Filomat 35:12 (2021), 4239–4255
https://doi.org/10.2298/FIL2112239B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Tensor Product of Kantorovich-Stancu Type Operators with Shifted Knots and their k^{th} Order Generalization

Behar Baxhaku[a], **Rahul Shukla**[b], **P. N. Agrawal**[b]

[a] *Department of Mathematics, University of Prishtina, Prishtina, Kosovo*
[b] *Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, India*

Abstract. In this paper, we introduce a tensor product of the Stancu-Kantorovich type operators defined by Içöz [11]. The rate of convergence of these operators is obtained in terms of the modulus of continuity and the Peetre's K-functional. Further, we consider a generalization of the above operators via Taylor's polynomials and examine their approximation behavior. Some applications of these two dimensional generalized Stancu-Kantorovich type polynomials are also discussed. Finally, we present some numerical examples and illustrations to show the convergence behavior of the operators under study using MATLAB algorithms.

1. Introduction

For $f \in C(I)$, the space of all continuous functions on $I = [0,1]$ with sup-norm, Stancu [15] proposed a sequence of polynomials

$$S_m^{(\alpha,\beta)}(f;x) = \sum_{j=0}^{m} f\Big(\frac{j+\alpha}{m+\beta}\Big) b_{m,j}(x), \tag{1}$$

where the Bernstein basis functions $b_{m,j}(x)$ are defined by

$$b_{m,j}(x) = \binom{m}{j} x^j (1-x)^{m-j}; \quad x \in I, \tag{2}$$

and showed that these polynomials converge to the function $f(x)$ uniformly in $x \in I$.
It is obvious that whenever $\alpha = \beta = 0$, the operators defined by equation (1) reduce to the classical Bernstein operators defined by Bernstein [6]. Gadjiev and Ghorbanalizadeh [10] constructed Bernstein-Stancu type polynomials with shifted knots involving some non-negative real numbers θ and $\theta_i, i = 1,2,3$, as

$$G_{m,\theta}^{(\theta_i)}(f;x) = \Big(\frac{m+\theta}{m}\Big)^m \sum_{j=0}^{m} \Omega_{m,j}^{(\theta,\theta_2)}(x) f\Big(\frac{j+\theta_3}{m+\theta_1}\Big), \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 41A10, 41A25, 41A36, 41A63.
Keywords. Kantorovich operators, Lipschitz-type space, K-functional, Modulus of continuity
Received: 03 August 2020; Accepted: 21 October 2020
Communicated by Miodrag Spalević
Email addresses: behar.baxhaku@uni-pr.edu (Behar Baxhaku), rshukla@ma.iitr.ac.in (Rahul Shukla), pnappfma@gmail.com (P. N. Agrawal)

where the basis functions $\Omega_{m,j}^{(\theta,\theta_2)}(x)$ are defined by

$$\Omega_{m,j}^{(\theta,\theta_2)}(x) = \binom{m}{j}\left(x - \frac{\theta_2}{m+\theta}\right)^j\left(\frac{m+\theta_2}{m+\theta} - x\right)^{m-j}, \tag{4}$$

$x \in [\frac{\theta_2}{m+\theta}, \frac{m+\theta_2}{m+\theta}]$ and $0 \leq \theta_3 \leq \theta_2 \leq \theta_1 \leq \theta$. It is obvious that whenever $\theta = \theta_i = 0\,; i = 1,2,3$, the operators defined by equation (1.3) include the classical Bernstein operators. Wang et al. [17] obtained some direct results and a converse result in approximation by the polynomials defined in (3). To make it possible to approximate the Lebesgue integrable functions on I, Kantorovich [12] proposed a modification of the Bernstein polynomials as

$$K_m(f;x) = (m+1)\sum_{j=0}^{m} b_{m,j}(x)\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} f(t)dt.$$

Inspired by the above idea, Içöz [11] introduced a Kantorovich variant of the Bernstein-Stancu type polynomials with shifted knots given by (3) as follows:

$$K_{m,\theta}^{(\theta_i)}(f;x) = \left(\frac{m+\theta}{m}\right)^m (m+\theta_1+1)\sum_{s=0}^{m}\Omega_{m,s}^{(\theta,\theta_2)}(x)\int_{\frac{s+\theta_3}{m+\theta_1+1}}^{\frac{s+\theta_3+1}{m+\theta_1+1}} f(t)dt, \tag{5}$$

where the basis functions $\Omega_{m,j}^{(\theta,\theta_2)}(x)$ are defined in (4) and $x \in [\frac{\theta_2}{m+\theta}, \frac{m+\theta_2}{m+\theta}]$. Evidently, in the particular case, $\theta = \theta_i = 0; i = 1,2,3$, the operators $K_{m,\theta}^{(\theta_i)}$ reduce to the operators K_m. The author [11] established some approximation results for the operators (5) in the continuous functions space with the aid of the usual modulus of continuity and the Peetre's K-functional and also investigated the approximation properties of a k^{th} order generalization of these operators. For other contributions, in the direction of the above study, we refer the reader to (cf. [1], [3] -[5], [14] etc.).

In this article, we introduce the following tensor product of Kantorovich-Stancu type polynomials on the rectangle $\square = [\frac{\theta_2}{n+\theta}, \frac{n+\theta_2}{n+\theta}] \times [\frac{\phi_2}{m+\phi}, \frac{m+\phi_2}{m+\phi}]$ as:

$$\mathfrak{R}_{n,m,\theta,\phi}^{\theta_i,\phi_i}(f(t_1,t_2);x,y) = \left(\frac{n+\theta}{n}\right)^n\left(\frac{m+\phi}{m}\right)^m (n+\theta_1+1)(m+\phi_1+1)$$
$$\sum_{s=0}^{n}\sum_{r=0}^{m}\Omega_{n,s}^{(\theta,\theta_2)}(x)\,\Omega_{m,r}^{(\phi,\phi_2)}(y)\int_{\frac{s+\theta_3}{n+\theta_1+1}}^{\frac{s+\theta_3+1}{n+\theta_1+1}}\int_{\frac{r+\phi_3}{m+\phi_1+1}}^{\frac{r+\phi_3+1}{m+\phi_1+1}} f(t_1,t_2)dt_1dt_2, \tag{6}$$

where the basis functions $\Omega_{n,s}^{(\theta,\theta_2)}(x)$ and $\Omega_{m,r}^{(\phi,\phi_2)}(y)$ are as defined in (4).

We investigate the uniform convergence of these operators in the space $C(I^2)$ where $I^2 = I \times I$ and then determine the degree of convergence by these operators using the modulus of continuity and the Peetre's K-functional. We also define a k^{th} order generalization of these operators to study the approximation of continuous functions having k^{th} order continuous partial derivatives on I^2 and present some applications of this study to bi-variate Bernstein type operators on a simplex. Finally, we validate the results of this paper by some graphs and error estimation tables using MATLAB.

2. Auxiliary results

In our future consideration, $\|\cdot\|_{C(I^2)}$ denotes the sup-norm on I^2.

Lemma 2.1. *Let $e_{ij}(t_1,t_2) = t_1^i t_2^j$ where $i,j \in \mathbb{N} \cup \{0\}$. For $x,y \in \square$, the Kantorovich type generalized Bernstein-Stancu operators $\mathfrak{R}_{n,m,\theta,\phi}^{\theta_i,\phi_i}(f;x,y)$, defined by* (6)*, possess the following properties:*

(i) $\mathfrak{R}_{n,m,\theta,\phi}^{\theta_i,\phi_i}(e_{00};x,y) = 1;$

(ii) $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{10};x,y) = \frac{n+\theta}{n+\theta_1+1}x + \frac{\theta_3-\theta_2}{n+\theta_1+1} + \frac{1}{2(n+\theta_1+1)}$;

(iii) $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{01};x,y) = \frac{m+\phi}{m+\phi_1+1}y + \frac{\phi_3-\phi_2}{m+\phi_1+1} + \frac{1}{2(m+\phi_1+1)}$;

(iv) $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{20};x,y) = \left(1-\frac{1}{n}\right)\left(\frac{n+\theta}{n+\theta_1+1}\right)^2\left(x-\frac{\theta_2}{n+\theta}\right)^2 + (2\theta_3+1)\frac{n+\theta}{(n+\theta_1+1)^2}\left(x-\frac{\theta_2}{n+\theta}\right)$
$+ \frac{n+\theta}{(n+\theta_1+1)^2}x + \frac{\theta_3^2-\theta_2+\theta_3}{(n+\theta_1+1)^2} + \frac{1}{3(n+\theta_1+1)^2}$;

(v) $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{02};x,y) = \left(1-\frac{1}{m}\right)\left(\frac{m+\phi}{m+\phi_1+1}\right)^2\left(y-\frac{\phi_2}{m+\phi}\right)^2 + \left(2\phi_3+1\right)\frac{m+\phi}{(m+\phi_1+1)^2}\left(y-\frac{\phi_2}{m+\phi}\right)$
$+ \frac{m+\phi}{(m+\phi_1+1)^2}y + \frac{\phi_3^2-\phi_2+\phi_3}{(m+\phi_1+1)^2} + \frac{1}{3(m+\phi_1+1)^2}$.

