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Positive Solutions for Second-Order Impulsive Time Scale Boundary Value Problems on Infinite Intervals

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Abstract. In this paper, we investigate the existence of at least one, two and three positive solutions to the nonlinear second order m-point impulsive time scale boundary value problems on infinite intervals by using the Krasnosel'skii fixed point theorem, Avery-Henderson fixed point theorem and the five functionals fixed point theorem, respectively.

1. Introduction

The study of dynamic equations on time scales goes back to its founder Hilger [1]. A result for a dynamic equation contains simultaneously a corresponding result for a differential equation, one for a difference equation, as well as results for other dynamic equations in arbitrary time scales. Time scales theory presents us with the tools necessary to understand and explain the mathematical structure underpinning the theories of discrete and continuous dynamical systems and allows us to connect them. We refer the reader to the excellent introductory text by Bohner and Peterson [2] as well as their recent research monograph [3].

The theory of impulsive differential equations describe processes with experience a sudden change of their state at certain moments. The theory of impulsive differential equation has become important in recent years in mathematical model of real processes rising in phenomena studied in physics, chemical technology, population dynamics, ecology, biological systems, biotechnology, industrial robotics, optimal control, economics, and so forth. For the introduction of the theory of impulsive differential equations, we refer to the books [4–6]. Especially, the study of impulsive dynamic equations on time scales has also attracted much attention since it provides an unifying structure for differential equations in the continuous cases and finite difference equations in the discrete cases, see [7–25] and references therein. In recent years, there are a few authors studied the existence of positive solutions for time scale boundary value problems on infinite intervals.

Zhao, Ge [26] discussed the existence of at least three positive solutions for the nonlinear time scale boundary value problems

$$\begin{cases} (\varphi_p(u^\Delta(t)))^\nabla + q(t)f\left(u(t), u^\Delta(t)\right) = 0, \quad t \in [0,\infty)_{\mathbb{T}} \\ u(0) = \beta u^\Delta(\eta), \quad \lim_{t\to\infty} u^\Delta(t) = 0 \end{cases}$$

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by using Leggett-Williams fixed point theorem, where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$.

Zhao, Ge [27] considered the following m-point boundary value problem on time scale

$$\begin{cases} (\varphi_p(u^{\Delta}(t)))^{\nabla} + h(t)f\left(t, u(t), u^{\Delta}(t)\right) = 0, \ \ t \in [0,\infty)_{\mathbb{T}} \\ u(0) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i), \quad u^{\Delta}(\infty) = \sum\limits_{i=1}^{m-2} \beta_i u^{\Delta}(\eta_i), \end{cases}$$

where $u^{\Delta}(\infty) = \lim\limits_{t\to\infty} u^{\Delta}(t)$,$\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\eta_1, \eta_2, \ldots, \eta_{m-2} \in \mathbb{T}$, $\sigma(0) < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < \infty$, $\alpha_i \geq 0$, $\beta_i \geq 0$ for $i = 1, 2, \ldots, m-2$. They established the sufficient conditions for the existence of positive solutions by using Avery-Peterson theorem.

Karaca, Tokmak [28] studied the nonlinear p-Laplacian impulsive time scale boundary value problems

$$\begin{cases} (\varphi(x^{\Delta}(t)))^{\nabla} + \phi(t)f\left(t, x(t), x^{\Delta}(t)\right) = 0, \ \ t \in (0,\infty)_{\mathbb{T}} \\ x(0) = \sum\limits_{i=1}^{m-2} \alpha_i x^{\Delta}(\eta_i), \quad \lim\limits_{t\to\infty} x^{\Delta}(t) = 0, \end{cases}$$

where $f \in C\left([0,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty), [0,\infty)\right)$. $\alpha_i \geq 0$ $(1 \leq i \leq m-2)$ $0 < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < \infty$, $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\varphi(0) = 0$. They obtained the criteria for the existence of three positive solutions for m-point time scale boundary value problems on infinite intervals by using the Leggett-Williams fixed point theorem and five functionals fixed point theorem.

Yaslan, Haznedar [29] investigated the criteria for the existence of at least one, two and three positive solutions to the nonlinear impulsive time scale boundary value problems

$$\begin{cases} (\varphi(y^{\Delta}(t))^{\nabla} + h(t)f\left(t, y(t), y^{\Delta}(t)\right) = 0, \ \ t \in [a,\infty)_{\mathbb{T}}, \ \ t \neq t_k, \ \ k = 1, 2, ..., n \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \ \ k = 1, 2, ..., n \\ y(a) - \beta y^{\Delta}(a) = \sum\limits_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i), \quad \lim\limits_{t\to\infty} y^{\Delta}(t) = 0, \quad m \geq 3 \end{cases}$$

by using Leray-Schauder fixed point theorem, Avery-Henderson fixed point theorem and the five functional fixed point theorem, respectively, where $\beta \geq 0$, $\alpha_i \geq 0$ $(1 \leq i \leq m-2)$, $0 \leq a < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < \infty$, $f \in C\left([a,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty), [0,\infty)\right)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\varphi(0) = 0$.

