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Further Mean Inequalities via some Integral Inequalities

Mustapha Raïssouli[a], Mohamed Chergui[b]

[a]*Department of Mathematics, Faculty of Science, Moulay Ismail University, Meknes, Morocco.*
[b]*Department of Mathematics, CRMEF-Rabat-Sale-Kenitra, EREAM Team, LaREAMI-Lab, Kenitra, Morocco.*

Abstract. Recently intensive efforts are deployed to establish inequalities involving some bivariate means. In this paper we aim to investigate further mean inequalities. The approach employed here is based on a combination of the integral representations of the involved means with some advanced integral inequalities known in the literature. In particular, applying the Grüss inequality and the Ostrowski inequality we derive a lot of estimations for differences between some standard means and weighted means.

1. Introduction

Means theory is attracting researchers increasingly by virtue of its theoretical and practical utility. By a (bivariate) mean we understand a binary map $m(a,b)$, with $a,b>0$, satisfying the following condition

$$\forall a,b>0 \qquad \min(a,b)\le m(a,b)\le \max(a,b). \tag{1}$$

Obviously, every mean satisfies $m(a,a)=a$ for any $a>0$. Symmetric and homogeneous means are defined in the habitual way.

• Let $a,b>0$. Among the standard means we mention the following

$$a\nabla b=\frac{a+b}{2};\ a!b=\frac{2ab}{a+b};\ a\sharp b=\sqrt{ab};$$
$$L(a,b)=\frac{a-b}{\log a-\log b},\ L(a,a)=a;\ I(a,b)=\frac{1}{e}\Big(\frac{b^b}{a^a}\Big)^{1/(b-a)},\ I(a,a)=a,$$

which are known in the literature as the arithmetic mean, harmonic mean, geometric mean, logarithmic mean and identric mean, respectively. All the previous means are symmetric and homogeneous. Further, these means satisfy the following chain of inequalities

$$\forall a,b>0 \qquad a!b\le a\sharp b\le L(a,b)\le I(a,b)\le a\nabla b, \tag{2}$$

with equalities if and only if $a=b$. For further other means, not needed here, we can consult [22, 23] for instance.

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Email addresses: raissouli.mustapha@gmail.com (Mustapha Raïssouli), chergui_m@yahoo.fr (Mohamed Chergui)

• Let $\lambda \in [0,1]$. The following expressions

$$a\nabla_\lambda b = (1-\lambda)a + \lambda b, \quad a!_\lambda b = \Big(a^{-1}\nabla_\lambda b^{-1}\Big)^{-1}, \quad a\sharp_\lambda b = a^{1-\lambda}b^\lambda \tag{3}$$

are known as the $\lambda-$weighted arithmetic mean, the $\lambda-$weighted harmonic mean and the $\lambda-$weighted geometric mean, respectively. These means are homogeneous but not symmetric unless $\lambda = 1/2$ that corresponds to the previous symmetric means, respectively. These weighted means satisfy the so-called weighted arithmetic-geometric-harmonic mean inequality, namely the following inequalities

$$a!_\lambda b \le a\sharp_\lambda b \le a\nabla_\lambda b \tag{4}$$

hold for any $a, b > 0$ and $\lambda \in [0,1]$. For the construction of some weighted logarithmic means as well as some weighted identric means we can consult [9, 21, 26].

• The right inequality in (4), also known as Young's inequality, has been refined and reversed in the literature. For instance, the following result has been proved in [12],

$$2r_\lambda\Big(a\nabla b - a\sharp b\Big) \le a\nabla_\lambda b - a\sharp_\lambda b \le 2(1-r_\lambda)\Big(a\nabla b - a\sharp b\Big), \tag{5}$$

where we set $r_\lambda =: \min(\lambda, 1-\lambda)$. A refinement of (5) has been established in [17] as recited in the following

$$2r_\lambda\Big(a\nabla b - a\sharp b\Big) + A(\lambda)\Big(\log(a/b)\Big)^2 \le a\nabla_\lambda b - a\sharp_\lambda b \le 2(1-r_\lambda)\Big(a\nabla b - a\sharp b\Big) + B(\lambda)\Big(\log(a/b)\Big)^2, \tag{6}$$

where we set

$$A(\lambda) =: \frac{\lambda(1-\lambda)}{2} - \frac{r_\lambda}{4}, \quad B(\lambda) =: \frac{\lambda(1-\lambda)}{2} - \frac{1-r_\lambda}{4}. \tag{7}$$

Remark that, if $\lambda = 1/2$ then (5) and (6) are both reduced to an equality.

For further refinements and reverses of the Young's inequality, we refer the interested reader to [8, 16, 28] and the related references cited therein.

• Another mean playing an important place in mean-theory is the so-called Heinz mean defined by

$$HZ_\lambda(a,b) = \frac{a\sharp_\lambda b + a\sharp_{1-\lambda} b}{2}. \tag{8}$$

Such mean interpolates the arithmetic and geometric means in the sense that the following inequalities

$$a\sharp b \le HZ_\lambda(a,b) \le a\nabla b \tag{9}$$

hold true for all $a, b > 0$ and $\lambda \in [0,1]$. A reverse of the right inequality in (9) has been proved in [13] and reads as follows

$$HZ_\lambda(a,b) \ge a\nabla b - \frac{1}{2}\lambda(1-\lambda)(a-b)\log(a/b). \tag{10}$$

• The idea of representing means with integrals was very useful in determining strong results in the case of real arguments, as well as in extending the study to operators and convex functionals, see [7, 14, 22–25, 27, 31] for instance.

The following integral formulas, that can be found in several references like [22], will be needed in the sequel.

$$L(a,b) = \int_0^1 a\sharp_t b \, dt, \tag{11}$$

$$L(a,b) = \Big(\int_0^1 \Big(a\nabla_t b\Big)^{-1} dt\Big)^{-1}, \tag{12}$$

$$I(a,b)=\exp\left(\int_0^1 \log(a\nabla_t b)\,dt\right), \tag{13}$$

$$a\sharp_\lambda b=\frac{\sin(\lambda\,\pi)}{\pi}\int_0^1 \frac{a!_t b}{t\sharp_\lambda(1-t)}\,dt. \tag{14}$$

This paper will be organized as follows: Section 2 provides some needed notions about the mean-theory involving matrix/operator argument. In section 3 we collect some integral inequalities, namely the Chebychev inequality, the Grüss inequality and the Ostrowski inequality that will be used for establishing our main results. Section 4 is devoted to give a lot of inequalities involving the logarithmic mean $L(a,b)$. Section 5 is focused to investigate some estimations about the mean-differences $L(a,b)-a\sharp_\lambda b$ and $L(a,b)-a\nabla_\lambda b$ by using the integral forms of $L(a,b)$, namely (11) and (12), respectively. In Section 6 we present some estimations about the difference $a\nabla_\lambda b-a\sharp_\lambda b$ when we use the integral formula (14). Section 7 deals with some bounds for the ratio $a\nabla_\lambda b/I(a,b)$ by utilizing (13). Finally, in section 8 we present some operator mean-inequalities.
Some obtained results in this paper are refinements and reverses for some mean-inequalities existing in the literature.

2. About operator means

For over the last half century the mean-theory has been extended from the case where the variables are positive real numbers to the case where the variables are positive operators. In this section we will recall some basic notions about the mean-theory involving matrix/operator argument that will be needed in the sequel.