Following ([11], Thm.1), the proof of this lemma easily follows. Consequently, in view of Theorems 2 and 4 of [11], we are led to;

Lemma 2.2. *For the operator $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y)$, following hold good:*

(i) $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_1-x;x,y) = \frac{\theta-\theta_1-1}{n+\theta_1+1}x + \frac{2(\theta_3-\theta_2)+1}{2(n+\theta_1+1)}$. *Further, if $\theta_2-\theta_3 \geq 1$, we have*

$$\|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_1-x;x,y)\|_{C(I^2)} \leq \frac{\theta-\theta_1+\theta_2-\theta_3-\frac{1}{2}}{n+\theta_1+1}.$$

(ii) $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_2-y;x,y) = \frac{\phi-\phi_1-1}{m+\phi_1+1}y + \frac{2(\phi_3-\phi_2)+1}{2(m+\phi_1+1)}$. *Further, $\phi_2-\phi_3 \geq 1$, we have*

$$\|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_2-y;x,y)\|_{C(I^2)} \leq \frac{\phi-\phi_1+\phi_2-\phi_3-\frac{1}{2}}{m+\phi_1+1}.$$

(iii) $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_1-x)^2;x,y) = \frac{1}{(n+\theta_1+1)^2}\Big\{[x(\theta-\theta_1-1)-(\theta_2-\theta_3-1)]^2 + \frac{(n+\theta)^2}{n}\left(x-\frac{\theta_2}{n+\theta}\right)\left(\frac{n+\theta_2}{n+\theta}-x\right) - x(\theta-\theta_1-1)-(\theta_3-\theta_2+\frac{2}{3})\Big\}$.
Also, if $\theta_2-\theta_3 \geq 1$ and $\theta-\theta_1 \geq \theta_2-\theta_3$, we get

$$\|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_1-x)^2;x,y)\|_{C(I^2)} \leq \frac{(\theta-\theta_1)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2};$$

In the case $\theta-\theta_1 < \theta_2-\theta_3$ such that $\theta_2-\theta_3 \geq 1$, we obtain

$$\|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_1-x)^2;x,y)\|_{C(I^2)} \leq \frac{(\theta_2-\theta_3)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2}.$$

(iv) $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_2-y)^2;x,y) = \frac{1}{(m+\phi_1+1)^2}\Big\{[y(\phi-\phi_1-1)-(\phi_2-\phi_3-1)]^2 + \frac{(m+\phi)^2}{m}\left(y-\frac{\phi_2}{m+\phi}\right)\left(\frac{m+\phi_2}{m+\phi}-y\right) - y(\phi-\phi_1-1)-(\phi_3-\phi_2+\frac{2}{3})\Big\}$.
Also, if $\phi_2-\phi_3 \geq 1$ and $\phi-\phi_1 \geq \phi_2-\phi_3$, we get

$$\|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_2-y)^2;x,y)\|_{C(I^2)} \leq \frac{(\phi-\phi_1)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2};$$

In the case $\phi-\phi_1 < \phi_2-\phi_3$ such that $\phi_2-\phi_3 \geq 1$, we obtain

$$\|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_2-x)^2;x,y)\|_{C(I^2)} \leq \frac{(\phi_2-\phi_3)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2}.$$

3. Rate of convergence by $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y)$

In this section, we first give the following Korovkin type theorem on the convergence of $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y)$ to $f(x,y)$.

Theorem 3.1. *Let $f \in C(I^2)$. Then*

$$\lim_{n,m\to\infty} \max_{(x,y)\in\square} |\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y) - f(x,y)| = 0.$$

Proof. Taking into consideration the equalities in Lemma 2.1, we obtain

$$\lim_{n,m\to\infty} \max_{(x,y)\in\square} |\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{ij};x,y) - e_{ij}| = 0, \tag{7}$$

for $(i,j) \in \{(0,0),(1,0),(0,1)\}$. Further

$$\lim_{n,m\to\infty} \max_{(x,y)\in\square} |\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{20}+e_{02};x,y) - x^2 - y^2| = 0. \tag{8}$$

Let us define

$$\mathfrak{K}^{*\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y) = \begin{cases} \mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y) & \text{if} \quad (x,y) \in \square \\ f(x,y) & \text{if} \quad (x,y) \in I^2 \setminus \square. \end{cases}$$

Considering the above definition of the operators, we easily get

$$\|\mathfrak{K}^{*\theta_i,\phi_i}_{n,m,\theta,\phi}(f) - f\|_{C(I^2)} = \max_{(x,y)\in\square} |\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y) - f(x,y)|. \tag{9}$$

Now, using (7)-(8), we immediately get

$$\lim_{n,m\to\infty} \|\mathfrak{K}^{*\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{ij}) - e_{ij}\|_{C(I^2)} = 0,$$

for $(i,j) \in \{(0,0),(1,0),(0,1)\}$ and

$$\lim_{n,m\to\infty} \|\mathfrak{K}^{*\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{20}+e_{02}) - x^2 - y^2\|_{C(I^2)} = 0.$$

Applying the two dimensional Korovkin's type theorem (see [16]) to the sequence of operators $\mathfrak{K}^{*\theta_i,\phi_i}_{n,m,\theta,\phi}$, we obtain

$$\lim_{n,m\to\infty} \|\mathfrak{K}^{*\theta_i,\phi_i}_{n,m,\theta,\phi}(f) - f\|_{C(I^2)} = 0,$$

for every continuous function $f \in C(I^2)$. Therefore (9) gives

$$\lim_{n,m\to\infty} \max_{(x,y)\in\square} |\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y) - f(x,y)| = 0.$$

This completes the proof. □

In order to discuss the next results, we recall some definitions of the modulus of continuity.

Definition 3.2. *For $f \in C(I^2)$ and $\delta > 0$, the complete modulus of continuity is defined as*

$$\omega^{(c)}(f;\delta) = \sup_{\sqrt{(t_1-x)^2+(t_2-y)^2}\leq\delta} \{\left|f(t_1,t_2) - f(x,y)\right| : (t_1,t_2),(x,y) \in I^2\}. \tag{10}$$

The partial moduli of continuity of f with respect to x and y is given by

$$\omega^{(1)}(f;\delta) = \sup_{|x_1-x_2|\leq\delta}\ \sup_{y\in I}\{\left|f(x_1,y) - f(x_2,y)\right|\}$$

and

$$\omega^{(2)}(f;\delta) = \sup_{x\in I}\ \sup_{|y_1-y_2|\leq\delta}\{\left|f(x,y_1) - f(x,y_2)\right|\}, \tag{11}$$

respectively. We shall use the following property of the complete modulus of continuity:

$$|f(t_1,t_2) - f(x,y)| \leq \omega^{(c)}(f;\delta)\left(1 + \frac{\sqrt{(t_1-x)^2+(t_2-y)^2}}{\delta}\right). \tag{12}$$

It is known that these definitions satisfy the properties analogous to the usual modulus of continuity. For more details, we refer to [2].

In the next result, we obtain an estimate of the rate of convergence in terms of the complete modulus of continuity for the operators defined by (6).

Theorem 3.3. *Let $f \in C(I^2)$. If $\theta_2 - \theta_3 \geq 1$ and $\phi_2 - \phi_3 \geq 1$, then the following inequalities hold:*

$$\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f) - f\| \leq \begin{cases} \frac{3}{2}\,\omega^{(c)}\left(f;\sqrt{\frac{4(\theta-\theta_1)^2+n+8}{(n+\theta_1+1)^2}+\frac{4(\phi-\phi_1)^2+m+8}{(m+\phi_1+1)^2}}\right); & \text{if } \theta-\theta_1 \geq \theta_2-\theta_3 \text{ and } \phi-\phi_1 \geq \phi_2-\phi_3 \\ \frac{3}{2}\,\omega^{(c)}\left(f;\sqrt{\frac{4(\theta_2-\theta_3)^2+n+8}{(n+\theta_1+1)^2}+\frac{4(\phi_2-\phi_3)^2+m+8}{(m+\phi_1+1)^2}}\right); & \text{if } \theta-\theta_1 < \theta_2-\theta_3 \text{ and } \phi-\phi_1 < \phi_2-\phi_3. \end{cases}$$

Proof. From the linearity and positivity of the operators (6), Cauchy-Schwarz inequality and Lemma 1, the property (12) of the complete modulus of continuity gives

$$\left|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y) - f(x,y)\right| \leq \omega^{(c)}(f;\delta)\left(1 + \frac{\sqrt{\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_1-x)^2+(t_2-y)^2;x,y)}}{\delta}\right), \tag{13}$$

where $\delta > 0$. Therefore considering Lemma 2.2, for $\theta-\theta_1 \geq \theta_2-\theta_3 \geq 1$ and $\phi-\phi_1 \geq \phi_2-\phi_3 \geq 1$, we have

$$\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f) - f\| \leq \omega^{(c)}(f;\delta)\left(1 + \frac{\sqrt{\frac{(\theta-\theta_1)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2}+\frac{(\phi-\phi_1)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2}}}{\delta_{mn}}\right).$$

Now choosing $\delta = \sqrt{\frac{4(\theta-\theta_1)^2+n+8}{(n+\theta_1+1)^2}+\frac{4(\phi-\phi_1)^2+m+8}{(m+\phi_1+1)^2}}$, we obtain

$$\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f) - f\| \leq \frac{3}{2}\,\omega^{(c)}\left(f;\sqrt{\frac{4(\theta-\theta_1)^2+n+8}{(n+\theta_1+1)^2}+\frac{4(\phi-\phi_1)^2+m+8}{(m+\phi_1+1)^2}}\right).$$

Analogously, taking into account Lemma 2.2, for $\theta-\theta_1 < \theta_2-\theta_3$ and $\phi-\phi_1 < \phi_2-\phi_3$ such that $\theta_2-\theta_3, \phi_2-\phi_3 \geq 1$(with $\delta = \sqrt{\frac{4(\theta_2-\theta_3)^2+n+8}{(n+\theta_1+1)^2}+\frac{4(\phi_2-\phi_3)^2+m+8}{(m+\phi_1+1)^2}}$), we are led to

$$\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f) - f\| \leq \frac{3}{2}\,\omega^{(c)}\left(f;\sqrt{\frac{4(\theta_2-\theta_3)^2+n+8}{(n+\theta_1+1)^2}+\frac{4(\phi_2-\phi_3)^2+m+8}{(m+\phi_1+1)^2}}\right).$$

□

In the forthcoming result, the degree of approximation of f by the operators (6) is estimated by means of the partial moduli of continuity.

