



## Analysis of Open-Water Plankton-Fish-Mussel System Under the Influence of Environmental Noises

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**Abstract.** In this article, we consider a mathematical model consisting of fish and mussel population in presence of plankton as common food source in marine ecosystem. The model is constructed considering all possible biological relationships among plankton (nutrient), fish and mussel. Nonconstant death rate is introduced for mussel population. To avoid ignoring any environmental factors, we consider some important parameters as stochastic. Existence and uniqueness of global positive solutions, boundedness of solutions are proved mathematically. Conditions for extinction of fish and mussel population are derived. From these conditions, we have reached at some conclusion which are verified numerically. We have established the conditions for which the model is persistent in mean. We have proved that the solutions are globally attractive in mean. Numerical simulations are performed by using MATLAB to justify the mathematical findings. Numerically, we have investigated several useful facts and have achieved some conclusion. For extension of this study, a model is proposed at the end.

### 1. Introduction

Now a days, dynamical study of marine ecosystem and aquaculture has become very essential for livelihood and economics. There are a lot of factors to be concerned about and there are a lot of species which are dependent on each other for survival in this ecosystem. Our study mainly concerns itself with the coexistence of fish with mussel in the same habitat, which may contribute in bio-economic modelling. Existence of shellfish (mussel) in the same habitat where fish is available has both positive and negative effects on fish density.

The shellfish are usually treated as bio-filter because they play an important role in controlling the water quality. They can consume excess feed not dissolved into water, which enhances the water quality. The shellfish (barnacles, mussel etc.) are used in an innovative way so that they behave like a bio-filtering system to clean up fish farms [39]. Mussel can help in pollution problem in two ways: (i) mussel can digest the wastage particles of feed, (ii) they consume phytoplankton which thrive on inorganic nutrients such as phosphorous and nitrogen.

Since both fish and mussel have same food source, they compete against each other for food. In this way, existence of mussel has negative effect on fish biomass.

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Many researchers have studied various models to find out various facts on fish farming and harvesting. Potential impact of fish farming on nutrient content is studied by Karakassis et al. [14] and they have found that fish farm waste can cause 1% on nutrient concentration in contrast with other anthropogenic activities. Some researchers [22] have found that detritivorous fish has huge impact on the estuarine ecosystem. Samanta et al. [37] have found how the maximum amount of species which are economically important can be harvested from a fishery. Bandyopadhyay et al. [1] have investigated the role of nutrients in controlling the behavior of an ecological system. Gazi et al. [9] have considered fish and mussel in a same model in existence of external food source and analyzed the stability of the system. Gazi [10] have extended this study and analyzed for the direction of Hopf-bifurcation, stability of the Hopf-bifurcating periodic orbits.

Fish farming can differ depending upon different environment and different water. Sometimes, these farms depend on tidal flow, sometimes fish are allowed to move to search for natural food, in some cases, natural food such as plankton are provided in the fisheries and sometime, mussels are cultivated in the farm to balance the toxic of water. But, other than the fish farm, in coastal area-river-ocean, these species are found to coexist without any artificial force. In both the cases, both fish and mussel consume same food which may be provided externally or naturally (plankton), should be in high quantity.

From the belief that any substance can dilute in water, the lakes become the dumping grounds for different types of wastages and pollutants which cause presence of toxic substances in marine ecosystem. So, consideration of existence of toxic in a mathematical model of aquaculture is highly significant and necessary. Hallam and Clark [11], Hallam and De Luna [12], Dubey and Hussain [8], Kar and Chaudhuri [15], Kar et al. [16], Pal and Samanta [21], Sharma and Samanta [38], Pal et al. [20], etc. have involved toxic substance to study their mathematical models. In most of the models, general single species or two species communities without any special emphasis on aquatic environments are considered. Maynard Smith [40] and Samanta [35] have incorporated the effects of toxic substances in a two species Lotka-Volterra competitive system, by considering that in presence of other species, each species produces a toxic substance.

Some researchers [17, 19, 23, 25–37] have introduced Gaussian white noise as environmental variation model to find out the effects of environmental noise on dynamical systems. May [19] has found that the birth rates, death rates, carrying capacity, competition coefficients and all other parameters involved in a dynamical model lead to random fluctuation because of continuous fluctuation in the environment, to a great lesser extent. If the dynamical patterns revealed in deterministic system are still in evidence when stochastic effects are introduced, they happen to be very useful ecologically. Usually, uncertain growth and death rates of populations are considered as an effect of environmental stochasticity. So, it is better to consider such parameters as a stochastic process rather than a deterministic one.