Karaca, Sinanoglu [30] obtained the criteria for the existence of at least one positive solution to the m-point time scale boundary value problems

$$\begin{cases} (\varphi_p(u^{\Delta}(t))^{\nabla} + h(t)f\left(t, u(t), u^{\Delta}(t)\right) = 0, \ \ t \in (0,\infty)_{\mathbb{T}}, \ \ t \neq t_k, \ \ k = 1, 2, ..., n \\ u(0) = \sum\limits_{i=1}^{m-2} \alpha_i u^{\Delta}(\eta_i), \quad u^{\Delta}(\infty) = \sum\limits_{i=1}^{m-2} \beta_i u(\eta_i), \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \ \ \varphi_p(u^{\Delta}(t_k^+)) - \varphi_p(u^{\Delta}(t_k^-)) = -\bar{I}_k(u(t_k)), \ \ k \in \mathbb{N} \end{cases}$$

by using the four functionals fixed point theorem, where $u^{\Delta}(\infty) = \lim\limits_{t\to\infty} u^{\Delta}(t)$,$\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $I_k \in C([0,\infty), [0,\infty))$, $\bar{I}_k \in C([0,\infty), [0,\infty))$, $\eta_1, \eta_2, \ldots, \eta_{m-2} \in \mathbb{T}$, $\sigma(0) < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < \infty$.

We consider the following boundary value problem (BVP)

$$\begin{cases} y^{\Delta\nabla}(t) + h(t)f\left(t, y(t), y^{\Delta}(t)\right) = 0, \ \ t \in [a,\infty)_{\mathbb{T}}, \ \ t \neq t_k, \ \ k = 1, 2, ..., n \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \ \ k = 1, 2, ..., n \\ y(a) - \gamma y^{\Delta}(a) = \sum\limits_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i), \quad \lim\limits_{t\to\infty} y^{\Delta}(t) = \sum\limits_{i=1}^{m-2} \beta_i y(\eta_i), \quad m \geq 3 \end{cases} \tag{1}$$

where $\mathbb{T}$ is a time scale, $\alpha_i \geq 0$, $\beta_i \geq 0$ $(1 \leq i \leq m-2)$, $\gamma \geq 0$, $0 \leq a < \eta_1 < \ldots < \eta_{m-2} < \infty$ and $f \in C\left([a,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty), [0,\infty)\right)$.

We have organized the paper as follows. In Section 2, we give some preliminary lemmas which are key tools for our main results. In Section 3, we establish criteria for the existence of at least one positive solution for the BVP (1) by using the Krasnosel'skii fixed point theorem. In Section 4, Avery-Henderson fixed point theorem is used to investigate the existence of at least two positive solution of the BVP (1). Finally, we apply the five functionals fixed point theorem to prove the existence of at least three positive solutions to the BVP (1) in Section 5. The results are even new for the difference equations and differential equations as well as for dynamic equations on general time scales.

We will assume that the following conditions are satisfied:

(H1) $h \in C\left([a,\infty)_{\mathbb{T}}, [0,\infty)\right)$, $\int_a^\infty h(s)\nabla s < \infty$;

(H2) $f(t,(1+t)u,v) \leq \omega\left(\max\{|u|,|v|\}\right)$ with $\omega \in C\left([0,\infty),[0,\infty)\right)$ nondecreasing;

(H3) $\sum_{a<t_k<\infty} I_k(y(t_k)) < \infty$, $I_k \in C\left(\mathbb{R},\mathbb{R}^+\right)$, $t_k \in [a,\infty)_{\mathbb{T}}$ and $y(t_k^+) = \lim_{h\to 0} y(t_k+h)$, $y(t_k^-) = \lim_{h\to 0} y(t_k-h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1,...,n$.

2. Preliminaries

We now state and prove several lemmas to state the main results of this paper.

Lemma 2.1. *Assume (H3) holds. If $x \in C\left([a,\infty)_{\mathbb{T}}, [0,\infty)\right)$ and $\int_a^\infty x(t)\nabla t < \infty$, then the boundary value problem*

$$\begin{cases} y^{\Delta\nabla}(t) + x(t) = 0, \quad t \in [a,\infty)_{\mathbb{T}}, \quad t \neq t_k, \quad k = 1,2,...,n \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad k = 1,2,...,n \\ y(a) - \gamma y^{\Delta}(a) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i), \quad \lim_{t\to\infty} y^{\Delta}(t) = \sum_{i=1}^{m-2} \beta_i y(\eta_i), \quad m \geq 3 \end{cases}$$

has a unique solution

$$\begin{aligned} y(t) &= (\gamma - a)\int_a^\infty x(s)\nabla s + t\int_t^\infty x(s)\nabla s + \int_a^t s x(s)\nabla s + (\gamma + t - a)\sum_{i=1}^{m-2}\beta_i y(\eta_i) \\ &+ \sum_{i=1}^{m-2}\alpha_i\Big[\sum_{j=1}^{m-2}\beta_j y(\eta_j) + \int_{\eta_i}^\infty x(s)\nabla s\Big] + \sum_{a<t_k<t} I_k(y(t_k)). \end{aligned} \tag{2}$$

Proof. Since we have $y^{\Delta\nabla}(t) = -x(t)$ for $t \in [a,\infty)_{\mathbb{T}}$, we obtain

$$y^{\Delta}(t) = \lim_{t\to\infty} y^{\Delta}(t) + \int_t^\infty x(\xi)\nabla\xi. \tag{3}$$

From the second boundary condition we get

$$y^{\Delta}(t) = \sum_{i=1}^{m-2}\beta_i y(\eta_i) + \int_t^\infty x(\xi)\nabla\xi.$$

Integrating the above equality from a to t, we have

$$y(t) - y(a) - \sum_{a<t_k<t} I_k\left(y(t_k)\right) = (t-a)\sum_{i=1}^{m-2}\beta_i y(\eta_i) + \int_a^t\int_\xi^\infty x(s)\nabla s\Delta\xi.$$

From the first boundary condition we obtain

$$y(t)=\gamma y^{\Delta}(a)+\sum_{i=1}^{m-2}\alpha_i y^{\Delta}(\eta_i)+(t-a)\sum_{i=1}^{m-2}\beta_i y(\eta_i)+\int_a^t (s-a)x(s)\nabla s+(t-a)\int_t^{\infty} x(s)\nabla s+\sum_{a<t_k<t} I_k\left(y(t_k)\right).$$

Thus, from (3) we have (2).