Let $(H,\langle.,.\rangle)$ be a complex Hilbert space and let $\mathcal{B}(H)$ be the $\mathbb{C}^*$-algebra of bounded linear operators acting on H. As usual, a self-adjoint operator $A\in\mathcal{B}(H)$ is called positive (in short $A\geq 0$) if $\langle Ax,x\rangle\geq 0$ for any $x\in\mathbb{C}^n$. We say that A is strictly positive (in short $A>0$) if A is positive invertible. The notation $A\leq B$ refers to the so-called Löwner operator order which means that $A,B\in\mathcal{B}(H)$ are self-adjoint and $B-A\geq 0$. We denote by $\mathcal{B}^{+*}(H)$ the open cone of all (self-adjoint) strictly positive operators in $\mathcal{B}(H)$. A real-valued function f, defined on a nonempty interval J of $\mathbb{R}$, is said to be operator monotone if and only if $A\leq B$ implies $f(A)\leq f(B)$ for any self-adjoint operators A and B whose spectral $\sigma(A),\sigma(B)\subset J$. As usual, $f(A)$ is defined by the techniques of functional calculus. For further details about operator monotone functions we can consult [2, 11, 29, 30] and the related references cited therein.

Following the Kubo-Ando theory [14], there exists a unique one-to-one correspondence between operator means and operator monotone functions. An operator mean m in the Kubo-Ando sense is such that

$$AmB=A^{1/2}f_m\Big(A^{-1/2}BA^{-1/2}\Big)A^{1/2},\ f_m(1)=1 \tag{15}$$

for some positive monotone increasing function f_m defined on $(0,\infty)$. The function f_m is called the representing function of the operator mean m. An operator mean in the Kubo-Ando sense is usually called operator monotone mean.

Now, let us recall the operator versions of some standard means needed in this paper. Let $A,B\in\mathcal{B}^{+*}(H)$ and $\lambda\in[0,1]$. The following

$$A\nabla_\lambda B=(1-v)A+vB,\ A!_\lambda B=\Big((1-\lambda)A^{-1}+\lambda B^{-1}\Big)^{-1},\ A\sharp_\lambda B=A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{\lambda}A^{1/2},$$

are known in the literature as the λ-weighted arithmetic mean, the λ-weighted harmonic mean and the λ-weighted geometric mean of A and B, respectively. If $\lambda=1/2$ they are simply denoted by $A\nabla B$, $A!B$ and $A\sharp B$, respectively. These weighted operator means satisfy the following inequalities

$$A!_\lambda B\leq A\sharp_\lambda B\leq A\nabla_\lambda B. \tag{16}$$

Inspired by (11) and (12), the logarithmic mean of two strictly positive operators $A, B \in \mathcal{B}^{+*}(H)$ can be, alternatively, defined by one of the following formulas

$$L(A, B) = \int_0^1 A\sharp_t B\, dt, \tag{17}$$

$$L(A, B) = \Big(\int_0^1 \Big(A\nabla_t B\Big)^{-1} dt\Big)^{-1}. \tag{18}$$

In fact, since (11) and (12) coincide for all $a, b > 0$ then, by (15) and the techniques of Functional Calculus, we deduce that (17) and (18) coincide for any $A, B \in \mathcal{B}^{+*}(H)$ and we then have

$$L(A, B) = A^{1/2} f_L\Big(A^{-1/2}BA^{-1/2}\Big)A^{1/2},$$

with $f_L(T) = L\Big(\mathcal{I}, T\Big)$ for any $T \in \mathcal{B}^{+*}(H)$, where $\mathcal{I}$ refers to the identity operator of $\in \mathcal{B}(H)$.

By the same arguments as previous, the arithmetic-logarithmic-geometric inequalities (2), i.e. $a\sharp b \leq L(a, b) \leq a\nabla b$, persists for operator arguments and we have, for any $A, B \in \mathcal{B}^{+*}(H)$,

$$A\sharp B \leq L(A, B) \leq A\nabla B \tag{19}$$

We also have, for any $A, B \in \mathcal{B}^{+*}(H)$,

$$A\sharp_\lambda B = \frac{\sin(\lambda\,\pi)}{\pi} \int_0^1 \frac{A!_t B}{t\sharp_\lambda(1-t)}\, dt. \tag{20}$$

The following lemma will be needed in the sequel.

Lemma 2.1. *For any $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0, 1]$ we have*

$$A\nabla_\lambda B - A\sharp_\lambda B = \frac{\sin(\pi\,\lambda)}{\pi} \int_0^1 (1-t)\sharp_\lambda t\, \frac{A\nabla_t B - A!_t B}{t(t-1)}\, dt. \tag{21}$$

Proof. Let Γ and β denote the standard special functions Gamma and Beta, respectively. We have

$$\frac{\sin(\pi\lambda)}{\pi} \int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda} dt = \frac{\sin(\pi\lambda)}{\pi} \beta\Big(\lambda, 1-\lambda\Big) = \frac{\sin(\pi\lambda)}{\pi}\Gamma(\lambda)\Gamma(1-\lambda) = 1.$$

This, with the definition of $A\sharp_\lambda B$, yields

$$A\nabla_\lambda B - A\sharp_\lambda B = \frac{\sin(\pi\lambda)}{\pi} \int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda}\Big(A\nabla_\lambda B - A!_t B\Big)dt.$$

Since $A\nabla_\lambda B = (1-\lambda)A + \lambda B$, with the following

$$\frac{\sin(\pi\lambda)}{\pi} \int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda}(1-t)dt = \frac{\sin(\pi\lambda)}{\pi}\beta\Big(\lambda, 2-\lambda\Big)$$
$$= \frac{\sin(\pi\lambda)}{\pi}\Gamma(\lambda)\Gamma(2-\lambda) = \frac{\sin(\pi\lambda)}{\pi}(1-\lambda)\Gamma(\lambda)\Gamma(1-\lambda) = 1-\lambda$$

and by similar way,

$$\frac{\sin(\pi\lambda)}{\pi} \int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda}\, t\, dt = \lambda.$$

We then get

$$A\nabla_\lambda B - A\sharp_\lambda B = \frac{\sin(\pi\lambda)}{\pi} \int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda}\Big(A\nabla_t B - A\sharp_t B\Big)dt,$$

and (21) follows after a simple algebraic manipulation. □

Remark 2.2. *It was also possible to show* (21) *for scalar numbers $a, b > 0$ and then deduce* (21) *for operator arguments by using* (15) *and the standard techniques of Functional Calculus, which consist to multiply at left and at right by $A^{1/2}$ the related formulas obtained via* (15)*. The proof of the following lemma explains more this latter point.*

Lemma 2.3. *Let $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$. Then we have*

$$A\nabla_\lambda B - A!_p B = \lambda(1-\lambda)(A-B)\Big(A\nabla_{1-\lambda}B\Big)^{-1}(A-B). \tag{22}$$

Proof. It is easy to see that the equality

$$(1-\lambda) + \lambda x - \Big(1-\lambda+\lambda x^{-1}\Big)^{-1} = \lambda(1-\lambda)\frac{(x-1)^2}{\lambda+(1-\lambda)x}$$

holds for any $x > 0$ and $\lambda \in [0,1]$. Using the standard weighted means this latter equality becomes

$$1\nabla_\lambda x - 1!_\lambda x = \lambda(1-\lambda)(1-x)\Big(1\nabla_{1-\lambda}x\Big)^{-1}(1-x).$$