Theorem 3.4. *Let $f \in C(I^2)$. If $\theta_2-\theta_3 \geq 1$ and $\phi_2-\phi_3 \geq 1$, then the following inequalities hold:*

$$\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)-f\| \leq \begin{cases} 2\Big[\omega^{(1)}(f;\frac{\sqrt{4(\theta-\theta_1)^2+n+8}}{2(n+\theta_1+1)})+\omega^{(2)}(f;\frac{\sqrt{4(\phi-\phi_1)^2+m+8}}{2\delta_2(n+\phi_1+1)})\Big]; & \text{if } \theta-\theta_1 \geq \theta_2-\theta_3 \text{ and } \phi-\phi_1 \geq \phi_2-\phi_3 \\ 2\Big[\omega^{(1)}(f;\frac{\sqrt{4(\theta_2-\theta_3)^2+n+8}}{(n+\theta_1+1)^2})+\omega^{(2)}(f;\frac{\sqrt{4(\phi_2-\phi_3)^2+m+8}}{(m+\phi_1+1)^2})\Big]; & \text{if } \theta-\theta_1 < \theta_2-\theta_3 \text{ and } \phi-\phi_1 < \phi_2-\phi_3. \end{cases}$$

Proof. From linearity and monotonicity of the operators $\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}$ and the definitions of the partial moduli of continuity with respect to x and y as defined in (11), we have

$$\Big|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f(t_1,t_2);x,y)-f(x,y)\Big| \leq \Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\omega^{(1)}(f;|t_1-x|);x,y)+\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\omega^{(2)}(f;|t_2-y|);x,y).$$

Now using the property of modulus of continuity similar to (12) and the Cauchy-Schwarz inequality, for $\delta_1,\delta_2>0$, we get

$$\begin{aligned}\Big|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f(t_1,t_2);x,y)-f(x,y)\Big| &\leq \Big\{1+\frac{1}{\delta_1}\Big(\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_1-x)^2;x,y)\Big)^{\frac{1}{2}}\Big\}\omega^{(1)}(f;\delta_1)\\ &+\Big\{1+\frac{1}{\delta_2}\Big(\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_2-y)^2;x,y)\Big)^{\frac{1}{2}}\Big\}\omega^{(2)}(f;\delta_2),\end{aligned}$$

for all $(x,y)\in I^2$. Therefore, for $\phi-\phi_1 \geq \phi_2-\phi_3 \geq 1$ and $\theta-\theta_1 \geq \theta_2-\theta_3 \geq 1$, using Lemma 2.2 we obtain

$$\begin{aligned}\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)-f\| &\leq \Big\{1+\frac{1}{\delta_1}\Big(\frac{(\theta-\theta_1)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2}\Big)^{\frac{1}{2}}\Big\}\omega^{(1)}(f;\delta_1)\\ &+\Big\{1+\frac{1}{\delta_2}\Big(\frac{(\phi-\phi_1)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2}\Big)^{\frac{1}{2}}\Big\}\omega^{(2)}(f;\delta_2).\end{aligned}$$

Choosing $\delta_1=\frac{\sqrt{4(\theta-\theta_1)^2+n+8}}{2(n+\theta_1+1)}$ and $\delta_2=\frac{\sqrt{4(\phi-\phi_1)^2+m+8}}{2(m+\phi_1+1)}$, we obtain

$$\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)-f\| \leq 2\Big[\omega^{(1)}(f;\frac{\sqrt{4(\theta-\theta_1)^2+n+8}}{2(n+\theta_1+1)})+\omega^{(2)}(f;\frac{\sqrt{4(\phi-\phi_1)^2+m+8}}{2\delta_2(n+\phi_1+1)})\Big].$$

This proves the first assertion of our result. Similarly, for $\theta-\theta_1<\theta_2-\theta_3$ and $\phi-\phi_1<\phi_2-\phi_3$ such that $\theta_2-\theta_3,\phi_2-\phi_3\geq 1$, using Lemma 2 with $\delta_1=\frac{\sqrt{4(\theta_2-\theta_3)^2+n+8}}{2(n+\theta_1+1)}$ and $\delta_2=\frac{\sqrt{4(\phi_2-\phi_3)^2+m+8}}{2(m+\phi_1+1)}$, we immediately find the second assertion. □

We study the rate of convergence of the bi-variate Bernstein-Stancu-Kantorovich type operators $\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}$ for elements of the Lipschitz class $Lip_M(\gamma)$, for $0<\gamma\leq 1$. We recall the following definition:

Definition 3.5. *A function $f\in C(I^2)$ is said to be in $Lip_M(\gamma)$ if*

$$\Big|f(t_1,t_2)-f(x,y)\Big| \leq M\{(t_1-x)^2+(t_2-y)^2\}^{\frac{\gamma}{2}},$$

holds for all $(t_1,t_2),(x,y)\in I^2$.

Theorem 3.6. *If $\theta_2-\theta_3\geq 1$ and $\phi_2-\phi_3\geq 1$, then for all $f\in Lip_M(\gamma)$, the following inequalities hold:*

$$\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)-f\| \leq M\begin{cases}\Big(\frac{(\theta-\theta_1)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2}+\frac{(\phi-\phi_1)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2}\Big)^{\frac{\gamma}{2}} & \text{for } \theta-\theta_1\geq\theta_2-\theta_3 \text{ and } \phi-\phi_1\geq\phi_2-\phi_3\\ \Big(\frac{(\theta_2-\theta_3)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2}+\frac{(\phi_2-\phi_3)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2}\Big)^{\frac{\gamma}{2}} & \text{for } \theta-\theta_1<\theta_2-\theta_3 \text{ and } \phi-\phi_1<\phi_2-\phi_3.\end{cases}$$

where $0<\gamma\leq 1$ and M is a positive constant.

Proof. From the assumption $f \in Lip_M(\gamma)$, we have

$$\left|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f(t_1,t_2);x,y) - f(x,y)\right| \leq M\, \Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\{(t_1-x)^2+(t_2-y)^2\}^{\frac{\gamma}{2}};x,y).$$

Now, applying the Hölder's inequality with $p = \frac{2}{\gamma}, q = \frac{2}{2-\gamma}$ and Lemma 2.1, we obtain

$$\left|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f(t_1,t_2);x,y) - f(x,y)\right| \leq M\left(\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}((t_1-x)^2+(t_2-y)^2;x,y)\right)^{\frac{\gamma}{2}}.$$

Finally using Lemma 2.2 and considering sup-norm, we reach to the desired result. □

Let $C^2(I^2)$ be the space of all continuous function f having continuous partial derivatives upto the second order. We consider the following norm on $C^2(I^2)$:

$$\|f\|_{C^2(I^2)} = \|f\|_{C(I^2)} + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|_{C(I^2)} + \left\|\frac{\partial^i f}{\partial y^i}\right\|_{C(I^2)}\right) + \left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_{C(I^2)}.$$

We use the following definition in our upcoming result.

Definition 3.7. *Let $f \in C^2(I^2)$ and $\delta > 0$. The Peetre's K-functional and second-order modulus of smoothness of f are given by*

$$K(f;\delta) = \inf_{g\in C^2(I^2)}\left\{\left\|f-g\right\|_{C(I^2)} + \delta\left\|g\right\|_{C^2(I^2)}\right\},$$

and

$$\omega_2(f;\delta) = \sup_{\sqrt{t^2+s^2}\leq\delta}\left|\Delta^2_{t,s}f(x,y)\right|,$$

where $\Delta^2_{t,s}f(x,y) = \sum\limits_{j=0}^{2}(-1)^{2-j}\binom{2}{j}f(x+jt,y+js)$, respectively.

In the next result, we establish an order of approximation for the bi-variate operator $\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}$ in terms of the Peetre's K-functional and the complete modulus of continuity.