Only a few researches [9, 10, 41] have considered fish and mussel in the same model. But, to the best of our knowledge, no one has incorporated the factor in mathematical model that fish play an important role in reproduction process of mussel. So, their death rate may differ depending on density of fish, i.e., death rate of mussel should be a fish density dependent function. Consideration of such biological facts and inclusion of this fact into mathematical modelling make this work unique. Also, no one has studied such a model in a randomly fluctuating environment, which makes this work more realistic and meaningful.

We represent the article as follows: Section 2 contains biological facts and some assumptions to construct both the deterministic and stochastic models. In Section 3, existence and uniqueness of global positive solution is proved for both the deterministic and stochastic systems. Boundedness of solutions is proved in Section 4 followed by conditions for extinction for fish and mussel which are derived at in Section 5. Section 6 contains the most important theorems of persistence of the underlying system with some necessary lemmas and theorems. In section 7, we have proved that solutions of the system are globally attractive in mean. Section 8 justifies mathematical findings by numerical simulations and investigates some effects of different factors in this system. Stochastic asymptotic stability of the system is shown through numerical simulation in this section. In Section 9, we have discussed our mathematical and numerical findings and concluded some interesting and useful results and have proposed a model to extend this study in future.

**2. Mathematical model with facts and assumptions**

Aquatic environment such as brackish estuaries, the tidal zone, open sea, lakes and ponds contain various kinds of species and they have various types of food chain system. Here we take plankton, fish and mussel (shellfish) into our consideration to construct our mathematical model. Plankton plays a key role in the aquatic food web: fish, shellfish (mussel), etc. depend on phytoplankton for survival. Energy captured by phytoplankton is transferred to these species by zooplankton. Plankton must be produced in high quantities to support the entire food chain because it is the primary link. Disappearance of plankton can brake the chain and cause extinction of other marine life. Plankton suffer due to toxicant which are released by industries in marine water. On the other hand, some phytoplankton produce chemicals (toxic), which are harmful to human and marine lives. Considering all the facts mentioned above we propose the system of differential equations for plankton ( $P$ ) and fish ( $F$ ) as follows:

$$\begin{aligned} \frac{dP}{dt} &= a - \alpha FP - d_1P - c_1P^2 \\ \frac{dF}{dt} &= \gamma FP - \eta F^2 - d_2F - c_2F^3 \end{aligned} \tag{1}$$

Here,  $a$  represents the constant input rate of plankton in the system. Since it is a primary link in this food chain, it is considered as highly available in this ecosystem and that it has a constant input rate. The term  $\alpha$  is the coefficient of the rate of consumption of plankton by fish,  $\gamma$  is the rate at which the energy is transferred to fish. Parameters  $d_1$  and  $d_2$  represent the outflow or sedimentation rate of plankton and death rate of fish respectively. Here,  $c_1$  and  $c_2$  are called the coefficients of toxicity to plankton and fish respectively,  $\eta$  represents the coefficient of the intra-specific competition between fishes. The term  $c_1P^2$  comes directly through the toxic effect on plankton by some external toxic substance, such as, industrial wastes. Since,  $\frac{d(c_1P^2)}{dP} = 2c_2P > 0$  and  $\frac{d^2(c_1P^2)}{dP^2} = 2c_2 > 0$ , there is an accelerated growth in the production of the toxic substance to the density of the  $P$  as more and more of the species consumes the infected foods. Similar thing happens in the case for fish, except for the fact that the effect of toxicity on the fish is more than it is on the plankton and it is taken as  $c_2F^3$  because it consumes the toxic plankton and gets directly affected by toxic in water.

Mussel is also a species in marine ecology, which also uses plankton as their food. Mussels have a very interesting unusual and complex mode of reproduction, which includes obligatory as a parasite on a fish. The reproduction procedure of mussel is shown in Figure 1.

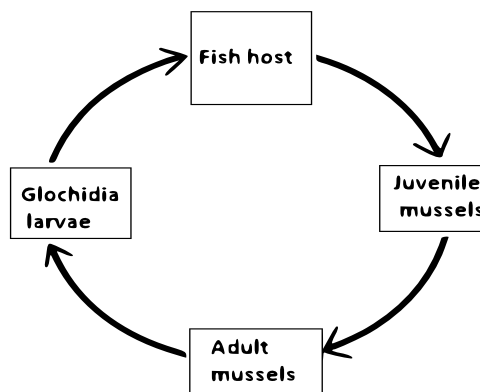


Figure 1: *Reproduction cycle of mussel.*

So, it is easy to observe that density of fish population affects mussel population and the effect is inversely proportional to the death rate of mussel, i.e., the death rate of this species becomes non constant,