□

By Lemma 2.1, the solutions of the BVP (1) are the fixed points of the operator A defined by

$$\begin{aligned} Ay(t) &= (\gamma-a)\int_a^{\infty} h(s)f\left(s,y(s),y^{\Delta}(s)\right)\nabla s+t\int_t^{\infty} h(s)f\left(s,y(s),y^{\Delta}(s)\right)\nabla s \\ &+ \int_a^t sh(s)f\left(s,y(s),y^{\Delta}(s)\right)\nabla s+(\gamma+t-a)\sum_{i=1}^{m-2}\beta_i y(\eta_i) \\ &+ \sum_{i=1}^{m-2}\alpha_i\Big[\sum_{j=1}^{m-2}\beta_j y(\eta_j)+\int_{\eta_i}^{\infty} h(s)f\left(s,y(s),y^{\Delta}(s)\right)\nabla s\Big]+\sum_{a<t_k<t} I_k(y(t_k)). \end{aligned}$$

Let $\mathcal{B}$ be the Banach space defined by

$$\mathcal{B}=\left\{y\in C^{\Delta}\left([a,\infty)\right):\sup_{t\in[a,\infty)_{\mathbb{T}}}\frac{y(t)}{1+t}<\infty,\ \ \lim_{t\to\infty} y^{\Delta}(t)=\sum_{i=1}^{m-2}\beta_i y(\eta_i)\right\}$$

with the norm $\|y\|=\max\left\{\|y\|_1,\|y^{\Delta}\|_{\infty}\right\}$, where

$$\|y\|_1=\sup_{t\in[a,\infty)_{\mathbb{T}}}\frac{|y(t)|}{1+t},\qquad \|y^{\Delta}\|_{\infty}=\sup_{t\in[a,\infty)_{\mathbb{T}}}|y^{\Delta}(t)|$$

and define the cone $P\subset\mathcal{B}$ by

$$\begin{aligned} P &= \Big\{y\in\mathcal{B}: y(a)-\gamma y^{\Delta}(a)=\sum_{i=1}^{m-2}\alpha_i y^{\Delta}(\eta_i),\ \ y \text{ is concave, non-decreasing and} \\ &\quad \text{nonnegative on } [a,\infty)_{\mathbb{T}}\Big\}. \end{aligned} \tag{4}$$

Lemma 2.2. *If $y\in P$, then we have $\|y\|_1\le M\|y^{\Delta}\|_{\infty}$, where*

$$M=\max\left\{\gamma-a+\sum_{i=1}^{m-2}\alpha_i,1\right\}. \tag{5}$$

Proof. For $y\in P$ and $t\in[a,\infty)_{\mathbb{T}}$, we have

$$\frac{y(t)}{1+t}=\frac{1}{1+t}\left(\int_a^t y^{\Delta}(s)\Delta s+\gamma y^{\Delta}(a)+\sum_{i=1}^{m-2}\alpha_i y^{\Delta}(\eta_i)\right)\le\frac{t-a+\gamma+\sum\limits_{i=1}^{m-2}\alpha_i}{1+t}\|y^{\Delta}\|_{\infty}\le M\|y^{\Delta}\|_{\infty}.$$

Hence, the proof is complete.

□

Lemma 2.3. *If (H1)-(H3) hold, then the operator $A : P \to P$ is completely continuous.*

Proof. First, we will show that $A : P \to P$. For $y \in P$, we have

$$(Ay)(a) - \gamma(Ay)^{\Delta}(a) = \sum_{i=1}^{m-2} \alpha_i (Ay)^{\Delta}(\eta_i),$$
$$(Ay)^{\Delta\nabla}(t) = -h(t) f\left(t, y(t), y^{\Delta}(t)\right) \le 0,$$
$$(Ay)^{\Delta}(t) = \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_t^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s \ge 0,$$
$$(Ay)(a) = \gamma\Big(\int_a^{\infty} x(s)\nabla s + \sum_{i=1}^{m-2} \beta_i y(\eta_i)\Big) + \sum_{i=1}^{m-2} \alpha_i \Big[\sum_{j=1}^{m-2} \beta_j y(\eta_j) + \int_{\eta_i}^{\infty} x(s)\nabla s\Big] + \sum_{a<t_k<t} I_k(y(t_k)) \ge 0.$$

Hence, $A : P \to P$.