Theorem 3.8. *For all $f \in C(I^2)$ and $\theta_2-\theta_3, \phi_2-\phi_3 \geq 1$, the following inequalities hold*

$$\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f) - f\| \leq \begin{cases} 4K(f,\delta_1) + \omega^{(c)}(f;\Delta), & \text{if } \theta-\theta_1 \geq \theta_2-\theta_3 \text{ and } \phi-\phi_1 \geq \phi_2-\phi_3 \\ 4K(f,\delta_2) + \omega^{(c)}(f;\Delta), & \text{if } \theta-\theta_1 \leq \theta_2-\theta_3 \text{ and } \phi-\phi_1 \leq \phi_2-\phi_3, \end{cases}$$

where

$$\delta_1 = \frac{1}{8}\left[\left(\sqrt{\frac{(\theta-\theta_1)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2}} + \sqrt{\frac{(\phi-\phi_1)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2}}\right)^2 + \left(\frac{\theta-\theta_1+\theta_2-\theta_3-\frac{1}{2}}{n+\theta_1+1} + \frac{\phi-\phi_1+\phi_2-\phi_3-\frac{1}{2}}{n+\phi_1+1}\right)^2\right],$$

$$\delta_2 = \frac{1}{8}\left[\left(\sqrt{\frac{(\theta_2-\theta_3)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2}} + \sqrt{\frac{(\phi_2-\phi_3)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2}}\right)^2 + \left(\frac{\theta-\theta_1+\theta_2-\theta_3-\frac{1}{2}}{n+\theta_1+1} + \frac{\phi-\phi_1+\phi_2-\phi_3-\frac{1}{2}}{n+\phi_1+1}\right)^2\right],$$

and

$$\Delta^2 = \left(\frac{\theta-\theta_1+\theta_2-\theta_3-\frac{1}{2}}{n+\theta_1+1}\right)^2 + \left(\frac{\phi-\phi_1+\phi_2-\phi_3-\frac{1}{2}}{n+\phi_1+1}\right)^2.$$

Proof. We consider the following auxiliary operators:

$$\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y)=\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y)+f(x,y)-f(\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_1;x,y),\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_2;x,y)).$$

From Taylor expansion, for any $h\in C^2(I^2)$, we have

$$\begin{aligned} h(t_1,t_2)-h(x,y) &= \frac{\partial h(x,y)}{\partial x}(t_1-x)+\int_x^{t_1}(t_1-\eta)\frac{\partial^2 h(\eta,y)}{\partial \eta^2}d\eta+\frac{\partial h(x,y)}{\partial y}(t_2-y)\\ &+ \int_y^{t_2}(t_2-\xi)\frac{\partial^2 h(x,\xi)}{\partial \xi^2}d\xi+\int_x^{t_1}\int_y^{t_2}\frac{\partial^2 h(u,v)}{\partial u\partial v}dudv, \end{aligned} \tag{14}$$

and, let $\psi_h^{i,j}(t_1,t_2)=\Big(\int_x^{t_1}(t_1-\eta)\frac{\partial^2 h(\eta,y)}{\partial\eta^2}d\eta\Big)^i\Big(\int_y^{t_2}(t_2-\xi)\frac{\partial^2 h(x,\xi)}{\partial\xi^2}d\xi\Big)^j$.

Applying the auxiliary operator $\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}$ on the equation (14) and taking $\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(1;x,y)=1;\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_1-x;x,y)=0=\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_2-y;x,y)$, we have

$$\begin{aligned} |\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(h;x,y)-h(x,y)| &\le |\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\psi_h^{1,0}(t_1,t_2);x,y)|+|\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\psi_h^{0,1}(t_1,t_2);x,y)|\\ &+ |\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\int_x^{t_1}\int_y^{t_2}\frac{\partial^2 h(u,v)}{\partial u\partial v}dudv;x,y)|. \end{aligned} \tag{15}$$

Further, applying the auxiliary operator $\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}$ on $\psi_h^{1,0}$ gives us

$$\begin{aligned} |\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\psi_h^{1,0}(t_1,t_2);x,y)| &\le \frac{\|h\|_{C^2(I^2)}}{2}\Big\{\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}\Big((t_1-x)^2;x,y\Big)+\Big(\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}\Big(t_1-x;x,y\Big)\Big)^2\Big\}\\ &= \frac{\|h\|_{C^2(I^2)}}{2}\{\mu_{2,x}+\mu_{1,x}^2\}, \end{aligned}$$

where $\mu_{2,x}$ and $\mu_{1,x}$ are the second and first order central moments, respectively. Similarly,

$$|\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\psi_h^{0,1}(t_1,t_2);x,y)| \le \frac{\|h\|_{C^2(I^2)}}{2}\{\nu_{2,y}+\nu_{1,y}^2\},$$

where $\nu_{2,y}$ and $\nu_{1,y}$ are the second and first order central moments, respectively. Also,

$$\begin{aligned} |\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\int_x^{t_1}\int_y^{t_2}\frac{\partial^2 h(u,v)}{\partial u\partial v}dudv;x,y)| &\le \|h\|_{C^2(I^2)}\Big\{\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}\Big(|t_1-x||t_2-y|;x,y\Big)\\ &+ |\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{1,0};x,y)-x||\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(e_{0,1};x,y)-y|\Big\}, \end{aligned}$$

hence using the Cauchy-Schwarz inequality

$$\begin{aligned} &|\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(\int_x^{t_1}\int_y^{t_2}\frac{\partial^2 h(u,v)}{\partial u\partial v}dudv;x,y)| \le \|h\|_{C^2(I^2)}\Big\{\Big(\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}\Big((t_1-x)^2;x,y\Big)\Big)^{\frac12}\\ &\times\Big(\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}\Big((t_2-y)^2;x,y\Big)\Big)^{\frac12}+|\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_1-x;x,y)||\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_2-y;x,y)|\Big\}\\ &= \|h\|_{C^2(I^2)}\{\mu_{2,x}^{1/2}\nu_{2,y}^{1/2}+|\mu_{1,x}||\nu_{1,y}|\}. \end{aligned} \tag{16}$$

Consequently, from the equation (14)

$$|\hat{\mathfrak{K}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(h;x,y)-h(x,y)| \le \frac{\|h\|_{C^2(I^2)}}{2}\Big\{(\mu_{2,x}^{1/2}+\nu_{2,y}^{1/2})^2+(|\mu_{1,x}|+|\nu_{1,y}|)^2\Big\}. \tag{17}$$

Now, from the definition of auxiliary operator and equation (17), we may write

$$\begin{aligned}|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y)-f(x,y)| &\leq |\hat{\mathfrak{R}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f-h;x,y)|+|\hat{\mathfrak{R}}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(h;x,y)-h(x,y)|+|(f-h)(x,y)|\\ &+ \left|f\Big(\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_1;x,y),\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(t_2;x,y)\Big)-f(x,y)\right|\\ &\leq 4\left\{\|f-h\|_{C(I^2)}+\frac{\|h\|_{C^2(I^2)}}{8}\left\{(\mu_{2,x}^{1/2}+\nu_{2,y}^{1/2})^2+(|\mu_{1,x}|+|\nu_{1,y}|)^2\right\}\right\}\\ &+\omega^{(c)}\left(f;\sqrt{\left(\mu_{1,x}\right)^2+\left(\nu_{1,y}\right)^2}\right).\end{aligned}$$

Now, for $\theta-\theta_1\geq\theta_2-\theta_3\geq 1$ and $\phi-\phi_1\geq\phi_2-\phi_3\geq 1$, using Lemma 2.2 and taking infimum over all $h\in C^2(I^2)$, we get

$$\|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)-f\| \leq 4K(f,\delta_1)+\omega^{(c)}(f;\Delta).$$

By a similar reasoning, for the other case $\theta-\theta_1\leq\theta_2-\theta_3$ and $\phi-\phi_1\leq\phi_2-\phi_3$ such that $\theta_2-\theta_3,\phi_2-\phi_3>1$, we have

$$\|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)-f\| \leq 4K(f,\delta_2)+\omega^{(c)}(f;\Delta).$$

This proves the required result. □

Corollary 3.9. *Considering the well-known relation [8] that*

$$K(f;\delta) \leq C\,\omega_2(f;\sqrt{\delta}),\quad \text{for any } \delta>0,$$

where C is some positive constant, the result of the Theorem 3.8 takes the following form:

$$\|\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)-f\| \leq \begin{cases}\frac{C}{4}\omega_2(f,\sqrt{\delta_1})+\omega^{(c)}(f;\Delta), & \text{if } \theta-\theta_1\geq\theta_2-\theta_3 \text{ and } \phi-\phi_1\geq\phi_2-\phi_3\\ \frac{C}{4}\omega_2(f,\sqrt{\delta_2})+\omega^{(c)}(f;\Delta), & \text{if } \theta-\theta_1\leq\theta_2-\theta_3 \text{ and } \phi-\phi_1\leq\phi_2-\phi_3,\end{cases}$$

4. A k^{th} order generalization of the operators $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}$

In this section, we use the method of Kirov and Popova [13] to introduce and investigate approximation properties of a k^{th} order generalization of our bi-variate Bernstein-Stancu-Kantorovich type operator $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(.;x,y)$ defined in (6). Let $C^k(I^2)$, $k\in\mathbb{N}\cup\{0\}$, denote the set of all functions $f:I^2\to\mathbb{R}$ having continuous partial derivatives upto the k^{th} $(s=0,1,2,...)$ order on the box I^2. We now define, for any function $f\in C^k(I^2)$,), the k^{th} order generalization of Bernstein-Stancu-Kantorovich type polynomials $\mathfrak{R}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(.;x,y)$ as

$$\begin{aligned}\mathfrak{R}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f(u,v);x,y) &= \left(\frac{n+\theta}{n}\right)^n\left(\frac{m+\phi}{m}\right)^m(n+\theta_1+1)(m+\phi_1+1)\\ &\times\sum_{s=0}^{n}\sum_{r=0}^{m}\Omega^{(\theta,\theta_2)}_{n,s}(x)\,\Omega^{(\phi,\phi_2)}_{m,r}(y)\int_{\frac{s+\theta_3}{n+\theta_1+1}}^{\frac{s+\theta_3+1}{n+\theta_1+1}}\int_{\frac{r+\phi_3}{m+\phi_1+1}}^{\frac{r+\phi_3+1}{m+\phi_1+1}}\sum_{l=0}^{k}\frac{d^l f(u,v)}{l!}dudv,\end{aligned} \tag{18}$$

where $d^l f(u,v)=\sum\limits_{i=0}^{l}\binom{l}{i}\frac{\partial^l f(u,v)}{\partial x^{l-i}\partial y^i}(x-u)^{l-i}(y-v)^i$.