dependent on fish density. We consider  $\mu(F, \theta)$  a function of  $F$  as the death rate of mussel, where  $\theta$  is the contact rate of glochidia larvae (of mussel) with fish. Now, combining the differential equation of mussel with system (1), we get

$$\begin{aligned} \frac{dP}{dt} &= a - \alpha FP - \beta MP - d_1 P - c_1 P^2 \\ \frac{dF}{dt} &= \gamma FP - \eta F^2 - d_2 F - c_2 F^3 \\ \frac{dM}{dt} &= \delta MP - \mu(F, \theta)M \end{aligned} \tag{2}$$

where  $\beta$  is the coefficient of the rate of consumption of plankton by mussel,  $\delta$  is the rate at which the energy is transferred to mussel.

We consider  $d_3$  and  $d$  are the minimum and maximum death rates of mussel respectively. From field experiments and reproduction process of mussel, one can assume that  $\mu(F, \theta)$  has the following properties:

$$\begin{aligned} (i) \mu(0, \theta) = d \quad (ii) \mu(F, 0) = d \quad (iii) \lim_{F \rightarrow \infty} \mu(F, \theta) = d_3 \\ (iv) \lim_{\theta \rightarrow \infty} \mu(F, \theta) = d_3 \quad (v) \frac{\partial \mu(F, \theta)}{\partial F} < 0 \quad (vi) \frac{\partial \mu(F, \theta)}{\partial \theta} < 0 \end{aligned} \tag{3}$$

Let us take  $\mu(F, \theta) = d_3 + \frac{d - d_3}{1 + \theta F}$  and it is easy to verify that the function  $\mu(F, \theta)$  satisfies all the properties described in (3). This function may be considered in many different ways satisfying the properties described in (3). Considering  $d - d_3 = d' > 0$  ( $\because d > d_3$ ) and substituting the function on system (2), we get the following system:

$$\begin{aligned} \frac{dP}{dt} &= a - \alpha FP - \beta MP - d_1 P - c_1 P^2 \\ \frac{dF}{dt} &= \gamma FP - \eta F^2 - d_2 F - c_2 F^3 \\ \frac{dM}{dt} &= \delta MP - \left( d_3 + \frac{d'}{1 + \theta F} \right) M \end{aligned} \tag{4}$$

with initial conditions  $P_0 > 0, F_0 > 0$  and  $M_0 > 0$ .

Since (i) production of plankton is directly dependent on water temperature and nutrient availability which is indirectly controlled by temperature-driven circulation patterns and (ii) metamorphosis of mussel depends also on temperature, environmental fluctuation is not a small fact to be ignored for this model.

We consider outflow or sedimentation rate of plankton ( $d_1$ ), death rate of fish ( $d_2$ ) and minimum death rate of mussel ( $d_3$ ) as stochastic parameters. We introduce environmental noise on system (4) by perturbing  $d_1, d_2$  and  $d_3$  with independent Gaussian white noise terms  $\gamma_1, \gamma_2$  and  $\gamma_3$  to obtain the following system:

$$\begin{aligned} \frac{dP}{dt} &= a - \alpha FP - \beta MP - (d_1 + \gamma_1)P - c_1 P^2 \\ \frac{dF}{dt} &= \gamma FP - \eta F^2 - (d_2 + \gamma_2)F - c_2 F^3 \\ \frac{dM}{dt} &= \delta MP - (d_3 + \gamma_3)M - \frac{d'M}{1 + \theta F} \end{aligned} \tag{5}$$

where  $\gamma_1, \gamma_2$  and  $\gamma_3$  are independent Gaussian white noises [24] satisfying the following characteristics:

$$\langle \gamma_j(t) \rangle = 0 \text{ and } \langle \gamma_j(t_1) \gamma_j(t_2) \rangle = \sigma_j^2 \delta_j(t_1 - t_2), \text{ for } j = 1, 2, 3.$$

Here,  $\sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0$  represent the respective intensities or strengths of the random perturbations,  $\delta_j$  ( $j = 1, 2, 3$ ) are the Dirac delta functions defined by

$$\begin{cases} \delta_j(x) = 0, \text{ for } x \neq 0, \\ \int_{-\infty}^{\infty} \delta_j(x) dx = \lim_{\epsilon \rightarrow 0^+} \int_{-\epsilon}^{\epsilon} \delta_j(x) dx = 1, \end{cases}$$

and  $\langle \cdot \rangle$  is the ensemble average of the considered stochastic process. Now, the system (5) can be represented as follows:

$$\begin{aligned} dP &= (a - \alpha FP - \beta MP - d_1 P - c_1 P^2) dt - \sigma_1 P dw_1 \\ dF &= F(\gamma P - \eta F - d_2 - c_2 F^2) dt - \sigma_2 F dw_2 \\ dM &= M\left(\delta P - d_3 - \frac{d'}{1 + \theta F}\right) dt - \sigma_3 M dw_3 \end{aligned} \tag{6}$$

with initial conditions  $P_0 > 0, F_0 > 0$  and  $M_0 > 0$ . Here  $\gamma_1 = \sigma_1 \frac{dw_1}{dt}, \gamma_2 = \sigma_2 \frac{dw_2}{dt}, \gamma_3 = \sigma_3 \frac{dw_3}{dt}$  and  $w = \{w_1, w_2, w_3, t \geq 0\}$  represents a three-dimensional standard Brownian motion.

### 3. Existence of unique global positive solution

**Theorem 3.1.** *If  $(P_0, F_0, M_0) \in \mathbb{R}_+^3$  be any initial value, then the deterministic system (4) has unique global positive solution  $(P(t), F(t), M(t))$  for all  $t \geq 0$ .*

*Proof.* For being RHS of system (4) continuous and locally Lipschitz on  $\mathbb{R}_+^3$ , the solution  $(P(t), F(t), M(t))$  of system (4) exists uniquely on  $[0, \tau)$ , where  $\tau \in (0, \infty]$ . From first equation of system (4), we have

$$\begin{aligned} \frac{dP}{dt} + (\alpha F + \beta M + d_1 + c_1 P) P &= a \\ \implies P(t) &= \left( P_0 + a \int_0^t \left( \exp \int_0^r (\alpha F(z) + \beta M(z) + d_1 + c_1 P(z)) dz \right) dr \right) \\ &\quad \times \exp \left( - \int_0^t (\alpha F(r) + \beta M(r) + d_1 + c_1 P(r)) dr \right) > 0, \quad \forall t \geq 0. \end{aligned}$$

From second equation of system (4), we have

$$F(t) = F_0 \exp \int_0^t (\gamma P(r) - \eta F(r) - d_2 - c_2 F^2(r)) dr > 0, \quad \forall t \geq 0.$$

From third equation of system (4), we have

$$M(t) = M_0 \exp \int_0^t \left( \delta P(r) - d_3 - \frac{d'}{1 + \theta F(r)} \right) dr > 0, \quad \forall t \geq 0.$$

Hence the theorem.  $\square$

**Lemma 3.1.** [5] *For all  $z > 0$ , the following inequality holds*

$$z \leq 2(z + 1 - \log(z)) - 2(2 - \log(2)).$$

**Theorem 3.2.** *For any initial value  $(P_0, F_0, M_0) \in \mathbb{R}_+^3$  of system (6), there is a unique positive solution  $(P(t), F(t), M(t))$  of system (6) on  $t \geq 0$ , and the solutions will remain in  $\mathbb{R}_+^3$  with probability 1 for all  $t \geq 0$  almost surely.*

*Proof.* Since coefficients of system (6) satisfy local Lipschitz condition, hence for any initial value  $(P_0, F_0, M_0) \in \mathbb{R}_+^3$  there is a unique local solution  $(P(t), F(t), M(t)) \in \mathbb{R}_+^3$  for  $t \in [0, \tau_e)$ , where  $\tau_e$  is the explosion time. To show this is a global positive solution, we need to show that  $\tau_e = \infty$ . Let  $r_0 > 0$  be sufficiently large so that both  $x_0, s_0$  and  $y_0$  lie in the interval  $\left[\frac{1}{r_0}, r_0\right]$ . We define stopping time  $(\tau_r)$  for each integer  $r \geq r_0$  such that

$$\tau_r = \inf \left\{ t \in [0, \tau_e) : \min \{P(t), F(t), M(t)\} < \frac{1}{k} \text{ or } \max \{P(t), F(t), M(t)\} > k \right\},$$

with  $\inf \phi = \infty$  ( $\phi$  denotes the empty set). It is easy to observe that  $\tau_r$  increases as  $t \rightarrow \infty$ . Here we set  $\tau_\infty = \lim_{r \rightarrow \infty} \tau_r$ , whence  $\tau_\infty \leq \tau_e$  a.s. If it can be proved that  $\tau_\infty = \infty$ , then it is easy to conclude that  $\tau_e = \infty$  and  $(P(t), F(t), M(t)) \in \mathbb{R}_+^3$  for all  $t \geq 0$  almost surely. So, to complete the proof all we need to do is to prove is that  $\tau_\infty = \infty$ . It can be proved by contradiction. Let if possible the statement is false, then there is a pair of constants  $T > 0$  and  $\epsilon \in (0, 1)$  such that

$$P \{ \tau_\infty \leq T \} > \epsilon.$$

So, there exists an integer  $r_1 \geq r_0$  such that

$$P \{ \tau_\infty \leq T \} > \epsilon, \quad \forall r \geq r_1. \tag{7}$$

Now, we define a  $C^3$ -function  $V : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  by

$$V(P, F, M) = (P + 1 - \log(P)) + (F + 1 - \log(F)) + (M + 1 - \log(M))$$

Since  $(z + 1 - \log(z)) \geq 0, \forall z > 0$ , so  $V(P, F, M)$  is positive.