By the principle of Functional Calculus, the operator equality

$$\mathcal{I}\nabla_\lambda X - \mathcal{I}!_\lambda X = \lambda(1-\lambda)(\mathcal{I}-X)\Big(\mathcal{I}\nabla_{1-\lambda}X\Big)^{-1}(\mathcal{I}-X)$$

holds for any $X \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$. If in this latter equality we replace X by $A^{-1/2}BA^{-1/2} \in \mathcal{B}^{+*}(H)$ and we apply the Kubo-Ando theory [14] then we get

$$A^{-1/2}\Big(A\nabla_\lambda B - A!_\lambda B\Big)A^{-1/2} = \lambda(1-\lambda)A^{-1/2}(A-B)A^{-1/2}A^{1/2}(A\nabla_{1-\lambda}B)^{-1}A^{1/2}A^{-1/2}(A-B)A^{-1/2}.$$

Hence (22) after simple reduction. □

The numerical radius of $A \in \mathcal{B}(H)$ is defined by

$$\omega(A) =: \sup\Big\{\big|\langle Ax, x\rangle\big|,\ x \in H,\ \|x\| = 1\Big\}.$$

If $A \geq 0$ then $\omega(A) = \|A\|$, where $\|A\| =: \sup\Big\{\|Ax\|,\ x \in H,\ \|x\| = 1\Big\}$ refers to the operator norm of A. It follows that, if $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}^{+*}(H)$ are such that $|\langle Ax, x\rangle| \leq \langle Bx, x\rangle$ for all $x \in H$, then $\omega(A) \leq \|B\|$. Otherwise, the following inequalities [10]

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$$

hold for any $A \in \mathcal{B}(H)$.

For further properties and applications of the numerical radius of matrices and operators we can consult [1, 10] for instance.

3. Some lemmas needed

In this section, we state some known integral inequalities in the aim to use them in the next sections for obtaining a lot of mean inequalities. We first state the following definition.

Definition 3.1. *Let f and g be two real functions defined on a nonempty interval I of $\mathbb{R}$. We say that f and g are synchronous (resp. asynchronous) on I, if they are monotonic in the same sense (resp. in the opposite sense) on I. That is, the following*

$$\Big(f(x) - f(y)\Big)\Big(g(x) - g(y)\Big) \geq (\leq)\ 0$$

holds for any $x, y \in I$.

This class of functions plays an important place in mathematical analysis. As an example, we mention the following result, known in the literature as the Chebychev's inequality [18, Theorem 8, p.39], which will be needed later.

Lemma 3.2. *Let I be a nonempty interval of $\mathbb{R}$ and let $f, g, h : I \longrightarrow \mathbb{R}$ be such that $h(x) \geq 0$ for all $x \in I$. We assume that h, hfg, hf and hg are integrable on I. If f and g are synchronous (resp. asynchronous) on I then the following inequality holds:*

$$\int_I h(t)\,dt \times \int_I h(t)f(t)g(t)\,dt \geq (\leq) \int_I h(t)f(t)\,dt \int_I h(t)g(t)\,dt. \tag{23}$$

Another result of interest which will be also needed later is the so-called Grüss integral inequality [18, Theorem 8, p.70] recited as follows.

Lemma 3.3. *Let f and g be two functions defined and integrable over (a, b). Assume that, for all $x \in (a, b)$, we have*

$$\phi \leq f(x) \leq \Phi \quad and \quad \gamma \leq g(x) \leq \Gamma,$$

where ϕ, Φ, γ and Γ are fixed real numbers. Then,

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t)\,dt - \frac{1}{(b-a)^2} \int_a^b f(t)\,dt \times \int_a^b g(t)\,dt \right| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma). \tag{24}$$

Further, the constant $1/4$ is the best possible.

A weighted version of the previous Grüss inequality [4, 5] is given in the following.

Lemma 3.4. *Let f and g be two functions defined and integrable over (a, b) and satisfying that*

$$\phi \leq f(x) \leq \Phi \quad and \quad \gamma \leq g(x) \leq \Gamma$$

for all $x \in (a, b)$, where ϕ, Φ, γ and Γ are fixed real constants. If h is a nonnegative function defined and integrable on (a, b), then we have the following inequality

$$\left| \int_a^b h(t)\,dt \times \int_a^b f(t)g(t)h(t)\,dt - \int_a^b f(t)h(t)\,dt \times \int_a^b g(t)h(t)\,dt \right| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma)\Big(\int_a^b h(t)\,dt \Big)^2. \tag{25}$$

Now, we recall another type of integral inequality, due to Ostrowski [3, Theorem 52], which will allow us to obtain some inequalities involving the logarithmic mean.

Lemma 3.5. *Let f be a bounded measurable function on $I = (a, b)$ such that $c_1 \leq f(t) \leq c_2$ for $t \in I$, and assume $g'(t)$ exists and is bounded on I. Then,*

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t)\,dt - \frac{1}{(b-a)^2} \int_a^b f(t)\,dt \times \int_a^b g(t)\,dt \right| \leq \frac{b-a}{8}(c_2 - c_1) \sup_{t \in I} |g'(t)|. \tag{26}$$

Further, $1/8$ is the best possible constant.

The following result, termed in the literature as a premature Gr*ü*ss inequality [3, Theorem 55], will be also needed throughout this paper.

Lemma 3.6. *Let f and g be two integrable functions defined on $[0, 1]$. Assume that, for all $t \in [0, 1]$, we have $d \leq g(t) \leq D$ for some constants $d, D \in \mathbb{R}$. Then we have*

$$\left| \int_0^1 f(t)g(t)\,dt - \int_0^1 f(t)\,dt \times \int_0^1 g(t)\,dt \right| \leq \frac{D-d}{2} \left(\int_0^1 f^2(t)\,dt - \Big(\int_0^1 f(t)\,dt \Big)^2 \right)^{1/2}. \tag{27}$$

Moreover, $1/2$ is the best possible constant.

We now state the Ostrowski's inequality [18, 19] as recited below.

Lemma 3.7. *Let f be a differentiable function on (a, b). Assume that, for all $x \in (a, b)$, we have $|f'(x)| \le M$ for some fixed real number $M > 0$. Then, for any $x \in (a, b)$, we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)M. \tag{28}$$

Further the constant $1/4$ is the best possible.

A special version of the Ostrowski's inequality [5, Corollary 11], recited below, will be used for bounding the difference between the logarithmic mean and the weighted geometric mean.

Lemma 3.8. *Let φ be a differentiable function on (a, b). Assume that φ is integrable on (a, b) and φ' is continuous on (a, b) with $\|\varphi'\| =: \int_a^b |\varphi'(t)|dt < \infty$. Then we have the following inequality*

$$\Big| \int_a^b \varphi(t)dt - \varphi(x)\Big(b-a\Big) \Big| \le \left(\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right) \|\varphi'\|. \tag{29}$$

The Ostrowski-Grüss inequality stated by S.Dragomir and S.Wang in [6, Corollary 2.2], is recalled in the following lemma in the purpose to derive more estimations.

Lemma 3.9. *Let I be a nonempty interval of $\mathbb{R}$. Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping in I and let $a, b \in I$ with $a < b$. Assume that $f \in L^1\Big([a, b]\Big)$ and for all $t \in [a, b]$ we have $|f'(t)| \le M$ for some $M > 0$. Then, for all $x \in [a, b]$, we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \le \frac{1}{2}(b-a)\, M. \tag{30}$$

Remark 3.10. *Extensions of some of the previous lemmas for matrix/operator arguments have been recently investigated in the literature. We omit here to state these extended lemmas, since they are not needed in the present paper. We just refer the interested reader to consult [15, 20, 27] for instance.*

4. Inequalities for the logarithmic mean

Our first main result in this section reads as follows.