Now, we will show that $A : P \to P$ is continuous. If $y_n \to y$ as $n \to \infty$ in P, then there exists τ such that $\sup_{n\in\mathbb{N}} \|y_n\| < \tau$. From (H2), for all $t \in [a,\infty)_{\mathbb{T}}$ we have $f\left(t, y_n(t), y_n^{\Delta}(t)\right) \le \omega\left(\max\left\{\frac{|y_n(t)|}{1+t}, |y_n^{\Delta}(t)|\right\}\right) \le \omega(\|y_n\|) < \omega(\tau)$ and $f\left(t, y(t), y^{\Delta}(t)\right) \le \omega(\|y\|) < \omega(\tau)$ by the continuity of norm function. Since

$$\int_t^{\infty} h(s)|f\left(s, y_n(s), y_n^{\Delta}(s)\right) - f\left(s, y(s), y^{\Delta}(s)\right)|\nabla s \le 2\omega(\tau) \int_a^{\infty} h(s)\nabla s < \infty$$

by using (H1), we get

$$\begin{aligned}\left|(Ay_n)^{\Delta}(t) - (Ay)^{\Delta}(t)\right| &\le \sum_{i=1}^{m-2} \left|\beta_i\Big(y_n(\eta_i) - y(\eta_i)\Big)\right| + \int_t^{\infty} h(s)\left|f\left(s, y_n(s), y_n^{\Delta}(s)\right) - f\left(s, y(s), y^{\Delta}(s)\right)\right|\nabla s \\ &\to 0, \quad n \to \infty\end{aligned}$$

by using the Lebesgue dominated convergence theorem. Hence, we obtain

$$\|(Ay_n)^{\Delta} - (Ay)^{\Delta}\|_{\infty} \to 0,$$

as $n \to \infty$. Since $\|Ay_n - Ay\| \le M\|(Ay_n)^{\Delta} - (Ay)^{\Delta}\|_{\infty} \to 0$, $A : P \to P$ is continuous.

Now we will show that the image of any bounded subset of P under A is relatively compact in P. If Ω is any bounded subset of P, then there exists $K > 0$ such that $\|y\| \le K$ for $\forall y \in \Omega$. By (H1) and (H2), for $\forall y \in \Omega$, we have

$$\|(Ay)^{\Delta}\|_{\infty} = \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s \le K \sum_{i=1}^{m-2} \beta_i(1+\eta_i) + \omega(K)\int_a^{\infty} h(s)\nabla s < \infty.$$

Since $\|A\Omega\| \le M\|(A\Omega)^{\Delta}\|_{\infty} < \infty$, $A\Omega$ is uniformly bounded.

Now, we show that $A\Omega$ is equicontinuous on $[a,\infty)_{\mathbb{T}}$. For any $R > 0$, $t, p \in [a,R]_{\mathbb{T}}$, and for all $y \in \Omega$, without loss of generality we may assume that $t < p$. By (H2), we have

$$\left|(Ay)^{\Delta}(t) - (Ay)^{\Delta}(p)\right| = \left|\int_t^{p} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right| \le \omega(K)\int_t^{p} h(s)\nabla s \to 0,$$

uniformly as $t \to p$. Since $\|(Ay)^{\Delta}(t) - (Ay)^{\Delta}(p)\|_{\infty} \to 0$, uniformly as $t \to p$, we obtain $\|(Ay)(t) - (Ay)(p)\| \to 0$, uniformly as $t \to p$, by Lemma 2.2. Thus, $A\Omega$ is equicontinuous on any compact interval of $[a,\infty)_{\mathbb{T}}$.

Now, we show that $A\Omega$ is equiconvergent on $[a,\infty)_{\mathbb{T}}$. For any $y \in \Omega$, we have

$$|(Ay)^{\Delta}(t) - (Ay)^{\Delta}(\infty)| = \left|\int_t^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right| \to 0$$

as $t \to \infty$. Then, we obtain $\|(Ay)(t)-(Ay)(\infty)\| \to 0$, as $t \to \infty$, by Lemma 2.2. Therefore $A\Omega$ is equiconvergent on $[a,\infty)_{\mathbb{T}}$.

Hence, the operator $A : P \to P$ is completely continuous.

□

3. Existence of at least one positive solution

To prove the existence of at least one positive solution for the BVP (1), we will apply the following Krasnosel'skii Fixed Point Theorem.

Theorem 3.1. *([31, Chapter 2]) (Krasnosel'skii Fixed Point Theorem) Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \ge \|u\|$ for $u \in K \cap \partial\Omega_2$;

or

(ii) $\|Au\| \ge \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \le \|u\|$ for $u \in K \cap \partial\Omega_2$

hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 3.2. *Suppose (H1)-(H3) hold. In addition, let there exist numbers $0 < r < R < \infty$ such that the function f satisfies the following conditions:*

(i) $f(t,(1+t)u,v) \le \frac{1}{N}\Big(\frac{1}{M} - \sum_{i=1}^{m-2}\beta_i(1+\eta_i)\Big)u(t)$ or $f(t,(1+t)u,v) \le \frac{1}{N}\Big(\frac{1}{M} - \sum_{i=1}^{m-2}\beta_i(1+\eta_i)\Big)v(t)$ for $(t,u,v) \in [a,\infty)_{\mathbb{T}} \times [0,r] \times [0,r]$;

(ii) $f(t,(1+t)u,v) \ge \frac{M}{N}v(a)$ for $(t,u,v) \in [a,\infty)_{\mathbb{T}} \times [0,R] \times [0,R]$,

where

$$N = \int_a^\infty h(s)\nabla s. \tag{6}$$

Then, the BVP (1) has at least one positive solution.

