = 0$, where $f : I \times R^2 \to R$ is a continuous function and $I = [0, 1]$. Note that the Green's function associated with the problem (6) is given by $G(t) = t^{\alpha-1}E_{\alpha-\beta}(-t^{\alpha-\beta})$.

Theorem 3.3. *Suppose that*

1. *there exists a function* $\xi : R^2 \to R$, $p \in C(I)$, $M \in (0, 1)$ *with* $P < \frac{1}{2}(1 - M)$ *and* $\psi \in \Psi$ *such that*

$$|f(t, a_1, a_2) - f(t, b_1, b_2)| \le p(t)|a_1 - b_1| + M|a_2 - b_2| \le \frac{\alpha}{2}\psi(|a_1 - b_1|) + \alpha\psi(|a_2 - b_2|),$$

for all $t \in I$ *and* $a, b \in R$ *with* $\xi(a_1, b_1) \ge 0$, $\xi(a_1, b_1) \ge 0$.

2. *there exists* $x_0 \in C(I)$ *such that* $\xi\Big(x_0, \int_0^1 G(t, s)f(s, x_0(s), D^{\alpha}x_0(s))ds\Big) \ge 0$ *for all* $t \in I$.
3. *for each* $t \in I$ *and* $x, y \in C(I)$, $\xi(x(t), y(t)) \ge 0$ *implies*

$$\psi\Big(\int_0^1 G(t, s)f(s, x(s), D^{\alpha}x(s))ds, \int_0^1 G(t, s)f(s, y(s), D^{\alpha}y(s))ds\Big) \ge 0;$$

and

4. *if* $\{x_n\}$ *is a sequence in* $C(I)$ *such that* $x_n \to x$ *in* $C(I)$ *and* $\xi(x_n, x_{n+1}) \ge 0$ *for all* n, *then* $\xi(x_n, x) \ge 0$ *for all* n.

Then the problem (6) has at least one solution.

Proof. We know that $x \in C(I)$ is a solution of (5) iff it is a solution of the integral equation

$$x(t) = \int_0^1 G(t - s)f(s, x(s), D^{\alpha}x(s))ds, \text{for all } t \in I.$$

Define an operator $F : C(I) \to C(I)$ by

$$F(x(t)) = \int_0^1 G(t - s)f(s, x(s), D^{\alpha}x(s))ds.$$

So finding a solution for (6), it is enough to show that F has a fixed point. Now, let $x, y \in C(I)$ be such that $\xi(x(t), y(t)) \ge 0$ for all $t \in I$. By using (1), we get

$$\begin{aligned}
|Fx(t) - Fy(t)| &= \Big| \int_0^1 G(t - s)f(s, x(s), D^{\alpha}x(s))ds - \int_0^1 G(t - s)f(s, y(s), D^{\alpha}y(s))ds \Big| \\
&\le \int_0^1 |G(t - s)| \; \Big|f(s, x(s), D^{\alpha}x(s)) - f(s, y(s), D^{\alpha}y(s))\Big| ds \\
&\le \int_0^1 |G(t - s)| \Big\{p(t)|x(s) - y(s)| + M|D^{\alpha}x(s) - D^{\alpha}y(s)|\Big\} ds \\
&\le \int_0^1 |G(t - s)| \Big\{\frac{\alpha}{2}\psi(|x(s) - y(s)|) + \frac{\alpha P}{1 - M}\psi(|x(s) - y(s)|)\Big\} ds \\
&\le \psi(\|x(s) - y(s)\|_{\infty}) \sup_{t \in I} \int_0^1 |G(t - s)| \Big[\frac{\alpha}{2} + \frac{P}{1 - M}\Big] ds
\end{aligned}$$

$$\leq \psi(\|x(s)-y(s)\|_\infty)\left[\frac{\alpha}{2}+\frac{\alpha P}{1-M}\right]$$
$$\leq \left[\frac{1}{2}+\frac{P}{1-M}\right]\psi(\|x(s)-y(s)\|_\infty)$$

Note that $G(t) = t^{\alpha-1}E_{\alpha-\beta}(-t^{\alpha-\beta}) \leq t^{\alpha-1} + |-t^{\alpha-\beta}| \leq t^{\alpha-1}$. Hence, $\sup_{t\in I}\int_0^t |G(t-s)|ds \leq \frac{1}{\alpha}$. This implies that $x, y \in C(I)$ with $\xi(x(t), y(t)) \geq 0$ for all $t \in I$, we have

$$\|Fx - Fy\|_\infty \leq \left[\frac{1}{2}+\frac{P}{1-M}\right]\psi(\|x-y\|_\infty).$$

Next, define the function
$\alpha : C(I)\times C(I) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} \left\{\frac{1}{2}+\frac{P}{1-M}\right\}^{-1} & \text{if } \xi(x(t),y(t)) \geq 0 \text{ for all } t\in I,\\ 0 & \text{otherwise.}\end{cases}$$

As $P < \frac{1}{2}(1-M)$, so we must have $\alpha(x,y) \geq 1$. Therefore, $\alpha(x,y)d(Fx,Fy) \leq \psi(d(x,y))$ for all $x,y \in C(I)$. This implies that F is $\alpha-\psi$-contractive mapping. By using the condition (3), we get

$$\alpha(x,y) \geq 1 \Rightarrow \psi(x(t),y(t)) \geq 0 \Rightarrow \psi(Fx(t),Fy(t)) \geq 0 \Rightarrow \alpha(Fx,Fy) \geq 1$$

for all $x,y \in C(I)$. Therefore, using (4) and Theorem 2.4, we get the existence of $x^\star \in C(I)$ such that $F(x^\star) = x^\star$. Hence, $x^\star$ is a solution to (6). □

4. Conclusion

In this article, we have proved three existence theorems for three nonlinear implicit fractional differential equations for various boundary conditions. Our results based on $\alpha-\psi$-contractive and α-admissible mappings. In the future, we will extend the results to other fractional derivatives.

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