Theorem 4.1. *Let $a, b, c, d > 0$. If $(a-1)(b-1) \le (\ge) 0$ then we have*

$$L(ac, bd) \times L(c, d) \ge (\le) L(ac, d) \times L(c, bd). \tag{31}$$

Proof. For fixed $a, b, c, d > 0$, we define for $t \in [0, 1]$ the following functions

$$f(t) = a^{1-t}, \;\; g(t) = b^t, \;\; h(t) = c^{1-t} d^t.$$

Obviously, $f'(t) = -a^{1-t} \log a$ and $g'(t) = b^t \log b$. Thus, f and g are synchronous (resp. asynchronous) if and only if $(\log a)(\log b) \le (\ge) 0$ or, equivalently, $(a-1)(b-1) \le (\ge) 0$. Further, $h(t) \ge 0$ for any $t \in [0, 1]$. By using the Chebychev's inequality (23) we get

$$\int_0^1 h(t)\, dt \times \int_0^1 h(t) f(t) g(t)\, dt \ge (\le) \int_0^1 f(t) h(t)\, dt \int_0^1 g(t) h(t)\, dt.$$

This, with the explicit expressions of $f(t)$, $g(t)$ and $h(t)$ and (11), gives (31) and the proof is complete. □

The previous theorem has many consequences. In particular, we mention the following corollaries.

Corollary 4.2. *For any $a, b > 0$ such that $(a-1)(b-1) \leq (\geq) 0$ one has*

$$L(a,b) \geq (\leq) L(a,1) \times L(1,b).$$

Proof. If we take $c = d = 1$ in (31) we immediately obtain the desired inequality. □

Corollary 4.3. *Let $a, b > 0$. If $(a-1)(b-1) \leq (\geq) 0$ then for every integer $n \geq 0$ we have*

$$L(a^n, b^n) \times L(a,b) \geq (\leq) L(a^n, b) \times L(a, b^n).$$

Proof. It is easy to see that if $(a-1)(b-1) \leq (\geq) 0$ then $(a^{n-1}-1)(b^{n-1}-1) \leq (\geq) 0$ for any $n \geq 0$. Now, replacing in Theorem 4.1, a, b, c and d by a^{n-1}, b^{n-1}, a and b, respectively, we get the desired result. □

Corollary 4.4. *For any $a, c, d > 0$ we have the following inequalities*

$$a\Big(L(c,d)\Big)^2 \leq L(ac,d) \times L(c,ad).$$

$$L(a^2c,d) \times L(c,d) \geq \Big(L(ac,d)\Big)^2.$$

Proof. If $a = b$ then $(a-1)(b-1) = (a-1)^2 \geq 0$. So, Theorem 4.1 gives

$$L(ac,ad) \times L(c,d) \leq L(ac,d) \times L(c,ad).$$

This, with the homogeneity of L, yields the first desired inequality. To prove the second inequality, we choose $a = 1/b$ for which we have $(a-1)(b-1) \leq 0$ and we conclude by the similar argument as in the previous case. □

Now, we will apply the Grüss inequality (24) for proving another main result as recited in what follows.

Theorem 4.5. *For all $a, b > 0$ we have*

$$\big|L(a,b) - L(a,1) \times L(1,b)\big| \leq \frac{1}{4}|a-1|\,|b-1|. \tag{32}$$

Proof. Let us consider the functions $f(t) = a^{1-t}$ and $g(t) = b^t$, for $t \in [0,1]$. Obviously, we have

$$\min_{t\in[0,1]} f(t) = \min(1,a),\ \max_{t\in[0,1]} f(t) = \max(1,a),\ \min_{t\in[0,1]} g(t) = \min(1,b),\ \max_{t\in[0,1]} g(t) = \max(1,b).$$

According to (24), we find

$$\left|L(a,b) - \int_0^1 a^{1-t}\,dt \times \int_0^1 b^t\,dt\right| \leq \frac{1}{4}\Big(\max(1,a) - \min(1,a)\Big)\Big(\max(1,b) - \min(1,b)\Big) = \frac{1}{4}|a-1||b-1|,$$

or, equivalently,

$$\big|L(a,b) - L(a,1) \times L(1,b)\big| \leq \frac{1}{4}|a-1|\,|b-1|,$$

which completes the proof. □

The weighted the Grüss inequality (25) gives more results as we will investigate in the following.

Theorem 4.6. *Let $a, b, c, d, e, k > 0$. We have,*

$$\big|L(e,k) \times L(ace,bdk) - L(ae,bk) \times L(ce,dk)\big| \leq \frac{1}{4}|b-a|\,|d-c|\Big(L(e,k)\Big)^2. \tag{33}$$

Proof. For $t \in [0,1]$ we consider the following functions,

$$f(t) = a^{1-t}b^t, \qquad g(t) = c^{1-t}d^t, \qquad h(t) = e^{1-t}k^t.$$

By (1), we have

$$\min(a,b) \le f(t) \le \max(a,b) \text{ and } \min(c,d) \le g(t) \le \max(c,d).$$

Combining these last inequalities with (25) we obtain

$$\left|\int_0^1 h(t)\,dt \times \int_0^1 f(t)g(t)h(t)\,dt - \int_0^1 f(t)h(t)\,dt \times \int_0^1 g(t)h(t)\,dt\right| \le \frac{1}{4}|b-a|\,|d-c|\Big(\int_0^1 h(t)\,dt\Big)^2.$$

This, when combined with the explicit expressions of f, g, h and the inequality (11), immediately yields (33). □

From Theorem 4.6 we will deduce some inequalities as recited in the following corollary.

Corollary 4.7. *Let $a,b,c,d > 0$. Then we have*

$$\Big|L(ac,bd) - L(a,b) \times L(c,d)\Big| \le \frac{1}{4}|a-b|\,|c-d|. \tag{34}$$

In particular, one has

$$|L(ac,1) - L(a,1) \times L(c,1)| \le \frac{1}{4}|a-1|\,|c-1|, \tag{35}$$

$$\Big|L(a^2,b^2) - \big(L(a,b)\big)^2\Big| \le \frac{1}{4}(a-b)^2. \tag{36}$$

Proof. If we take $e = k = 1$ in (33) we immediately get (34). Taking $b = d = 1$ in (34) we obtain (35). Finally, if we choose $a = c$ and $b = d$ then (34) becomes (36). □

Corollary 4.8. *For any $a,b > 0$ we have*

$$|L(ab,1) - L(a,b)| \le \frac{1}{2}|a-1|\,|b-1|. \tag{37}$$

Proof. If we combine (35) with (32) and we use the standard triangular inequality we get (37). □

Now, using the inequality (26) we will prove the following result.

Theorem 4.9. *Let $a,b > 0$. Then the following inequality holds true*

$$|L(a,b) - L(a,1) \times L(1,b)| \le \frac{1}{8}|a-1||\log b|\max(1,b). \tag{38}$$

Proof. Consider the following two functions defined on $[0,1]$ by, $f(t) = a^{1-t}$ and $g(t) = b^t$. We have $\min_{t\in[0,1]} f(t) = \min(1,a)$, $\max_{t\in[0,1]} f(t) = \max(1,a)$. Further, one has $|g'(t)| = |\log b| b^t$ and so, $\sup_{t\in[0,1]} |g'(t)| = |\log b| \max(1,b)$. Now, applying (26) we then infer that

$$\left|\int_0^1 f(t)g(t)\,dt - \int_0^1 f(t)\,dt \times \int_0^1 g(t)\,dt\right| \le \frac{1}{8}|a-1||\log b|\max(1,b).$$

This, with the explicit forms of f, g, h and the inequality (11), immediately gives (38). □

Before giving further results, let us observe the following remark which concerns a comparison about the previous estimations.