Proof. We apply the Krasnosel'skii Fixed Point Theorem to prove this theorem. Define the open bounded subsets of $\mathcal{B}$ by $\Omega_1 = \{y \in P : \|y\| < r\}$ and $\Omega_2 = \Big\{y \in P : \|y\| < R\Big\}$. $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is completely continuous operator from Lemma 2.3.

If $y \in P \cap \partial\Omega_1$, then $\|y\| = r$. Therefore, by using the hypothesis (i) and Lemma 2.2, we have

$$\begin{aligned}
\|Ay\| &\le M \sup_{t\in[a,\infty)_{\mathbb{T}}} |Ay^{\Delta}(t)| \\
&= M\Big(\sum_{i=1}^{m-2}\beta_i y(\eta_i) + \int_a^\infty h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\Big) \\
&\le M\Big(\|y\|\sum_{i=1}^{m-2}\beta_i(1+\eta_i) + \Big(\frac{1}{NM} - \frac{1}{N}\sum_{i=1}^{m-2}\beta_i(1+\eta_i)\Big)\|y\|_1 \int_a^\infty h(s)\nabla s\Big) \\
&\le \|y\|,
\end{aligned}$$

where $f(t,(1+t)u,v) \leq \left(\frac{1}{NM} - \frac{1}{N}\sum_{i=1}^{m-2}\beta_i(1+\eta_i)\right)u(t)$. If we get $f(t,(1+t)u,v) \leq \left(\frac{1}{NM} - \frac{1}{N}\sum_{i=1}^{m-2}\beta_i(1+\eta_i)\right)v(t)$, then by using the hypothesis (i) and Lemma 2.2, we have

$$\|Ay\| \leq M\left(\|y\|\sum_{i=1}^{m-2}\beta_i(1+\eta_i) + \frac{1}{N}\left(\frac{1}{M} - \sum_{i=1}^{m-2}\beta_i(1+\eta_i)\right)\|y^\Delta\|_\infty \int_a^\infty h(s)\nabla s\right) \leq \|y\|.$$

Thus, $\|Ay\| \leq \|y\|$ for $y \in P \cap \partial\Omega_1$. On the other hand, $y \in P \cap \partial\Omega_2$ implies $\|y\| = R$. Then, we have

$$\|Ay\| \geq \|(Ay)^\Delta\|_\infty = \sum_{i=1}^{m-2}\beta_i y(\eta_i) + \int_a^\infty h(s) f\left(s, y(s), y^\Delta(s)\right)\nabla s \geq \frac{M}{N} y^\Delta(a)\left(\int_a^\infty h(s)\nabla s\right) \geq \|y\|$$

from (ii) and Lemma 2.2. Consequently, $\|Ay\| \geq \|y\|$ for $y \in P \cap \partial\Omega_2$.

By the first part of Theorem 3.1, A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that $r \leq \|y\| \leq R$. Therefore BVP (1) has at least one positive solution.

□

4. Existence of at least two positive solutions

We will need also the following (Avery-Henderson) fixed point theorem [32] to prove the existence of at least two positive solutions for the BVP (1).

Theorem 4.1. *[32] Let P be a cone in a real Banach space E. Set*

$$P(\phi, r) = \{u \in P : \phi(u) < r\}.$$

If η and ϕ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive constants r and M,

$$\phi(u) \leq \theta(u) \leq \eta(u) \text{ and } \|u\| \leq M\phi(u)$$

for all $u \in \overline{P(\phi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \leq \lambda\theta(u), \text{ for all } 0 \leq \lambda \leq 1 \text{ and } u \in \partial P(\theta, q).$$

If $A : \overline{P(\phi, r)} \to P$ is a completely continuous operator satisfying
(i) $\phi(Au) > r$ for all $u \in \partial P(\phi, r)$,
(ii) $\theta(Au) < q$ for all $u \in \partial P(\theta, q)$,
(iii) $P(\eta, p) \neq \emptyset$ and $\eta(Au) > p$ for all $u \in \partial P(\eta, p)$,
then A has at least two fixed points u_1 and u_2 such that

$$p < \eta(u_1) \text{ with } \theta(u_1) < q \text{ and } q < \theta(u_2) \text{ with } \phi(u_2) < r.$$

Theorem 4.2. *Assume (H1)-(H3) hold. Suppose there exist numbers $0 < p < q < r$ such that the function f satisfies the following conditions:*
(i) $f(t,(1+t)u,v) > \frac{r}{N}$ for $(t,u,v) \in [a,\infty)_{\mathbb{T}} \times [0, Mr] \times [0, r]$,
(ii) $f(t,(1+t)u,v) < \frac{q}{MN}\left[1 - M\sum_{i=1}^{m-2}\beta_i(1+\eta_i)\right]$ for $(t,u,v) \in [a,\infty)_{\mathbb{T}} \times [0,q] \times [0,q]$,
(iii) $f(t,(1+t)u,v) > \frac{p}{N}$ for $(t,u,v) \in [a,\infty)_{\mathbb{T}} \times [0,p] \times [0,p]$,
where M and N are defined in (5) and (6), respectively. Then the BVP (1) has at least two positive solutions y_1 and y_2 such that

$$\|y_1\| > p \text{ with } \|y_1\| < q \text{ and } \|y_2\| > q \text{ with } y_2^\Delta(a) < r.$$

Proof. Define the cone P as in (4). From Lemma 2.3, $A : P \to P$ is completely continuous. Let the nonnegative increasing continuous functionals ϕ, θ and η be defined on the cone P by

$$\phi(y) := y^{\Delta}(a), \ \theta(y) := \|y\|, \ \eta(y) := \|y\|.$$

For each $y \in P$, we have $\phi(y) \leq \theta(y) = \eta(y)$ and from Lemma 2.2 we have

$$\|y\| \leq M\|y^{\Delta}\|_{\infty} = My^{\Delta}(a) = M\phi(y).$$

In addition, $\theta(0) = 0$ and for all $y \in P$, $\lambda \in [0, 1]$ we get $\theta(\lambda y) = \lambda\theta(y)$. We now verify that all of the conditions of Theorem 4.1 are satisfied.