Now, there is a unit vector (μ,η) for which $(x-u,y-v)=w(\mu,\eta)$ where $w>0$. Let

$$P(w)=f(u+w\mu,v+w\eta)=f(u+(x-u),v+(y-v))=f(x,y). \tag{19}$$

Following remarks can be made from the equations (18) and (19).

Remark 4.1. *Note that, when $k = 0$ in the equation (18), we immediately get the operator defined in (6), i.e.*

$$\Re^{\theta_i,\phi_i,0}_{n,m,\theta,\phi}(f;x,y) = \Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y)$$

Remark 4.2. *The k^{th} order derivative of the function $P(w)$ has the following form (See chapter 3 in [7])*

$$P^k(w) = \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^k f(u + w\mu, v + w\eta)}{\partial x^{k-i}\partial y^i} \mu^{k-i}\eta^i, \ (k \in N). \tag{20}$$

Also, using the equation (20), we can easily deduce that the Taylor's formula for $P(w)$ at $w = 0$ is the same as that of $f(x,y)$ at (u,v).

The following intermediate result is useful in the proof of some important corollaries which provide us a deeper insight into the approximation behavior of the operators defined by (18):

Theorem 4.3. *For each $m,n,k \in \mathbb{N}$, and for all $f \in C^k\left(I^2\right)$ such that $P^k(w) \in Lip_M(\gamma)$, we have*

$$\|f - \Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)\|_{C(I^2)} \leq \frac{M}{(k-1)!}\frac{\gamma}{\gamma+k} B(\gamma,k) \times \|\Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}\left(|\left(x-u, y-v\right)|^{k+\gamma}\right)\|_{C(I^2)},$$

where $0 < \gamma \leq 1, M > 0$ and $B(\gamma,k)$ denotes the usual Beta function.

Proof. Let $f \in C^k\left(I^2\right)$ and $(x,y) \in I^2$. By the definition of the operators $\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f;x,y)_k$ in (18), we see that for any $m,n,k \in \mathbb{N}$,

$$\begin{aligned} f(x,y) - \Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f(u,v);x,y) &= \left(\frac{n+\theta}{n}\right)^n \left(\frac{m+\phi}{m}\right)^m (n+\theta_1+1)(m+\phi_1+1) \\ &\times \sum_{s=0}^{n}\sum_{r=0}^{m} \Omega^{(\theta,\theta_2)}_{n,s}(x)\,\Omega^{(\phi,\phi_2)}_{m,r}(y) \int_{\frac{s+\theta_3}{n+\theta_1+1}}^{\frac{s+\theta_3+1}{n+\theta_1+1}} \int_{\frac{r+\phi_3}{m+\phi_1+1}}^{\frac{r+\phi_3+1}{m+\phi_1+1}} \left(f(x,y) - \sum_{l=0}^{k} \frac{d^l f(u,v)}{l!} dudv\right), \end{aligned} \tag{21}$$

It is known from Taylor's integral remainder formula for $f(x,y)$ at (u,v) (see[7]) that

$$f(x,y) - \sum_{l=0}^{k-1} \frac{d^l f(u,v)}{l!} = \frac{1}{(k-1)!}\int_0^1 (1-z)^{k-1} \times \left(\sum_{i=0}^{k}\binom{k}{i} \frac{\partial^k f(u+z(x-u), v+z(y-v))}{\partial x^{k-i}\partial y^i}(x-u)^{k-i}(y-v)^i\right) dz. \tag{22}$$

Using Remark 4.2, the equation (22) takes the form

$$P(u) - \sum_{l=0}^{k} P^l(0)w^l = \frac{w^k}{(k-1)!}\int_0^1 (1-z)^{k-1}[P^k(wz) - P^k(0)]dz.$$

Since $P^k(w) \in Lip_M(\gamma)$, it follows that

$$\left|f(x,y) - \sum_{l=0}^{k-1} \frac{d^l f(u,v)}{l!}\right| = \left|P(u) - \sum_{l=0}^{k} P^l(0)w^l\right| \leq \frac{M|w|^{k+\gamma}}{(k-1)!}\int_0^1 z^\gamma (1-z)^{k-1} dz. \tag{23}$$

From the definition of Beta function, we have

$$\int_0^1 z^\gamma (1-z)^{k-1} dz = B(1+\gamma,k) = \frac{\gamma B(\gamma,k)}{\gamma+k},$$

Hence, the equation (23) takes the following from

$$\left|f(x,y)-\sum_{l=0}^{k-1}\frac{d^l f(u,v)}{l!}\right| \quad \leq \quad \frac{M}{(k-1)!}\frac{\gamma B(\gamma,k)}{\gamma+k}\left|\left(x-u,y-v\right)\right|^{k+\gamma} \tag{24}$$

Finally, using (24) in (21) and taking supremum over all $(x,y)\in I^2$, we obtain the desired result. □

Let $g\in C(I^2)$ be a function defined by

$$g(u,v) \quad = \quad \left|(u,v)-(x,y)\right|^{k+\gamma} \tag{25}$$

Since $g\in C(I^2)$ and $g(x,y)=0$, Theorem 3.1 yields

$$\|\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(g;x,y)\|_{C(I^2)}\to 0 \ as \ m,n\to\infty.$$

Thus, Theorem 4.3 yields that for all $f\in C^k(I^2)$ such that $P^k(w)\in Lip_M(\gamma)$,

$$\|\Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f;x,y)-f(x,y)\|_{C(I^2)}\to 0 \ as \ m,n\to\infty.$$

Taking into consideration Theorem 2, one can deduce the following result from Theorem 4.3 immediately:

Corollary 4.4. *If $\theta_2-\theta_3\geq 1$ and $\phi_2-\phi_3\geq 1$, then for each $m,n\in\mathbb{N}$, and for all $f\in C^k\left(I^2\right)$ such that $P^k(w)\in Lip_M(\gamma)$ we have*

$$\|f-\Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)\|_{C(I^2)}\leq\frac{3M}{2(k-1)!}\frac{\gamma}{\gamma+k}B(\gamma,k)\times$$
$$\begin{cases}\omega^{(c)}\left(g;\sqrt{\frac{4(\theta-\theta_1)^2+n+8}{(n+\theta_1+1)^2}+\frac{4(\phi-\phi_1)^2+m+8}{(m+\phi_1+1)^2}}\right); & for\ \theta-\theta_1\geq\theta_2-\theta_3\ and\ \phi-\phi_1\geq\phi_2-\phi_3\\ \omega^{(c)}\left(g;\sqrt{\frac{4(\theta_2-\theta_3)^2+n+8}{(n+\theta_1+1)^2}+\frac{4(\phi_2-\phi_3)^2+m+8}{(m+\phi_1+1)^2}}\right); & for\ \theta-\theta_1<\theta_2-\theta_3\ and\ \phi-\phi_1<\phi_2-\phi_3.\end{cases}$$

where g is given by (25).

Applying Theorem 3.6, the following result is immediate from Theorem 4.3:

Corollary 4.5. *For each $m,n\in\mathbb{N},k\in\mathbb{N}\cup\{0\}$ and $f\in C^k\left(I^2\right)$ such that $f^{(k)}\in Lip_M(\gamma)$, and assuming that $g\in Lip_{2^{\frac{k}{2}}}(\gamma)$ in Theorem 3.6, we have*

$$\|f-\Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)\|_{C(I^2)}\leq\frac{2^{\frac{k}{2}}M}{(k-1)!}\frac{\gamma}{\gamma+k}B(\gamma,k)$$
$$\begin{cases}\left(\frac{(\theta-\theta_1)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2}+\frac{(\phi-\phi_1)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2}\right)^{\frac{\gamma}{2}} & for\ \ \theta-\theta_1\geq\theta_2-\theta_3\ and\ \phi-\phi_1\geq\phi_2-\phi_3\\ \left(\frac{(\theta_2-\theta_3)^2+\frac{n}{4}+2}{(n+\theta_1+1)^2}+\frac{(\phi_2-\phi_3)^2+\frac{m}{4}+2}{(m+\phi_1+1)^2}\right)^{\frac{\gamma}{2}} & for\ \theta-\theta_1<\theta_2-\theta_3\ and\ \phi-\phi_1<\phi_2-\phi_3.\end{cases}$$

Lastly, taking into account Theorem 3.8, we can easily deduce the following from Theorem 4.3:

Corollary 4.6. *For all $f\in C^k\left(I^2\right)$ such that $f^{(k)}\in Lip_M(\gamma)$, if $\theta_2-\theta_3,\phi_2-\phi_3\geq 1$, then we obtain*

$$\|f-\Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)\|\leq\frac{M}{(k-1)!}\frac{\gamma}{\gamma+k}B(\gamma,k)\times\begin{cases}4K(g,\delta_1)+\omega^{(c)}(f;\Delta), & if\ \theta-\theta_1\geq\theta_2-\theta_3\ and\ \phi-\phi_1\geq\phi_2-\phi_3\\ 4K(g,\delta_2)+\omega^{(c)}(f;\Delta), & if\ \theta-\theta_1\leq\theta_2-\theta_3\ and\ \phi-\phi_1\leq\phi_2-\phi_3,\end{cases}$$

where δ_1,δ_2,Δ are given in Theorem 3.8 and g is defined by (25).