Now, applying Itô formula, we get

$$\begin{aligned} dV(P, F, M) &= \left[ \left(1 - \frac{1}{P}\right) P \left( \frac{a}{P} - \alpha F - \beta M - d_1 - c_1 P \right) + \left(1 - \frac{1}{F}\right) F \right. \\ &\quad \times \left( \gamma P - \eta F - d_2 - c_2 F^2 \right) + \left(1 - \frac{1}{M}\right) M \left( \delta P - d_3 - \frac{d'}{1 + \theta F} \right) \\ &\quad \left. + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \right] dt - [(P - 1)\sigma_1 dw_1 + (F - 1)\sigma_2 dw_2 + (M - 1)\sigma_3 dw_3] \\ &= \left[ (P - 1) \left( \frac{a}{P} - \alpha F - \beta M - d_1 - c_1 P \right) + (F - 1) \right. \\ &\quad \times \left( \gamma P - \eta F - d_2 - c_2 F^2 \right) + (M - 1) \left( \delta P - d_3 - \frac{d'}{1 + \theta F} \right) \\ &\quad \left. + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \right] dt - [(P - 1)\sigma_1 dw_1 + (F - 1)\sigma_2 dw_2 + (M - 1)\sigma_3 dw_3] \\ &\leq \left( a - d_1 P + \alpha F + \beta M + d_1 + c_1 P - (\alpha - \gamma)FP - (\beta - \delta)MP - \gamma P \right. \\ &\quad \left. + \eta F + d_2 - d_2 F - d_3 M - \delta P + d_3 + d' + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \right) dt \\ &\quad - [(P - 1)\sigma_1 dw_1 + (F - 1)\sigma_2 dw_2 + (M - 1)\sigma_3 dw_3] \\ &\leq \left[ \left( a + d_1 + d_2 + d_3 + d' + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \right) + (-d_1 + c_1 - \gamma - \delta)P \right. \\ &\quad \left. + (\alpha + \eta - d_2)F + (\beta - d_3)M \right] dt \quad [\because \alpha > \gamma \text{ and } \beta > \delta] \\ &\quad - [(P - 1)\sigma_1 dw_1 + (F - 1)\sigma_2 dw_2 + (M - 1)\sigma_3 dw_3] \\ &= (b_1 + b_2 P + b_3 F + b_4 M) dt - (P - 1)\sigma_1 dw_1 - (F - 1)\sigma_2 dw_2 - (M - 1)\sigma_3 dw_3, \end{aligned}$$

where  $b_1 = \left(a + d_1 + d_2 + d_3 + d' + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2}\right)$ ,  $b_2 = (-d_1 + c_1 - \gamma - \delta)$ ,  $b_3 = (\alpha + \eta - d_2)$  and  $b_4 = (\beta - d_3)$   
 Using Lemma 3.1, we get

$$dV(P, F, M) \leq [b_1 + 2b_2(P + 1 - \log P) + 2b_3(F + 1 - \log F) + 2b_4(M + 1 - \log M)] dt - (P - 1)\sigma_1 dw_1 - (F - 1)\sigma_2 dw_2 - (M - 1)\sigma_3 dw_3$$

Let  $b_5 = \max\{b_1, 2b_2, 2b_3, 2b_4\}$  and define  $v_1 \wedge v_2 := \min\{v_1, v_2\}$ .

$$\therefore dV(P, F, M) \leq b_5(1 + V(P, F, M))dt - (P - 1)\sigma_1 dw_1 - (F - 1)\sigma_2 dw_2 - (M - 1)\sigma_3 dw_3$$

Hence, for  $t_1 \leq T$ ,

$$\int_0^{\tau_r \wedge t_1} d(V(P(t), F(t), M(t))) < \int_0^{\tau_r \wedge t_1} b_5(1 + V(P(t), F(t), M(t)))dt - \int_0^{\tau_r \wedge t_1} (P - 1)\sigma_1 dw_1 - \int_0^{\tau_r \wedge t_1} (F - 1)\sigma_2 dw_2 - \int_0^{\tau_r \wedge t_1} (M - 1)\sigma_3 dw_3$$