Remark 4.10. *(i) Numerical experiments show that there is no general comparison between* (32) *and* (38). *That is, neither* (32) *nor* (38) *is uniformly better than the other. More precisely, it is not hard to check that,* (32) *is better (resp. weaker) than* (38) *for any $a,b>0$ such that $L(a,b)\leq(\geq)\frac{1}{2}\max(a,b)$.*
(ii) Of course, since the quantity in the left hand-side of (38) *is symmetric in a and b then we also have*

$$\forall a,b>0 \qquad |L(a,b)-L(a,1)\times L(1,b)|\leq\frac{1}{8}|b-1||\log a|\max(a,1).$$

The inequality (26) will be employed in order to give another type of estimation for the left quantity in (38).

Theorem 4.11. *For any $a,b>0$ the following inequality is satisfied*

$$|L(a,b)-L(a,1)\times L(1,b)|\leq\frac{|b-1|}{2}\Big(L(a^2,1)-\big(L(a,1)\big)^2\Big)^{1/2}. \tag{39}$$

Proof. Let us consider the functions $f(t)=a^{1-t}$ and $g(t)=b^t$ for $t\in[0,1]$. It is clear that f and g are integrable on $[0,1]$ and for every $t\in[0,1]$ we have $\min(1,b)\leq g(t)\leq\max(1,b)$. Using the inequality (27) we find,

$$\left|\int_0^1 f(t)g(t)\,dt-\int_0^1 f(t)\,dt\times\int_0^1 g(t)\,dt\right|\leq\frac{|b-1|}{2}\left(\int_0^1 f^2(t)\,dt-\Big(\int_0^1 f(t)\,dt\Big)^2\right)^{1/2}. \tag{40}$$

By (11), with the explicit expressions of f and g, we have

$$\int_0^1 f^2(t)\,dt=L(a^2,1),\ \int_0^1 f(t)\,dt=L(a,1),\ \int_0^1 g(t)\,dt=L(1,b).$$

Substituting these in (40) we get (39), so completes the proof. □

5. Estimations about the difference $L(a,b)-a\sharp_\lambda b$ and $L(a,b)-a\nabla_\lambda b$

In this section we focus on giving some estimations for the difference between the logarithmic mean $L(a,b)$ and the weighted geometric mean $a\sharp_\lambda b$. Our first main result here reads as follows.

Theorem 5.1. *Let $a,b>0$. For any $\lambda\in[0,1]$, we have the following inequality*

$$\big|L(a,b)-a\sharp_\lambda b\big|\leq\big|\log(a/b)\big|\max(a,b)\Big(\frac{1}{4}+\big(\lambda-\frac{1}{2}\big)^2\Big). \tag{41}$$

In particular

$$0\leq L(a,b)-a\sharp b\leq\frac{1}{4}\big|\log(a/b)\big|\max(a,b). \tag{42}$$

Proof. Let us consider the function $\phi(t)=a^{1-t}b^t$ for which we have $\big|\phi'(t)\big|=\big|\log(a/b)\big|a^{1-t}b^t\leq|\log(a/b)|\max(a,b)$, for all $t\in[0,1]$. This when substituted in the Ostrowski's inequality (28) gives

$$\left|\phi(\lambda)-\int_0^1\phi(t)\,dt\right|\leq|\log(a/b)|\max(a,b)\Big(\frac{1}{4}+\big(\lambda-\frac{1}{2}\big)^2\Big),$$

which is exactly (41). Taking $\lambda=1/2$ in (41) we get (42), so completing the proof. □

Utilizing the Ostrowski's inequality (29), we obtain another estimation of $L(a,b)-a\sharp_\lambda b$ as recited in what follows.

Theorem 5.2. *Let $a, b > 0$. For every $\lambda \in [0,1]$ we have*

$$\Big|L(a,b) - a\sharp_\lambda b\Big| \le \Big(\frac{1}{2} + \Big|\lambda - \frac{1}{2}\Big|\Big)|b-a|. \tag{43}$$

In particular,

$$0 \le L(a,b) - a\sharp b \le \frac{1}{2}|b-a|. \tag{44}$$

Proof. We consider the function $\varphi(t) = a^{1-t}b^t$, defined for $t \in [0,1]$, for which we have $|\varphi'(t)| = |\log(a/b)|a^{1-t}b^t$. Then we get,

$$\| \varphi' \| = \int_0^1 |\varphi'(t)|\, dt = |\log(a/b)|\, L(a,b) = |b-a| < \infty.$$

According to the Ostrowski's inequality (29), we obtain for every $\lambda \in [0,1]$,

$$\Big|\int_0^1 \varphi(t)\, dt - \varphi(\lambda)\Big| \le \Big(\frac{1}{2} + \Big|\lambda - \frac{1}{2}\Big|\Big) \| \varphi' \|,$$

hence (43). The proof is finished. □

Remark 5.3. *As in Remark 4.10, there is no general uniform comparison between* (41) *and* (43) *(resp.* (42) *and* (44)*). We can also check that, if $L(a,b) \le (\ge)\frac{1}{2}\max(a,b)$ then* (44) *is better (resp. weaker) than* (42)*. This means that* (42) *and* (44) *can be reduced to the following inequality*

$$0 \le L(a,b) - a\sharp b \le \frac{1}{4}\Big|\log(a/b)\Big| \min\Big(2L(a,b), \max(a,b)\Big).$$

Utilizing the inequality (30), we have the following result as well.

Theorem 5.4. *Let $a, b > 0$ and $\lambda \in [0,1]$. We have the following inequality,*

$$\Big|L(a,b) - a\sharp_\lambda b + (b-a)\Big(\lambda - \frac{1}{2}\Big)\Big| \le \frac{1}{2}\Big|\log(a/b)\Big| \max(a,b). \tag{45}$$

Proof. Let us consider again $\varphi(t) = a^{1-t}b^t$ and $|\varphi'(t)| = |\log(a/b)|\varphi(t) \le |\log(a/b)|\max(a,b)$, for all $t \in [0,1]$. Substituting these in (30), we obtain

$$\Big|\varphi(\lambda) - \int_0^1 \varphi(t)\, dt - (b-a)\Big(\lambda - \frac{1}{2}\Big)\Big| \le \frac{1}{2}\max(a,b)|\log(a/b)|,$$

whence (45). The proof is complete. □

Remark 5.5. *(i) Since the right hand-sides of* (41) *and* (43) *are both symmetric in λ and $1-\lambda$ then it is easy to see that these two inequalities imply, respectively,*

$$|L(a,b) - HZ_\lambda(a,b)| \le \Big|\log(a/b)\Big| \max(a,b)\Big(\frac{1}{4} + \Big(\lambda - \frac{1}{2}\Big)^2\Big), \tag{46}$$

$$|L(a,b) - HZ_\lambda(a,b)| \le \Big(\frac{1}{2} + \Big|\lambda - \frac{1}{2}\Big|\Big)|b-a|. \tag{47}$$

(ii) Otherwise, (45) *implies that*

$$|L(a,b) - HZ_\lambda(a,b)| \le \frac{1}{2}\Big|\log(a/b)\Big| \max(a,b). \tag{48}$$

It is not hard to verify that (46) *is better than* (48)*. However, as in Remark 5.3, there is no general comparison between* (46) *and* (47)*.*

In what precedes we gave some estimations of $L(a,b) - a\sharp_\lambda b$ by using (11). In what follows we will use (12) for obtaining some estimations about $L(a,b) - a\nabla_\lambda b$.