Example 1. Let $f(x,y) = (x+2)^3y^4$ and $\theta_3 = 1$, $\theta_2 = 2$, $\theta_1 = 3$, $\theta = 4$ and $\phi_3 = 1$, $\phi_2 = 2$, $\phi_1 = 3$, $\phi = 4$. The convergence of the operators $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)$ and $\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)$ to the function f for $n = m = 5$ and $k = 2$ and $k = 5$ is illustrated in Figure 1 and Figure 2 respectively. It is seen that if f is differentiable k times then $\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)$ yields a better convergence in comparison to the classical Bernstein-Stancu-Kantorovich operator $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)$. In Table 1, we obtain estimates of the maximum absolute errors in the approximation of the $f(x,y) = (x+2)^3y^4$ by using the operators $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)$ as defined in (6) and $\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)$ as given in (18), namely $\mathcal{E}^{\theta_i,\phi_i}_{n,m,\theta,\phi} = \|\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}f - f\|_{C(I^2)}$ and $\mathcal{E}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi} = \|\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}f - f\|_{C(I^2)}$, respectively.

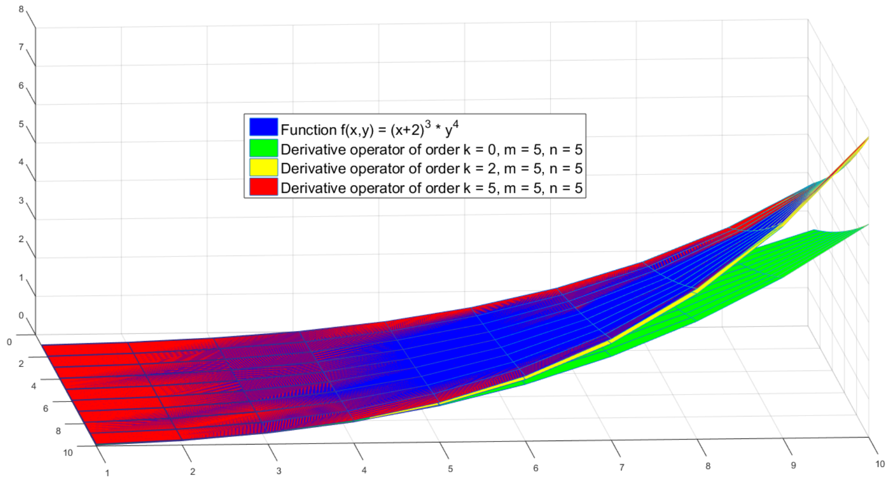

Figure 1: $\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)$ approximates $f(x,y)$ much better than $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)$

Table 1: Comparison of $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}$ and $\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}$ for $n = m = 5$ and some values of k

m,n	Error bound for $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}$	Derivative order k	Error bound $\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}$
5,5	2.2878	2	0.1323
5,5	2.2878	3	0.0263
5,5	2.2878	4	0.0015
5,5	2.2878	5	0.0002

Example 2. For $m = n = 5$ and $\theta_3 = 1$, $\theta_2 = 2$, $\theta_1 = 3$, $\theta = 4$ and $\phi_3 = 1$, $\phi_2 = 2$, $\phi_1 = 3$, $\phi = 4$, the estimates of the maximum absolute errors in the approximation of the function $f(x,y) = (x+3)^{\frac{5}{2}}e^{-y}$ by using operators $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)$ and $\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)$ are listed in Table 2. The convergence of the operators $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)$ and $\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)$ to the function f for $k = 2$ and $k = 5$ is illustrated in Figure 2. Further from the figure 2 and Table 2 it follows that, depending on the order of the derivative k, $\mathfrak{K}^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)$ gives better approximation to the function f in comparison to the Bernstein-Stancu-Kantorovich operators $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)$.

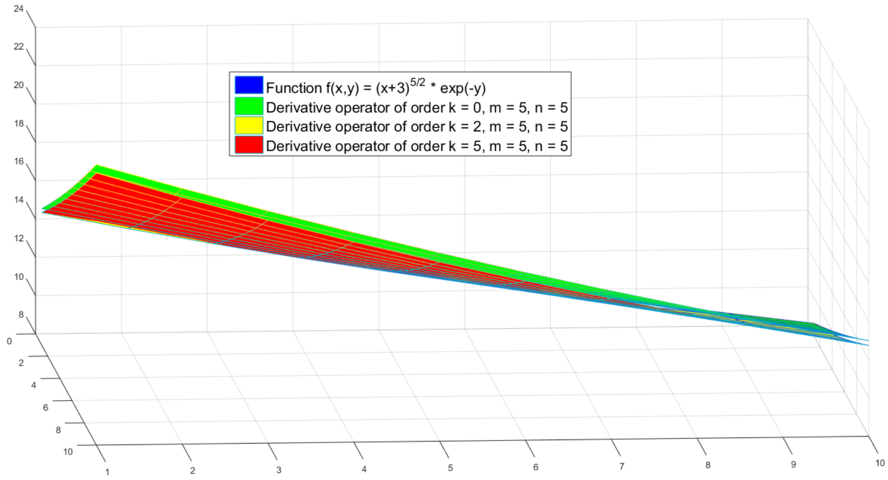

Figure 2: $\Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}(f)$ approximates $f(x,y)$ much better than $\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}(f)$

Table 2: Comparison of $\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}$ and $\Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}$ for $n = m = 5$ and some values of k

m,n	Error bound for $\Re^{\theta_i,\phi_i}_{n,m,\theta,\phi}$	Derivative order k	Error bound for $\Re^{\theta_i,\phi_i,k}_{n,m,\theta,\phi}$
5,5	0.4851	2	0.0058
5,5	0.4851	3	0.00081
5,5	0.4851	5	0.00003595

5. Applications

We shall now consider some further generalized Bernstein type polynomials. To obtain an approximation process for k^{th} order generalization of the operator of Bernstein-type, we introduce some examples;

5.1. Bivariate Bernstein operators in rectangle

In [10], Gadjiev and Ghorbanalizadeh also introduced two dimensional Bernstein polynomials on the rectangle $\square = [\frac{\theta_2}{m+\theta}, \frac{m+\theta_2}{m+\theta}] \times [\frac{\phi_2}{n+\phi}, \frac{n+\phi_2}{n+\phi}]$ and the polynomials $B^{(\theta_i,\phi_i)}_{m,n}$ defined as follows:

$$B^{(\theta_i,\phi_i)}_{m,n}(f;x,y) = \left(\frac{m+\theta}{m}\right)^m \left(\frac{n+\phi}{n}\right)^n \sum_{s=0}^{m}\sum_{r=0}^{n} \Omega^{(\theta,\theta_2)}_{m,s}(x)\Omega^{(\phi,\phi_2)}_{n,r}(y) f\left(\frac{s+\theta_3}{m+\theta_1}, \frac{r+\phi_3}{n+\phi_1}\right),$$

where the basis functions $\Omega^{(\theta,\theta_2)}_{m,s}(x), \Omega^{(\phi,\phi_2)}_{n,r}(y); (x,y) \in \square$ are as defined in (4) and $\theta, \phi, \theta_i, \phi_i, i = 1,2,3$ are non-negative real numbers satisfying $0 \le \theta_3 \le \theta_2 \le \theta_1 \le \theta$ and $0 \le \phi_3 \le \phi_2 \le \phi_1 \le \phi$.
We consider the following generalization $B^{(\theta_i,\phi_i,k)}_{m,n}(f;x,y)$ of the above linear positive operators:

$$B^{(\theta_i,\phi_i,k)}_{m,n}(f;x,y) = \left(\frac{m+\theta}{m}\right)^m \left(\frac{n+\phi}{n}\right)^n \sum_{s=0}^{m}\sum_{r=0}^{n} \Omega^{(\theta,\theta_2)}_{m,s}(x)\Omega^{(\phi,\phi_2)}_{n,r}(y) \times \sum_{l=0}^{k} \frac{d^l f\left(\frac{s+\theta_3}{m+\theta_1}, \frac{r+\phi_3}{n+\phi_1}\right)}{l!}, \tag{26}$$

where

$$d^l f\left(\frac{s+\theta_3}{m+\theta_1}, \frac{r+\phi_3}{n+\phi_1}\right) = \sum_{i=0}^{l} \binom{l}{i} \frac{\partial^l f\left(\frac{s+\theta_3}{m+\theta_1}, \frac{r+\phi_3}{n+\phi_1}\right)}{\partial x^{l-i}\partial y^i} \times \left(x - \frac{s+\theta_3}{m+\theta_1}\right)^{l-i} \left(y - \frac{r+\phi_3}{n+\phi_1}\right)^i. \tag{27}$$

Example 3. For $\theta_3 = 1$, $\theta_2 = 2$, $\theta_1 = 3$, $\theta = 4$ $f(x,y) = (x+3)^{\frac{5}{2}}e^{-y}$ and $\phi_3 = 1$, $\phi_2 = 2$, $\phi_1 = 3$, $\phi = 4$, the convergence of the operators $B_{m,n}^{(\theta_i,\phi_i,k)}(f)$ towards the function $f(x,y)$ for $k = 0,2,5$ is illustrated in Fig.3. From Fig 3 it is clear that the operators $B_{m,n}^{(\theta_i,\phi_i,k)}(f)$ provides better approximation than the operator $B_{m,n}^{(\theta_i,\phi_i,0)}(f)$ for both $k = 2,5$. In Table 3, we observe that as the value of the order k of the derivative increases, the error in the approximation of function f by the operator $B_{m,n}^{(\theta_i,\phi_i,k)}(f)$ becomes smaller.