Taking expectation on both sides, we get

$$\begin{aligned} & E(V(P(t), F(t), M(t))) \Big|_{t=\tau_r \wedge t_1} \\ & \leq V(P_0, F_0, M_0) + E \int_0^{\tau_r \wedge t_1} b_5(1 + V(P(t), F(t), M(t)))dt \\ & \leq V(P_0, F_0, M_0) + b_5 t_1 + a_5 E \int_0^{\tau_r \wedge t_1} V(P(t), F(t), M(t))dt \\ & \leq V(P_0, F_0, M_0) + b_5 T + b_5 \int_0^{t_1} E[V(P(\tau_r \wedge t_1), F(\tau_r \wedge t_1), M(\tau_r \wedge t_1))] dt \end{aligned}$$

By Gronwall inequality [18]:

$$E(V(P(\tau_r \wedge t_1), F(\tau_r \wedge t_1), M(\tau_r \wedge t_1))) \leq b_6, \tag{8}$$

where  $b_6 = (V(P_0, F_0, M_0) + b_5 T) e^{b_5 T}$ .

Define  $\Omega_r = \{\tau_r \leq T\}$  for  $r \geq r_1$  and by (7),  $P(\Omega_r) \geq \epsilon$ . Note that for every  $\tau' \in \Omega_r$  there is at least one of  $P(\tau_r, \tau')$ ,  $F(\tau_r, \tau')$ ,  $M(\tau_r, \tau')$  which is equal either  $r$  or  $\frac{1}{r}$ . So,  $V(P(\tau_r, \tau'), F(\tau_r, \tau'), M(\tau_r, \tau'))$  is not less than the smallest of

$$r + 1 - \log(r) \text{ and } \frac{1}{r} + 1 - \log\left(\frac{1}{r}\right) = \frac{1}{r} + 1 + \log(r).$$

Consequently,

$$V(P(\tau_r, \tau'), F(\tau_r, \tau'), M(\tau_r, \tau')) \geq (r + 1 - \log(r)) \wedge \left(\frac{1}{r} + 1 + \log(r)\right).$$

Now from (7) and (8), we get

$$\begin{aligned} b_6 & \geq E[1_{\Omega_r} V(P(\tau_r, \tau'), F(\tau_r, \tau'), M(\tau_r, \tau'))] \\ & \geq \epsilon \left[ (r + 1 - \log(r)) \wedge \left(\frac{1}{r} + 1 + \log(r)\right) \right], \end{aligned}$$

where  $1_{\Omega_r}$  is the indicator function of  $\Omega_r$ . Therefore,  $r \rightarrow \infty$  leads towards the contradiction  $\infty > b_6 = \infty$ . Hence, our assumption was wrong. So,  $\tau_\infty = \infty$ .  $\square$

**4. Boundedness of solutions**

In this section, we shall discuss the boundedness property of solutions of system (6).

Now, we define  $M_1(t) := \int_0^t \sigma_1 dw_1$ ,  $M_2(t) := \int_0^t \sigma_2 dw_2$  and  $M_3(t) := \int_0^t \sigma_3 dw_3$  as real valued continuous local martingales. Applying strong law of large numbers:  $\lim_{t \rightarrow \infty} \frac{M_1(t)}{t} = \lim_{t \rightarrow \infty} \frac{M_2(t)}{t} = \lim_{t \rightarrow \infty} \frac{M_3(t)}{t} = 0$ .

Boundedness of solutions is proved followed by the following Lemma.

**Lemma 4.1.** For all  $t \geq 0$ ,  $E(e^{-w(t)}) = e^{\frac{t}{2}}$ .

*Proof.* From Taylor expansion of exponential function, we have

$$e^{-w(t)} = \sum_{i=0}^{\infty} (-1)^i \frac{(w(t))^i}{i!}$$

$$\therefore e^{-w(t)} = 1 - w(t) + \frac{(w(t))^2}{2!} - \frac{(w(t))^3}{3!} + \frac{(w(t))^4}{4!} + \dots \tag{9}$$

We know that,  $E(w(t))^{2k} = (1.3.5 \dots (2k - 1))t^k$  and  $E(w(t))^{2k+1} = 0$   
 Taking expectation on both sides of (9), we get

$$E(e^{-w(t)}) = 1 - E(w(t)) + \frac{E(w(t))^2}{2!} - \frac{E((w(t))^3)}{3!} + \frac{E((w(t))^4)}{4!} + \dots$$

$$= 1 - 0 + \frac{t}{2!} - 0 + \frac{1.3.t^2}{4!} + \dots$$

$$= 1 + \frac{t}{2} + \frac{\left(\frac{t}{2}\right)^2}{2!} + \dots = e^{\frac{t}{2}}$$