Theorem 5.6. *Let $a, b > 0$ and $\lambda \in [0,1]$. Then we have*

$$\left|\Big(L(a,b)\Big)^{-1} - \Big(a\nabla_\lambda b\Big)^{-1}\right| \le \left(\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right) \frac{|a-b|}{\min\left(a^2, b^2\right)}. \tag{49}$$

Proof. We consider the function f and its derivative f' such that

$$\forall t \in [0,1] \qquad f(t) = \frac{1}{(1-t)a + tb}, \;\; f'(t) = \frac{a-b}{\Big((1-t)a+tb\Big)^2}.$$

By (1) we immediately deduce that $|f'(t)| \le \frac{|a-b|}{\min(a^2,b^2)}$ for all $t \in [0,1]$. Now, by the Ostrowski's inequality (28) we obtain (49). □

We have the following result as well.

Theorem 5.7. *Let $a, b > 0$ and $\lambda \in [0,1]$. The following inequality holds*

$$\left|\Big(L(a,b)\Big)^{-1} - \Big(a\nabla_\lambda b\Big)^{-1}\right| \le \left(\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right) \frac{|a-b|}{ab}. \tag{50}$$

Proof. With the same function f as in the proof of Theorem 5.6, it is easy to verify that,

$$\|f'\| =: \int_0^1 |f'(t)|\, dt = \left|\frac{1}{a} - \frac{1}{b}\right| = \frac{|a-b|}{ab}.$$

Using the Ostrowski's inequality (29) we conclude the proof. □

Finally, the following result may be stated.

Theorem 5.8. *For any $a, b > 0$ and $\lambda \in [0,1]$ we have*

$$\left|\Big(L(a,b)\Big)^{-1} - \Big(a\nabla_\lambda b\Big)^{-1} + \left(\frac{1}{b} - \frac{1}{a}\right)\left(\lambda - \frac{1}{2}\right)\right| \le \frac{|a-b|}{2\ \min\left(a^2, b^2\right)}. \tag{51}$$

Proof. We consider the same function as in the proof of Theorem 5.6 and we use the Ostrowski-Grüss inequality (30). We then get

$$\left|f(\lambda) - \int_0^1 f(t)\, dt - \Big(f(1) - f(0)\Big)\left(\lambda - \frac{1}{2}\right)\right| \le \frac{|a-b|}{2\ \min\left(a^2, b^2\right)},$$

or, equivalently,

$$\left|\Big(a\nabla_\lambda b\Big)^{-1} - \Big(L(a,b)\Big)^{-1} - \left(\frac{1}{b} - \frac{1}{a}\right)\left(\lambda - \frac{1}{2}\right)\right| \le \frac{|a-b|}{2\ \min\left(a^2, b^2\right)}.$$

Hence (51) after a simple manipulation. □

6. Estimations about the difference $a\nabla_\lambda b - a\sharp_\lambda b$

This section deals with some estimations about the weighted arithmetic-geometric difference. Our first main result in this section reads as follows.

Theorem 6.1. *Let $a, b > 0$ and $\lambda \in [0,1]$. We have the following inequality,*

$$\left| a\nabla_\lambda b - a\sharp_\lambda b - \frac{\lambda(1-\lambda)}{2}(a-b)\log(a/b) \right| \leq \frac{\sin(\pi\lambda)}{4\pi}\frac{(a-b)^2}{\min(a,b)}. \tag{52}$$

Proof. First, we notice that if $\lambda \in \{0,1\}$ then (52) is reduced to an equality. We then assume $\lambda \in (0,1)$ throughout the following. We consider the functions f and g defined for $t \in (0,1)$ by:

$$f(t) = (1-t)\sharp_\lambda t, \;\; g(t) = \frac{a\nabla_t b - a!_t b}{t(t-1)}.$$

For any $t \in (0,1)$ we have $0 \leq f(t) \leq 1$ and one can check that $g(t) = \frac{(a-b)^2}{ta+(1-t)b}$. By (1) we immediately deduce that $|g(t)| \leq \frac{(a-b)^2}{\min(a,b)}$ for all $t \in (0,1)$. Hence, applying the Grüss inequality (24) we obtain

$$\left| \int_0^1 f(t)\, g(t)\, dt - \int_0^1 f(t)\, dt \times \int_0^1 g(t)\, dt \right| \leq \frac{(a-b)^2}{4\, \min(a,b)}. \tag{53}$$

Now, (21) yields

$$\int_0^1 f(t)\, g(t)\, dt = \frac{\pi}{\sin(\pi\, \lambda)}\Big(a\nabla_\lambda b - a\sharp_\lambda b\Big).$$

Otherwise, by the celebrated relationships $\Gamma(x+1) = x\Gamma(x)$ and $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, valid for any $x, y > 0$, we obtain

$$\int_0^1 f(t)\, dt = \int_0^1 (1-t)^{1-\lambda} t^\lambda\, dt = B(\lambda+1, 2-\lambda) = \frac{\Gamma(\lambda+1)\,\Gamma(2-\lambda)}{\Gamma(3)}$$
$$= \frac{\lambda(1-\lambda)\Gamma(\lambda)\,\Gamma(1-\lambda)}{2!} = \frac{\pi}{\sin(\pi\, \lambda)}\frac{\lambda(1-\lambda)}{2}.$$

We also have

$$\int_0^1 g(t)\, dt = \int_0^1 \frac{(a-b)^2}{ta+(1-t)b}\, dt = (a-b)\;\log(a/b).$$

Substituting these expressions in (53) we get (52), so completing the proof. □

Now, we will present the following remark which concerns a comparison between (52) and (7).

Remark 6.2. *If $\lambda \in \{0,1\}$ then* (52) *remains an equality while* (6) *is far from to be an equality. This means that if λ is enough small or λ is close to* 1 *then* (52) *is better than* (7)*. However, for $\lambda = 1/2$,* (6) *is an equality while* (52) *is not. This means that if λ is close to* $1/2$ *then* (6) *is better than* (52)*. In a numerical point of view, see TABLE 1 which explains this latter situation.*

Let l_{RC} and U_{RC} denote respectively the lower and upper bounds obtained in the inequality (52) for the difference $a\nabla_\lambda b - a\sharp_\lambda b$ and let l_M and U_M represent those in (6).

a	b	λ	l_{RC}	U_{RC}	l_M	U_M
20	35	0,05	0,0593157	0,3394105	0,1077718	1,9137850
41	58	0,09	0,0849814	0,3979671	0,1345664	1,3157251
55	60	0,1	0,0083999	0,0307552	0,0110261	0,0965096
45	85	0,2	0,3720720	3,6982561	1,2735007	4,9969272
42	90	0,3	0,3095090	7,3728628	2,7284030	6,2849535
100	210	0,4	0,6359695	18,9511764	8,0799955	12,0869649
300	486	0,5	2,0395339	20,3932820	11,1623382	11,1623382
105	203	0,6	0,8302868	14,6751704	6,4114319	9,5910715
157	314	0,7	1,3189455	21,5341170	8,0954960	18,8222273
315	458	0,8	1,2454730	7,3184319	2,6726394	10,6569341
500	954	0,9	3,0618043	23,3360265	7,2779746	65,3515114
50	4500	0,95	-4454,7179497	5405,8652187	180,2936283	3416,9227955
1200	32150	0,99	-1491,5584393	2499,0459296	209,3008871	20715,5431392

Table 1: Some numerical values for bounds of $a\nabla_\lambda b - a\sharp_\lambda b$

Further, from Theorem 6.1 we can deduce the following corollary which brings us an inequality reversing (10).