If $y \in \partial P(\phi, r)$, for $s \in [a, \infty)_{\mathbb{T}}$ we have $0 \leq y^{\Delta}(s) \leq r$ and $0 \leq \frac{y(s)}{1+s} \leq Mr$ from Lemma 2.2. Then, from the hypothesis (i) and (6), we find

$$\phi(Ay) = \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s > \frac{r}{N} \int_a^{\infty} h(s) \nabla s = r.$$

Thus, the condition (i) of Theorem 4.1 holds.

If $y \in \partial P(\theta, q)$, we have $0 \leq \frac{y(s)}{1+s} \leq q$ and $0 \leq y^{\Delta}(s) \leq q$ for $s \in [a, \infty)_{\mathbb{T}}$. Then, we obtain

$$\begin{aligned}
\theta(Ay) &\leq M \sup_{t \in [a,\infty)_{\mathbb{T}}} |Ay^{\Delta}(t)| \\
&= M\Big(\sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s\Big) \\
&\leq M\Big(q \sum_{i=1}^{m-2} \beta_i (1 + \eta_i) + \frac{q}{MN}\Big(1 - M \sum_{i=1}^{m-2} \beta_i (1 + \eta_i)\Big) \int_a^{\infty} h(s) \nabla s\Big) \\
&< q
\end{aligned}$$

by hypothesis (ii), (6) and Lemma 2.2. Hence the condition (ii) of Theorem 4.1 is satisfied.

Since $0 \in P$ and $p > 0$, $P(\eta, p) \neq \emptyset$. If $y \in \partial P(\eta, p)$, we have $0 \leq \frac{y(s)}{1+s} \leq p$ and $0 \leq y^{\Delta}(s) \leq p$ for $s \in [a, \infty)_{\mathbb{T}}$. Then, we get

$$\eta(Ay) \geq \left\|(Ay)^{\Delta}\right\|_{\infty} = \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s > \frac{p}{N}\Big(\int_a^{\infty} h(s) \nabla s\Big) = p$$

using hypothesis (iii) and (6). Since all the conditions of Theorem 4.1 are fulfilled, the BVP (1) has at least two positive solutions y_1 and y_2 such that

$$\|y_1\| > p \text{ with } \|y_1\| < q \ \text{ and } \ \|y_2\| > q \text{ with } y_2^{\Delta}(a) < r.$$

□

5. Existence of at least three positive solutions

We will present the five functionals fixed point theorem. Let φ, η, θ be nonnegative continuous convex functionals on the cone P, and γ, ψ nonnegative continuous concave functionals on the cone P. For nonnegative numbers h, p, q, r and d, define the following convex sets:

$$\begin{aligned} P(\varphi, r) &= \{x \in P : \varphi(x) < r\}, \\ P(\varphi, \gamma, p, r) &= \{x \in P : p \leq \gamma(x), \varphi(x) \leq r\}, \\ Q(\varphi, \eta, d, r) &= \{x \in P : \eta(x) \leq d, \varphi(x) \leq r\}, \\ P(\varphi, \theta, \gamma, p, q, r) &= \{x \in P : p \leq \gamma(x), \theta(x) \leq q, \varphi(x) \leq r\}, \\ Q(\varphi, \eta, \psi, h, d, r) &= \{x \in P : h \leq \psi(x), \eta(x) \leq d, \varphi(x) \leq r\}. \end{aligned} \tag{7}$$

Theorem 5.1. *([33])(Five Functionals Fixed Point Theorem) Let P be a cone in a real Banach space E. Suppose that there exist nonnegative numbers r and M, nonnegative continuous concave functionals γ and ψ on P, and nonnegative continuous convex functionals φ, ϑ and θ on P, with*

$$\gamma(x) \leq \vartheta(x), \|x\| \leq M\varphi(x), \forall x \in \overline{P(\varphi, r)}.$$

Suppose that $A : \overline{P(\varphi, r)} \to \overline{P(\varphi, r)}$ is a completely continuous and there exist nonnegative numbers h, p, k, q, with $0 < p < q$ such that

(i) $\{x \in P(\varphi, \theta, \gamma, q, k, r) : \gamma(x) > q\} \neq \emptyset$ and $\gamma(Ax) > q$ for $x \in P(\varphi, \theta, \gamma, q, k, r)$,

(ii) $\{x \in Q(\varphi, \vartheta, \psi, h, p, r) : \vartheta(x) < p\} \neq \emptyset$ and $\vartheta(Ax) < p$ for $x \in Q(\varphi, \vartheta, \psi, h, p, r)$,

(iii) $\gamma(Ax) > q$, for $x \in P(\varphi, \gamma, q, r)$, with $\theta(Ax) > k$,

(iv) $\vartheta(Ax) < p$, for $x \in Q(\varphi, \vartheta, p, r)$, with $\psi(Ax) < h$,

then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, r)}$ such that

$$\vartheta(x_1) < p, \quad \gamma(x_2) > q, \quad \vartheta(x_3) > p \quad \text{with } \gamma(x_3) < q.$$

Define the constant

$$\lambda = \int_{\frac{1}{k}}^{k} h(s)\nabla s. \tag{8}$$

Now, we will apply the five functionals fixed point theorem to investigate the existence of at least three positive solutions for the BVP (1).