So, proved.  $\square$

**Theorem 4.1.** Let  $(P(t), F(t), M(t))$  be a solution of system (6) with initial conditions  $(P_0, F_0, M_0) \in \mathbb{R}_+^3$ , then  $E(P^q(t)) \leq K_1(q)$ ,  $E(F^q(t)) \leq K_2(q)$ ,  $\forall q \geq 1$ , where

$$K_1(q) = \max \left\{ P_0^q, q \left( \frac{a}{q} \right)^q \left[ \frac{1 - q}{\frac{1}{q} - d_1 + \frac{q-1}{2} \sigma_1^2} \right]^{q-1} \right\}, \text{ i.e., } K_1(1) = P_0$$

$$\text{and } K_2(q) = \max \left\{ F_0^q, q \left( \frac{q}{\eta} \right)^q \left( \frac{\frac{1}{q} + \gamma P_0 - d_2 + \frac{q-1}{2} \sigma_2^2}{q + 1} \right)^{q+1} \right\}$$

and for  $\delta P_0 + \frac{q-1}{2} \sigma_3^2 \leq d_3$ ,  $E(M^q(t)) \leq M_0^q$ , so, for  $\delta P_0 \leq d_3$ ,  $E(M(t)) \leq M_0$ .

*Proof.* From first equation of system (6), we have

$$dP = (a - \alpha FP - \beta MP - d_1 P - c_1 P^2) dt - \sigma_1 P dw_1$$

Let us take  $V(t) = e^t P^q$  and apply Itô formula:



$$\begin{aligned} dV_1 &= \left[ e^t P^q + qe^t P^{q-1}(a - \alpha FP - \beta MP - d_1 P - c_1 P^2) + \frac{q(q-1)}{2} \sigma_1^2 e^t P^q \right] dt \\ &\quad - qe^t P^q \sigma_1 dw_1 \\ &= qe^t P^{q-1} \left[ \frac{P}{q} + a - \alpha FP - \beta MP - d_1 P - c_1 P^2 + \frac{q-1}{2} \sigma_1^2 P \right] dt - qe^t P^q \sigma_1 dw_1 \\ &\leq qe^t P^{q-1} \left[ a + P \left( \frac{1}{q} - d_1 P + \frac{q-1}{2} \sigma_1^2 \right) \right] dt - qe^t P^q \sigma_1 dw_1 \end{aligned}$$

Integrating and taking expectation on both sides, we get

$$E(V_1(t)) \leq P_0^q + q \int_0^t e^r E \left[ P^{q-1} \left\{ a + P \left( \frac{1}{q} - d_1 P + \frac{q-1}{2} \sigma_1^2 \right) \right\} \right] dr$$

Let us consider,  $f_1(P) = P^{q-1} \left\{ a + P \left( \frac{1}{q} - d_1 P + \frac{q-1}{2} \sigma_1^2 \right) \right\}$

$$\therefore f_1'(P) = 0 \Rightarrow P = \frac{(1-q)a}{q \left( \frac{1}{q} - d_1 P + \frac{q-1}{2} \sigma_1^2 \right)} = P^* \text{ (say).}$$

After some calculations, it is observed that  $f_1''(P^*) < 0$ , so,  $f_1(P)$  occurs to it's maximum at  $P = P^*$ .

$$\therefore f_1(P)|_{\max} = f_1(P^*) = \left( \frac{a}{q} \right)^q \left[ \frac{1-q}{\frac{1}{q} - d_1 + \frac{q-1}{2} \sigma_1^2} \right]^{q-1}$$

Hence,

$$\begin{aligned} E(e^t P^q) &\leq P_0^q + q \left( \frac{a}{q} \right)^q \left[ \frac{1-q}{\frac{1}{q} - d_1 + \frac{q-1}{2} \sigma_1^2} \right]^{q-1} (e^t - 1) \\ \Rightarrow E(P^q) &\leq q \left( \frac{a}{q} \right)^q \left( \frac{1-q}{\frac{1}{q} - d_1 + \frac{q-1}{2} \sigma_1^2} \right)^{q-1} + \left[ P_0^q - q \left( \frac{a}{q} \right)^q \left( \frac{1-q}{\frac{1}{q} - d_1 + \frac{q-1}{2} \sigma_1^2} \right)^{q-1} \right] e^{-t} \end{aligned}$$

Now, for  $t = 0$ ,  $E(P^q) \leq P_0^q$  and for  $t \rightarrow \infty$ ,  $E(P^q) \leq q \left( \frac{a}{q} \right)^q \left( \frac{1-q}{\frac{1}{q} - d_1 + \frac{q-1}{2} \sigma_1^2} \right)^{q-1}$ .