Corollary 6.3. *For any $a, b > 0$ and $\lambda \in [0,1]$, we get*

$$HZ_\lambda(a,b) \le a\nabla b - \frac{\lambda(1-\lambda)}{2}(a-b)\log(a/b) + \frac{\sin(\pi\lambda)}{4\pi}\frac{(a-b)^2}{\min(a,b)}. \tag{54}$$

Proof. If in (52) we replace λ by $1-\lambda$ we get

$$\left| a\nabla_{1-\lambda} b - a\sharp_{1-\lambda} b - \frac{\lambda(1-\lambda)}{2}(a-b)\log(a/b)\right| \le \frac{\sin(\pi\lambda)}{4\pi}\frac{(a-b)^2}{\min(a,b)}. \tag{55}$$

Now, adding (52) and (55) side by side and using the classical triangular inequality, with the definition of $HZ_\lambda(a,b)$ and the fact that $a\nabla_\lambda b + a\nabla_{1-\lambda} b = 2(a\nabla b)$, we obtain

$$\left| a\nabla b - HZ_\lambda(a,b) - \frac{\lambda(1-\lambda)}{2}(a-b)\log(a/b)\right| \le \frac{\sin(\pi\lambda)}{4\pi}\frac{(a-b)^2}{\min(a,b)}. \tag{56}$$

According to (10), the expression inside the absolute value in (56) is negative and so (54) follows. □

7. Estimations about the ratio $a\nabla_\lambda b/I(a,b)$

In the ongoing section, we will present some lower and upper bounds for the ratio between the weighted arithmetic mean and the identric mean.

Theorem 7.1. *For any $a, b > 0$ and $\lambda \in [0,1]$ we have*

$$\exp\left(-\Big(\frac{1}{4} + \big(\lambda - \frac{1}{2}\big)^2\Big)\frac{|b-a|}{\min(a,b)}\right) \le \frac{a\nabla_\lambda b}{I(a,b)} \le \exp\left(\Big(\frac{1}{4} + \big(\lambda - \frac{1}{2}\big)^2\Big)\frac{|b-a|}{\min(a,b)}\right). \tag{57}$$

Proof. Let us consider the function f and its derivative f' defined by

$$\forall t \in [0,1] \quad f(t) = \log\Big((1-t)a + tb\Big), \;\; f'(t) = \frac{b-a}{(1-t)a+tb}.$$

It is clear that $|f'(t)| \leq \frac{|b-a|}{\min(a,b)}$ for all $t \in [0,1]$. By virtue of the Ostrowski's inequality (28) we obtain, for any $\lambda \in [0,1]$,

$$\left|\int_0^1 f(t)\,dt - f(\lambda)\right| \leq \Big(\frac{1}{4} + \Big(\lambda - \frac{1}{2}\Big)^2\Big)\frac{|b-a|}{\min(a,b)}.$$

This, with the integral representation (13), becomes equivalent to

$$\Big|\log I(a,b) - \log a\nabla_\lambda b\Big| \leq \Big(\frac{1}{4} + \Big(\lambda - \frac{1}{2}\Big)^2\Big)\frac{|b-a|}{\min(a,b)},$$

and whence the inequality (57). $\square$

Theorem 7.2. *For any $a,b>0$ and $\lambda \in [0,1]$, the following inequality holds true*

$$\left(\frac{\min(a,b)}{\max(a,b)}\right)^{\frac{1}{2}+|\lambda-\frac{1}{2}|} \leq \frac{a\nabla_\lambda b}{I(a,b)} \leq \left(\frac{\max(a,b)}{\min(a,b)}\right)^{\frac{1}{2}+|\lambda-\frac{1}{2}|}. \tag{58}$$

Proof. We consider the same function f as in the proof of the previous theorem. We have

$$\|f'\| =: |b-a|\int_0^1 \frac{1}{(1-t)a+tb}\,dt = |\log(a/b)|.$$

According to the inequality (29), for all $\lambda \in [0,1]$ we have

$$\Big|\log I(a,b) - \log a\nabla_\lambda b\Big| \leq \left(\frac{1}{2} + \Big|\lambda - \frac{1}{2}\Big|\right)\Big|\log(a/b)\Big|.$$

This, with the equality $\big|\log(a/b)\big| = \log\frac{\max(a,b)}{\min(a,b)}$, implies (58). $\square$

We end this section by stating the following remark which is of interest.

Remark 7.3. *As in Remark 4.10, there is no general comparison between the bounds of $a\nabla_\lambda b/I(a,b)$ given in the previous inequalities* (57) *and* (58)*. TABLE 2 explains more this latter situation.*

Let l_1 and l_2 denote the lower bounds and L_1 and L_2 be the upper bounds obtained respectively in the inequalities (57) and (58).

a	b	λ	l_1	l_2	L_1	L_2
10	11	0,1	0,959829	0,875085	1,010212	1,089566
23	25	0,2	0,970868	0,897272	1,011525	1,068980
35	41	0,3	0,951501	0,827068	1,032533	1,117123
42	400	0,4	0,109024	0,083812	12,897167	3,866219
56	500	0,5	0,137774	0,112000	7,258280	2,988072
70	700	0,6	0,096328	0,125893	6,050493	3,981072
120	900	0,7	0,151829	0,199504	2,658932	4,097726
500	1000	0,8	0,711770	0,615572	1,266015	1,741101
1452	3154	0,95	0,588364	0,652691	1,298104	2,089538

Table 2: Some numerical values for bounds of $a\nabla_\lambda b/I(a,b)$

8. Some operator mean-inequalities

This section deals with some matrix/operator versions of some of the previous scalar mean-inequalities. As already pointed before, although some integral inequalities of Section 3 have been extended for matrix/operator arguments, we omit to pursue this way in this section in the aim to not lengthen the present paper. We then restrict ourselves to explain to the reader how to deduce some matrix/operator inequalities from their related scalar mean-inequalities when the positive real numbers a and b are replaced by two strictly positive operators A and B, respectively. We then preserve the notations and notions that we pointed in Section 2.

We start by stating an analog of Theorem 5.1 for operator arguments as recited in the following.