Theorem 5.2. *Assume (H1)-(H3) hold and $\gamma - a \geq 1$, $\frac{1}{k} \in \mathbb{T}$. Suppose that there exist constants $0 < p < q < r$ such that the function f satisfies the following conditions:*

(i) $f(t, (1+t)u, v) > \frac{q(k+1)}{\lambda[k(\gamma - a) + 1]}$ for $(t, u, v) \in [\frac{1}{k}, k]_{\mathbb{T}} \times [\frac{q}{k}, r] \times [0, r]$,

(ii) $f(t, (1+t)u, v) < \frac{p}{MN}\Big[1 - M\sum_{i=1}^{m-2} \beta_i(1+\eta_i)\Big]$ for $(t, u, v) \in [a, \infty)_{\mathbb{T}} \times [0, p] \times [0, p]$,

(iii) $f(t, (1+t)u, v) \leq \frac{r}{MN}\Big[1 - M\sum_{i=1}^{m-2} \beta_i(1+\eta_i)\Big]$ for $(t, u, v) \in [a, \infty)_{\mathbb{T}} \times [0, r] \times [0, r]$,

where M and N are defined in (5) and (6), respectively. Then the BVP (1) has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\|y_1\| < p < \|y_3\|, \text{ and } \frac{k}{k+1} \min_{t \in [\frac{1}{k}, \infty)_{\mathbb{T}}} y_3(t) < q < \frac{k}{k+1} \min_{t \in [\frac{1}{k}, \infty)_{\mathbb{T}}} y_2(t).$$

Proof. Define the cone P as in (4) and define these maps

$$\gamma(y)=\frac{k}{k+1}\min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}}y(t),\quad \varphi(y)=\vartheta(y)=\theta(y)=\|y\|,\quad \psi(y)=0.$$

Then γ and ψ are nonnegative continuous concave functionals on P, and φ,ϑ and θ are nonnegative continuous convex functionals on P. Let $P(\varphi,r)$, $P(\varphi,\gamma,p,r)$, $Q(\varphi,\vartheta,d,r)$, $P(\varphi,\theta,\gamma,p,q,r)$ and $Q(\varphi,\vartheta,\psi,h,d,r)$ be defined by (7). We have

$$\gamma(y)=\frac{k}{k+1}\min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}}y(t)=\frac{1}{1+\frac{1}{k}}\min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}}y(t)\le\sup_{t\in[a,\infty)_{\mathbb{T}}}\frac{|y(t)|}{1+t}=\|y\|_1\le\|y\|=\vartheta(y)$$

and $\|y\|=\varphi(y)$ for all $y\in\overline{P(\varphi,r)}$.

If $y\in\overline{P(\varphi,r)}$, then we have $0\le\frac{y(t)}{1+t}\le r$ and $0\le y^{\Delta}(t)\le r$ for all $t\in[a,\infty)_{\mathbb{T}}$. By hypothesis (*iii*) and Lemma 2.2, we find

$$\begin{aligned}\varphi(Ay) &\le M\sup_{t\in[a,\infty)_{\mathbb{T}}}|Ay^{\Delta}(t)|\\ &= M\Big(\sum_{i=1}^{m-2}\beta_i y(\eta_i)+\int_a^{\infty}h(s)f\left(s,y(s),y^{\Delta}(s)\right)\nabla s\Big)\\ &\le M\Big(r\sum_{i=1}^{m-2}\beta_i(1+\eta_i)+\frac{r}{MN}\Big(1-M\sum_{i=1}^{m-2}\beta_i(1+\eta_i)\Big)\int_a^{\infty}h(s)\nabla s\Big)\\ &= r.\end{aligned}$$

Then, we have $A:\overline{P(\varphi,r)}\to\overline{P(\varphi,r)}$.

Now we verify that the remaining conditions of Theorem 5.1.

Let $y_1(t)=\frac{q+r}{2}(t+1)+\frac{q+r}{2}\Big(\sum_{i=1}^{m-2}\alpha_i+\gamma-(a+1)\Big)$ for $t\in[a,\infty)_{\mathbb{T}}$. Since we get $\gamma(y_1)=\frac{k}{k+1}\Big[\frac{q+r}{2}\Big(\frac{1}{k}+1\Big)+\frac{q+r}{2}\Big(\sum_{i=1}^{m-2}\alpha_i+\gamma-(a+1)\Big)\Big]\ge\frac{q+r}{2}>q$, $\theta(y_1)\le\frac{q+r}{2}$ and $\varphi(y_1)<r$, we obtain $\{y\in P(\varphi,\theta,\gamma,q,\frac{q+r}{2},r):\gamma(y)>q\}\neq\emptyset$.