Table 3: Comparison of $B_{m,n}^{(\theta_i,\phi_i)}$ and $B_{m,n}^{(\theta_i,\phi_i,k)}$ for $n = m = 5$ and some values k

m,n	Error bound for $B_{m,n}^{(\theta_i,\phi_i)}$	Derivative order k	Error bound $B_{m,n}^{(\theta_i,\phi_i,k)}$
5,5	0.4079	2	0.0103
5,5	0.4079	5	0.000004456

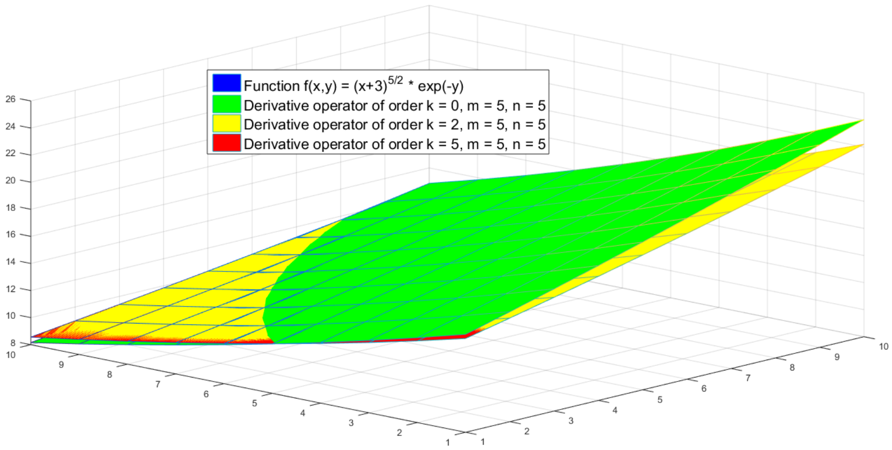

Figure 3: $\mathfrak{K}_{n,m,\theta,\phi}^{\theta_i,\phi_i,k}(f)$ approximates $f(x,y)$ much better than $\mathfrak{K}_{n,m,\theta,\phi}^{\theta_i,\phi_i}(f)$

5.2. *Bivariate-Stancu type operators in a triangle*

Gadjiev and Ghorbanalizadeh [10] defined another bivariate Bernstein-Stancu type operators on the triangle Δ for the functions $f : \Delta = \left\{(x,y) : x+y \leq \frac{m+2\theta_2}{m+\theta}; x,y \geq \frac{\theta_2}{m+\theta}\right\} \to \mathbb{R}$. More precisely, in [10], they considered $\mathfrak{B}_{m,\theta,\phi}^{\theta_i,\phi_i}$ with:

$$\mathfrak{B}_{m,\theta,\phi}^{\theta_i,\phi_i}(f;x,y) = \left(\frac{m+\theta}{m}\right)^m \sum_{s=0}^{m}\sum_{r=0}^{m-s} \Omega_{m,s,r}^{(\theta,\theta_2)}(x,y) f\left(\frac{s+\theta_3}{m+\theta_1}, \frac{r+\phi_3}{m+\phi_1}\right),$$

where the basis functions $\Omega_{m,s,r}^{(\theta,\theta_2)}(x)$ are defined by

$$\Omega_{m,s,r}^{(\theta,\theta_2)}(x,y) = \binom{m}{s}\binom{m-s}{r}\left(x - \frac{\theta_2}{m+\theta}\right)^s \left(y - \frac{\theta_2}{m+\theta}\right)^r \left(\frac{m+2\theta_2}{m+\theta} - x - y\right)^{m-s-r}, \tag{28}$$

and $\theta, \phi, \theta_i, \phi_i$, $i = 1,2$ are the positive numbers satisfying $0 < \theta_2 \leq \theta_3 \leq \theta_1 \leq \theta$ and $0 < \phi_2 \leq \phi_3 \leq \phi_1 \leq \phi$.

The authors [10] derived the rate of convergence in terms of the complete and partial moduli of continuity for operators $\mathfrak{B}_{m,\theta,\phi}^{\theta_i,\phi_i}$.

We now introduce the k^{th} order generalization of the operators $\mathfrak{B}^{\theta_i,\phi_i}_{m,\theta,\phi}$:

$$\mathfrak{B}^{\theta_i,\phi_i,k}_{m,\theta,\phi}(f;x,y)=\left(\frac{m+\theta}{m}\right)^m\sum_{s=0}^{m}\sum_{r=0}^{m-s}\Omega^{(\theta,\theta_2)}_{m,s,r}(x,y)\sum_{l=0}^{k}\frac{d^l f\left(\frac{s+\theta_3}{m+\theta_1},\frac{r+\phi_3}{m+\phi_1}\right)}{l!},\tag{29}$$

where $d^l f\left(\frac{s+\theta_3}{m+\theta_1},\frac{r+\phi_3}{m+\phi_1}\right)$ is given by (27).

Example 4. Let $\theta_3=1$, $\theta_2=2$, $\theta_1=3$, $\theta=4$, $f(x,y)=y^3e^{-2x}$ and $\phi_3=1$, $\phi_2=2$, $\phi_1=3$, $\phi=4$, and $m=5$. In Fig. 4, the comparison of convergence of the operators $\mathfrak{B}^{\theta_i,\phi_i}_{m,\theta,\phi}$ and $\mathfrak{B}^{\theta_i,\phi_i,k}_{m,\theta,\phi}$, $k=2,5$ towards the function $f(x,y)$ is illustrated. From Table 4, it is clear that the Bernstein-Stancu-Taylor operators $\mathfrak{B}^{\theta_i,\phi_i,k}_{m,\theta,\phi}$ give us a better approximation to $f(x,y)$ compared to Bernstein-Stancu operators $\mathfrak{B}^{\theta_i,\phi_i}_{m,\theta,\phi}$. Further, it may be remarked that the parameters $\theta_3,\theta_2,\theta_1,\theta$ and $\phi_3,\phi_2,\phi_1,\phi$, play an important role to achieve a better approximation.

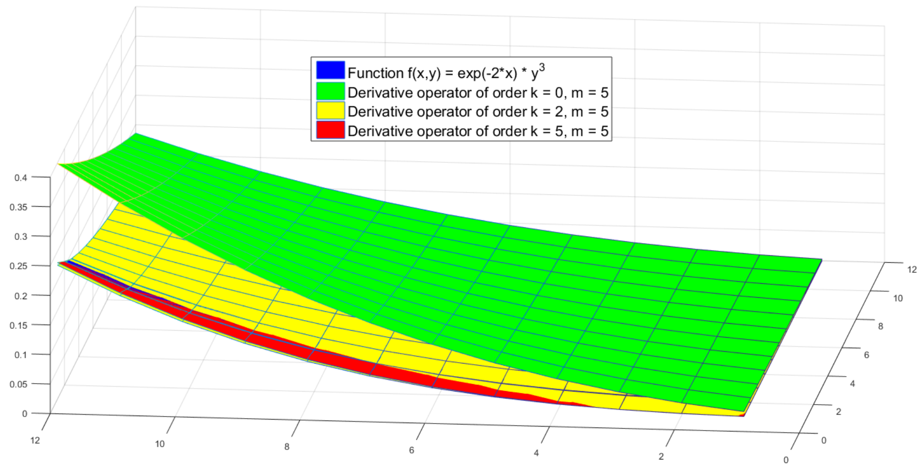

Figure 4: $\mathfrak{B}^{\theta_i,\phi_i,k}_{m,\theta,\phi}(f)$ approximates $f(x,y)$ much better than $\mathfrak{B}^{\theta_i,\phi_i}_{m,\theta,\phi}(f)$

Table 4: Comparison of $\mathfrak{B}^{\theta_i,\phi_i}_{m,\theta,\phi}(f)$ and $\mathfrak{B}^{\theta_i,\phi_i,k}_{m,\theta,\phi}(f)$ for $m=5$ and some values of k

m	Error bound for $\mathfrak{B}^{\theta_i,\phi_i}_{m,\theta,\phi}$	Derivative order k	Error bound $\mathfrak{B}^{\theta_i,\phi_i,k}_{m,\theta,\phi}$
5	0.1675	2	0.0340
5	0.1675	5	0.0004425

5.3. *Bivariate Stancu-Kantorovich operators in a triangle*

Inspired by the work [10], we present the following bivariate extension of the operators (28) on the triangle $\Delta=\left\{(x,y):x+y\leq\frac{m+2\theta_2}{m+\theta};x,y\geq\frac{\theta_2}{m+\theta}\right\}$:

$$\mathfrak{P}^{\theta_i,\phi_i}_{m,\theta,\phi}(f;x,y)=(m+\phi_1+1)(m+\theta_1+1)\left(\frac{m+\theta}{m}\right)^m\sum_{j=0}^{m}\sum_{l=0}^{m-j}\Omega^{(\theta,\theta_2)}_{m,j,l}(x,y)\times\int_{\frac{j+\theta_3}{m+\theta_1+1}}^{\frac{j+\theta_3+1}{m+\theta_1+1}}\int_{\frac{l+\phi_3}{m+\phi_1+1}}^{\frac{l+\phi_3+1}{m+\phi_1+1}}f(u,v)dudv,\tag{30}$$

where the basis functions $\Omega^{(\theta,\theta_2)}_{m,j,l}(x,y)$ are as defined by (28). At last, we define the Bernstein-Stancu-Kantorovich-Taylor extension of these operators as follows:

For $f \in C^k(I^2), k \in \mathbb{N} \cup \{0\}$, we propose

$$\mathfrak{P}^{*\theta_i,\phi_i,k}_{m,\theta,\phi}(f(u,v);x,y) = (m+\phi_1+1)(m+\theta_1+1)\left(\frac{m+\theta}{m}\right)^m \sum_{j=0}^{m}\sum_{l=0}^{m-j} \Omega^{(\theta,\theta_2)}_{m,j,l}(x,y) \times \int_{\frac{j+\theta_3}{m+\theta_1+1}}^{\frac{j+\theta_3+1}{m+\theta_1+1}} \int_{\frac{l+\phi_3}{m+\phi_1+1}}^{\frac{l+\phi_3+1}{m+\phi_1+1}} \sum_{r=0}^{k} \frac{d^r f(u,v)}{r!} dudv, \quad (31)$$

where $d^r f(u,v) = \sum_{i=0}^{r} \binom{r}{i} \frac{\partial^r f(u,v)}{\partial x^{r-i}\partial y^i}(x-u)^{r-i}(y-v)^i$.

Remark 5.1. *It is remarked that the results analogous to Theorem 4.3 and the resulting corollaries can be easily deduced for the above k^{th} order generalizations* (26), (29) *and* (31).

Example 5. Since $f(x,y) = e^{-2x}y^3$ is infinitely continuously differentiable on $\mathbb{R}^2$, we can use Bernstein-Stancu-Kantorovich-Taylor operators to study the approximation of f on I^2. It is observed that, we achieve a better approximation by these operators in comparison to Bernstein-Stancu-Kantorovich operators, if we make a suitable choice of the parameters. For $m = 5$, $k = 2, 5$ and $\theta_3 = 1$, $\theta_2 = 2$, $\theta_1 = 3$, $\theta = 4$ and $\phi_3 = 1$, $\phi_2 = 2$, $\phi_1 = 3$, $\phi = 4$, the illustrative graphics of $\mathfrak{P}^{\theta_i,\phi_i}_{m,\theta,\phi}$, $\mathfrak{P}^{*\theta_i,\phi_i,2}_{m,\theta,\phi}$; $\mathfrak{P}^{*\theta_i,\phi_i,5}_{m,\theta,\phi}$ and the function $f(x,y) = e^{-2x}y^3$ are shown together in Fig. 5. From the estimates of the absolute maximum errors in the approximation of $f(x,y)$ by the operators $\mathfrak{P}^{\theta_i,\phi_i}_{m,\theta,\phi}$ in (30) and $\mathfrak{P}^{*\theta_i,\phi_i,k}_{m,\theta,\phi}$ in (31) for $m = 5$ and $k = 2, 5$ presented in Table 5, it turns out that as the value of k increases, the error becomes smaller.

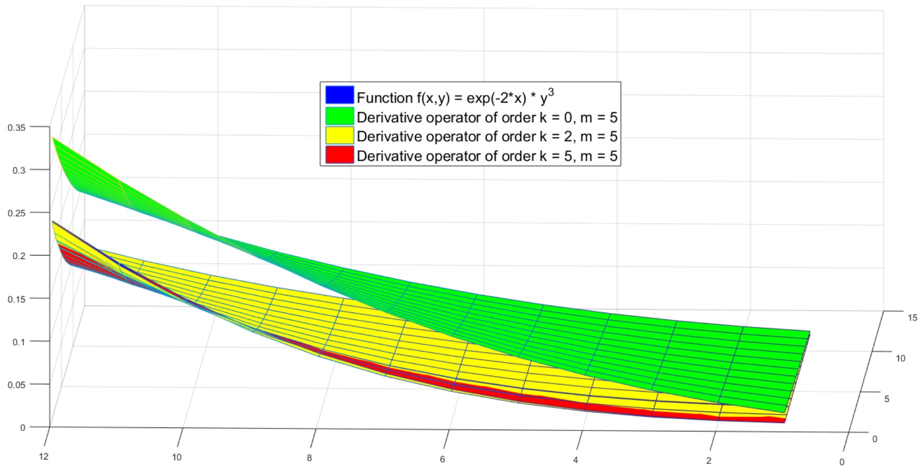

Figure 5: $\mathfrak{P}^{*\theta_i,\phi_i,k}_{m,\theta,\phi}(f)$ approximates $f(x,y)$ much better than $\mathfrak{P}^{\theta_i,\phi_i}_{m,\theta,\phi}(f)$

Table 5: Comparison of $\mathfrak{P}^{\theta_i,\phi_i}_{m,\theta,\phi}(f)$ and $\mathfrak{P}^{*\theta_i,\phi_i,k}_{m,\theta,\phi}(f)$ for $m = 5$ and some values of k

m	Error bound for $\mathfrak{P}^{\theta_i,\phi_i}_{m,\theta,\phi}$	Derivative order k	Error bound $\mathfrak{P}^{*\theta_i,\phi_i,k}_{m,\theta,\phi}$
5	0.1030	2	0.0228
5	0.1030	5	0.0002907

6. Conclusion

The Stancu-Kantorovich operators and the k^{th} order generalization of Bernstein-Stancu-Kantorovich type operators for functions of two variables are constructed with the help of modified Bernstein basis functions with shifted knots for $x, y \in [\frac{\theta_2}{n+\theta}, \frac{n+\theta_2}{n+\theta}] \times [\frac{\phi_2}{m+\phi}, \frac{m+\phi_2}{m+\phi}]$. By introducing the parameters $\theta, \phi, \theta_i, \phi_i, i = 1, 2, 3$ we enable the shift of Bernstein basis functions over the subintervals of I. A simulation was performed through MATLAB and it was shown that depending on the order of the derivative k, the k^{th} order generalization of Bernstein-Stancu-Kantorovich type polynomials $\mathfrak{K}^{\theta_i,\phi_i}_{n,m,\theta,\phi}(.; x, y)$ shows much better approximation results to a function compared to Bernstein-Stancu-Kantorovich operators which are presented in Examples 1 and 2. Finally, the k^{th} order generalizations of the generalized bivariate Bernstein type polynomials are studied and elaborated by means of some examples.

References

[1] P. N. Agrawal, B. Baxhaku, R. Chauhan, The approximation of bivariate Chlodowsky-Szasz-Kantorovich-Charlier-type operators, J Inequal Appl 195 (2017), https://doi.org/10.1186/s13660-017-1465-1.

[2] G. A. Anastassiou, S. G. Gal, Approximation Theory: Moduli of Continuity and Global Smoothness Preservation., Birkhäuser Boston, Inc., Boston, MA (2000).

[3] D. Bărbosu, Kantorovich-Schurer bivariate operators, Miskolc Math Notes 5 (2004) 129-136.

[4] B. Baxhaku, P. N. Agrawal, Degree of approximation for bivariate extension of Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators, Appl. Math. Comput. 306 (2017), 56-72.

[5] B. Baxhaku, A. Kajla, Blending type approximation by bivariate generalized Bernstein type operators. Quaes. Math., (2019) 1-17.

[6] S. N. Bernstein, Demonstration du theoreme de Weierstrass fondee sur le calcul de probabilities., Commun. Soc. Math Kharkow. 13(2) (1912) 1-2.

[7] J. Callahan, Advanced Calculus: A Geometric View. Undergraduate Texts in Mathematics Springer (2010).

[8] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, (1993).

[9] E. Dobrescu, I. Matei, The approximation by Bernstein type polynomials of bidimensional continuous functions., An. Univ. Timişoara Ser. Şti. Mat.-Fiz. 4 (1966) 85-90.

[10] A. D. Gadjiev, A. M. Ghorbanalizadeh, Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables., Appl. Math Comput. 216(3) (2010) 890-901.

[11] G. Içöz, A Kantorovich variant of a new type Bernstein-Stancu polynomials, Appl. Math. Comput. 218(17) (2012) 8552-8560.

[12] L. V. Kantorovich, Sur certains développements suivant les polynômes de la forme de S. bernstein., C. R Acad URSS. 20 (1930) 563-568.

[13] G. Kirov, I. Popova, A generalization of liner positive operators, Math Balkanica, 7 (1993) 149-162.

[14] S. Rahman, M. Mursaleen, A. M. Acu, Approximation properties of λ-Bernstein-Kantorovich operators with shifted knots., Math Meth. Appl. Sci. (2019) 1-12.

[15] D. D. Stancu, Approximation of functions by a new class of linear polynomial operators. Rev. Roumaine Math. Pures Appl, 13 (1968) 1173-1194.

[16] V. I. Volkov, On the convergence of sequences of linear positive operators in the space of continuous functions of two variables., Dokl. Akad. Nauk. SSSR (N.S) 115 (1957) 17-19.

[17] M. Wang, D. Yu, P. Zhou, On the approximation by operators of Bernstein-Stancu types. Appl. Math. Comput. 246(1) (2014) 79-87.