Theorem 8.1. *Let $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$. Then the following inequality*

$$\Big|\Big\langle\Big(L(A,B) - A\sharp_\lambda B\Big)x, x\Big\rangle\Big| \leq \Big(\frac{1}{4} + \Big(\lambda - \frac{1}{2}\Big)^2\Big)\|A\|\Big\|\log\Big(A^{-1/2}BA^{-1/2}\Big)\Big\|\max\Big(1, \|A^{-1}\|\|B\|\Big)\|x\|^2 \tag{59}$$

holds for any $x \in H$. In particular, we have

$$\Big\|L(A,B) - A\sharp B\Big\| \leq \frac{1}{4}\|A\|\Big\|\log\Big(A^{-1/2}BA^{-1/2}\Big)\Big\|\max\Big(1, \|A^{-1}\|\|B\|\Big). \tag{60}$$

Proof. Let $x \in H$ be fixed. For $t \in [0,1]$ we define

$$\phi_x(t) =: \langle (A\sharp_t B)x, x\rangle =: \Big\langle A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^t A^{1/2}x, x\Big\rangle.$$

By arguments of linearity we have

$$\frac{d}{dt}\phi_x(t) = \Big\langle A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^t \log\Big(A^{-1/2}BA^{-1/2}\Big)A^{1/2}x, x\Big\rangle.$$

If we recall that $\|X^t\| = \|X\|^t$ for any $X \geq 0$ and $t \in [0,1]$, then we get

$$\Big|\frac{d}{dt}\phi_x(t)\Big| \leq \|A\|\Big\|\log\Big(A^{-1/2}BA^{-1/2}\Big)\Big\|\max_{0\leq t\leq 1}\Big\|A^{-1/2}BA^{-1/2}\Big\|^t\|x\|^2. \tag{61}$$

The real function $t \longmapsto \Big\|A^{-1/2}BA^{-1/2}\Big\|^t$ is convex and so we have

$$\max_{0\leq t\leq 1}\Big\|A^{-1/2}BA^{-1/2}\Big\|^t = \max\Big(1, \Big\|A^{-1/2}BA^{-1/2}\Big\|\Big) \leq \max\Big(1, \|A^{-1}\|\|B\|\Big). \tag{62}$$

Now, by the Ostrowski's inequality (28) we obtain

$$\left|\phi_x(\lambda) - \int_0^1 \phi_x(t)\,dt\right| \leq \Big(\frac{1}{4} + \Big(\lambda - \frac{1}{2}\Big)^2\Big)\sup_{t\in[0,1]}\Big|\frac{d}{dt}\phi_x(t)\Big|,$$

which, with (61), (62) and (17), implies (59).

Taking $\lambda = 1/2$ in (59), with the help of the left inequality in (19), we get (60), so completing the proof. □

Remark 8.2. *We left to the reader the routine task for formulating the statement of an analog of Theorem 5.2 whose the proof is similar to that of Theorem 8.1*

Finally, we state an analog of Theorem 6.1 for operator arguments which reads as follows.

Theorem 8.3. *Let $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$. Then the inequality*

$$\left|\langle\big(A\nabla_\lambda B\big)x, x\rangle - \langle\big(A\sharp_\lambda B\big)x, x\rangle - \frac{\lambda(1-\lambda)}{2}\Big\langle(A-B)\big(L(A,B)\big)^{-1}(A-B)x, x\Big\rangle\right|$$
$$\leq \frac{\sin(\pi\lambda)}{4\pi}\max\Big(\langle(A-B)A^{-1}(A-B)x, x\rangle, \langle(A-B)B^{-1}(A-B)x, x\rangle\Big) \tag{63}$$

holds for any $x \in H$.

Proof. We assume below $\lambda \in (0,1)$, since for $\lambda \in \{0,1\}$, (63) is an equality. For $t \in [0,1]$, we consider $f(t) =: (1-t)\sharp_\lambda t$ for which we have $0 \leq f(t) \leq 1$ for any $t \in [0,1]$. Now, let $x \in H$ be fixed. For $t \in (0,1)$ we set

$$g_x(t) = \frac{\Big\langle\big(A\nabla_t B - A\sharp_t B\big)x, x\Big\rangle}{t(1-t)}. \tag{64}$$

According to (22) we have

$$g_x(t) = \Big\langle(A-B)\big(tA + (1-t)B\big)^{-1}(A-B)x, x\Big\rangle. \tag{65}$$

Thus, g_x defined by (64) on the open interval $(0,1)$ can be extended on the closed interval $[0,1]$ by setting

$$g_x(0) =: \Big\langle(A-B)B^{-1}(A-B)x, x\Big\rangle, \quad g_x(1) =: \Big\langle(A-B)A^{-1}(A-B)x, x\Big\rangle.$$

Since the map $t \longmapsto tA + (1-t)B$ is affine and $z \longmapsto 1/z$ is operator convex on $(0,\infty)$, then we deduce that the map $t \longmapsto \big(tA + (1-t)B\big)^{-1}$ is convex on $[0,1]$ for the Löwner operator order, i.e. for any $y \in H$, the real-function $t \longmapsto \langle\big(tA + (1-t)B\big)^{-1}y, y\rangle$ is convex on $[0,1]$. Letting $y = (A-B)x$ we then infer that, for any $x \in H$, the real-function g_x defined by (65) is convex for $t \in [0,1]$. Remark that $g_x(t) \geq 0$ for any $t \in [0,1]$. This, with the convexity of $t \longmapsto g_x(t)$ on $[0,1]$, allows us to write

$$\sup_{t\in[0,1]}|g_x(t)| = \sup_{t\in[0,1]} g_x(t) = \sup_{t\in\{0,1\}} g_x(t) = \max\big(g_x(0), g_x(1)\big)$$
$$= \max\Big(\Big\langle(A-B)A^{-1}(A-B)x, x\Big\rangle, \Big\langle(A-B)B^{-1}(A-B)x, x\Big\rangle\Big).$$

Applying the Grüss inequality (24), the following inequality

$$\left|\int_0^1 f(t)\, g_x(t)\, dt - \int_0^1 f(t)\, dt \times \int_0^1 g_x(t)\, dt\right|$$
$$\leq \frac{1}{4}\max\Big(\Big\langle(A-B)A^{-1}(A-B)x, x\Big\rangle, \Big\langle(A-B)B^{-1}(A-B)x, x\Big\rangle\Big) \tag{66}$$

holds for any $x \in H$. According to (21) we have

$$\int_0^1 f(t)\, g_x(t)\, dt = \frac{\pi}{\sin(\pi\,\lambda)}\Big\langle\big(A\nabla_\lambda B - A\sharp_\lambda B\big)x, x\Big\rangle.$$

As shown in the proof of Theorem 6.1 one has

$$\int_0^1 f(t)\, dt = \frac{\pi}{\sin(\pi\,\lambda)}\frac{\lambda(1-\lambda)}{2}.$$

Otherwise, by (65) with an argument of linearity we can write

$$\int_0^1 g_x(t)\, dt = \Big\langle(A-B)\Big(\int_0^1 \big(tA + (1-t)B\big)^{-1}\, dt\Big)(A-B)x, x\Big\rangle,$$

which, with (18) and the change of variables $t = 1 - u$, yields

$$\int_0^1 g_x(t)\, dt = \Big\langle (A-B)\Big(L(A,B)\Big)^{-1}(A-B)x, x\Big\rangle.$$

Substituting these expressions in (66) we get (63), so concludes the proof. □

From the previous theorems we immediately deduce the following results which involve the numerical radius.

Corollary 8.4. *For any $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$, we have*

$$\omega\Big(L(A,B) - A\sharp_\lambda B\Big) \leq \Big(\frac{1}{4} + \Big(\lambda - \frac{1}{2}\Big)^2\Big)\|A\|\Big\|\log\Big(A^{-1/2}BA^{-1/2}\Big)\Big\|\max\Big(1, \|A^{-1}\|\|B\|\Big),$$

$$\omega\left(A\nabla_\lambda B - A\sharp_\lambda B - \frac{\lambda(1-\lambda)}{2}(A-B)\Big(L(A,B)\Big)^{-1}(A-B)\right) \leq \frac{\sin(\pi\lambda)}{4\pi}\max\Big(\|(A-B)A^{-1}(A-B)\|, \|(A-B)B^{-1}(A-B)\|\Big).$$

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