If $y\in P(\varphi,\theta,\gamma,q,\frac{q+r}{2},r)$, then we have $\frac{q}{k}\le\frac{y(t)}{1+t}\le r$ and $0\le y^{\Delta}(t)\le r$ for all $t\in[\frac{1}{k},k]$. By the hypothesis (*i*), we obtain

$$\gamma(Ay)\ge\frac{k}{k+1}(\gamma+\frac{1}{k}-a)\int_{\frac{1}{k}}^{\infty}h(s)f(s,y(s),y^{\Delta}(s))\nabla s>\frac{k}{k+1}(\gamma+\frac{1}{k}-a)\frac{q(k+1)}{\lambda[k(\gamma-a)+1]}\int_{\frac{1}{k}}^{k}h(s)\nabla s=q.$$

Then, we have

$$\gamma(Ay)>q. \tag{9}$$

Thus, the condition (*i*) of Theorem 5.1 is fulfilled.

Let $y_2(t)=\frac{p}{2}(t+1)+\frac{p}{2}\Big(\sum_{i=1}^{m-2}\alpha_i+\gamma-(a+1)\Big)$ for $t\in[a,\infty)_{\mathbb{T}}$. Since $\vartheta(y_2)<p$, $\varphi(y_2)<r$ and $\psi(y_2)=0=l$, we find $\{y\in Q(\varphi,\vartheta,\psi,l,p,r):\vartheta(y)<p\}\neq\emptyset$. If $y\in Q(\varphi,\vartheta,\psi,l,p,r)$, then we obtain $0\le\frac{y(t)}{1+t}\le p$ and $0\le y^{\Delta}(t)\le p$ for $t\in[a,\infty)_{\mathbb{T}}$. Hence, we have

$$\begin{aligned}\vartheta(Ay) &\leq M \sup_{t\in[a,\infty)_{\mathbb{T}}} |Ay^{\Delta}(t)| \\ &\leq M\Big(p\sum_{i=1}^{m-2}\beta_i(1+\eta_i)+\frac{p}{MN}\Big(1-M\sum_{i=1}^{m-2}\beta_i(1+\eta_i)\Big)\int_a^{\infty}h(s)\nabla s\Big) \\ &= p.\end{aligned}$$

by the hypothesis (*ii*) and Lemma 2.2. It follows that condition (*ii*) of Theorem 5.1 holds.

Now, we shall show that the condition (*iii*) of Theorem 5.1 is satisfied. If $y \in P(\varphi,\gamma,q,r)$, then for all $t \in [\frac{1}{k},k]_{\mathbb{T}}$ we have $\frac{q}{k} \leq \frac{y(t)}{1+t} \leq r$ and $0 \leq y^{\Delta}(t) \leq r$. According to (9), we have $\gamma(Ay) > q$. Thus, the condition (*iii*) of Theorem 5.1 holds.

Finally, we shall verify that the condition (*iv*) of Theorem 5.1 holds. Since $\psi(Ay) < l = 0$ is impossible, we omit the condition (*iv*) of Theorem 5.1.

Since all the conditions of Theorem 5.1 are satisfied, the BVP (1) has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\|y_1\| < p < \|y_3\|, \text{ and } \frac{k}{k+1}\min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}} y_3(t) < q < \frac{k}{k+1}\min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}} y_2(t).$$

This completes the proof.

□

Example 5.3. *Let $\mathbb{T} = [0,4] \cup \{5,6\} \cup [7,\infty)$. Consider the following boundary value problem:*

$$\begin{cases} y^{\Delta\nabla}(t) + \frac{1}{(1+t)^2} f\left(t, y(t), y^{\Delta}(t)\right) = 0, \ \ t \neq \frac{1}{3}, \ \ t \in [0,\infty) \subset \mathbb{T} \\ y(\frac{1}{3}^{+}) - y(\frac{1}{3}^{-}) = 4, \\ y(0) - y^{\Delta}(0) = \frac{1}{10} y^{\Delta}\left(\frac{1}{2}\right) + \frac{1}{10} y^{\Delta}(\frac{1}{3}), \ \ \lim_{t\to\infty} y^{\Delta}(t) = \frac{1}{4} y\left(\frac{1}{2}\right) + \frac{1}{3} y(\frac{1}{3}), \end{cases}$$

where

$$f(t,(1+t)u,v) = \begin{cases} \frac{t}{1+t^2}\Big(u^4 + \frac{v}{3.10^4}\Big); u < 1, \ v \geq 0, \ t \in \mathbb{T}, \\ \frac{t}{1+t^2}\Big(480 + \frac{v}{3.10^4}\Big); u \geq 1, \ v \geq 0 \ t \in \mathbb{T}. \end{cases}$$

Taking $h(t) = \frac{1}{(1+t)^2}$, $a = 0$, $\gamma = 1$, $\alpha_1 = \alpha_2 = \frac{1}{10}$, $\eta_1 = \frac{1}{2}$, $t_1 = \eta_2 = \beta_2 = \frac{1}{3}$, $\beta_1 = \frac{1}{4}$ and $k = 4$, we have $M = 1.2$, $N \approx 0.9888$ and $\lambda = 0.6$. If we take $p = 0.1, q = 4$ and $r = 3.10^5$, then all the conditions in Theorem 5.2 are verified. Thus, the BVP has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\|y_1\| < 0.1 < \|y_3\|, \textit{ and } \frac{1}{5}\min_{t\in[\frac{1}{4},\infty)_{\mathbb{T}}} y_3(t) < 1 < \frac{1}{5}\min_{t\in[\frac{1}{4},\infty)_{\mathbb{T}}} y_2(t).$$

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