Hence,  $E(P^q) \leq K_1(q)$ , where  $K_1(q) = \max \left\{ P_0^q, q \left( \frac{a}{q} \right)^q \left[ \frac{1-q}{\frac{1}{q} - d_1 + \frac{q-1}{2} \sigma_1^2} \right]^{q-1} \right\}$ .

From second equation of system (6), we have

$$dF = F(\gamma P - \eta F - d_2 - c_2 F^2) dt - \sigma_2 F dw_2$$

Let us consider  $V_2(t) = e^t F^q$  and apply Itô formula:

$$\begin{aligned} d(V_2(t)) &= \left[ e^t F^q + qe^t F^{q-1} F(\gamma P - \eta F - d_2 - c_2 F^2) + \frac{q(q-1)}{2} \sigma_2^2 e^t F^q \right] dt \\ &\quad - qe^t F^q \sigma_2 dw_2 \\ &= qe^t F^q \left[ \frac{1}{q} + \gamma P - \eta F - d_2 - c_2 F^2 + \frac{q-1}{2} \sigma_2^2 \right] dt - qe^t F^q \sigma_2 dw_2 \end{aligned}$$

Integrating and taking expectation on both sides, we get

$$\begin{aligned} E(e^t F^q) &= F_0^q + q \int_0^t e^r E \left[ F^q \left( \frac{1}{q} + \gamma P - \eta F - d_2 - c_2 F^2 + \frac{q-1}{2} \sigma_2^2 \right) \right] dr \\ &\leq F_0^q + q \int_0^t e^r E \left[ F^q \left( \frac{1}{q} + \gamma P_0 - \eta F - d_2 + \frac{q-1}{2} \sigma_2^2 \right) \right] dr \end{aligned}$$

Now, considering  $f_2(F) = F^q \left( \frac{1}{q} + \gamma P_0 - \eta F - d_2 + \frac{q-1}{2} \sigma_2^2 \right)$  and calculating as previously, it can be derived that

$$E(F^q) \leq K_2(q), \text{ where } K_2(q) = \max \left\{ F_0^q, q \left( \frac{q}{\eta} \right)^q \left( \frac{\frac{1}{q} + \gamma P_0 - d_2 + \frac{q-1}{2} \sigma_2^2}{q+1} \right)^{q+1} \right\}$$

From third equation of the underlying system (6), we have

$$dM = M \left( \delta P - d_3 - \frac{d'}{1 + \theta F} \right) dt - \sigma_3 M dw_3$$

Let us take,  $V_3(t) = \log M(t)$  and apply Itô formula:

$$\begin{aligned} d(V_3(t)) &= \left( \delta P - d_3 - \frac{d'}{1 + \theta F} - \frac{\sigma_3^2}{2} \right) dt - \sigma_3 dw_3 \\ &\leq \left( \delta P_0 - d_3 - \frac{\sigma_3^2}{2} \right) dt - \sigma_3 dw_3 \end{aligned}$$

Integrating both sides, we get

$$\begin{aligned} \log M(t) &\leq \log M_0 + \left( \delta P_0 - d_3 - \frac{\sigma_3^2}{2} \right) t - M_3 \\ \Rightarrow M(t) &\leq M_0 e^{\left( \delta P_0 - d_3 - \frac{\sigma_3^2}{2} \right) t - M_3} \\ \Rightarrow M^q(t) &\leq M_0^q e^{q \left( \delta P_0 - d_3 - \frac{\sigma_3^2}{2} \right) t - q M_3} \end{aligned}$$

Taking expectation on both sides and using Lemma 4.1, we get

$$E(M^q(t)) \leq M_0^q e^{q \left( \delta P_0 - d_3 + \frac{q-1}{2} \sigma_3^2 \right) t}$$

So, for  $\left( \delta P_0 + \frac{q-1}{2} \sigma_3^2 \right) < d_3$ ,  $E(M^q(t)) \leq M_0^q$ . For  $\delta P_0 < d_3$ ,  $E(M(t)) \leq M_0$ . Hence the theorem.  $\square$

Since, there is a constant input rate “a” of plankton in the system (6), this species should not extinct and a lower bound should exist. Let us find, the lower bound of  $P(t)$  in the next theorem.

**Theorem 4.2.** Let  $(P(t), F(t), M(t))$  be a solution of system (6) with initial conditions  $(P_0, F_0, M_0) \in \mathbb{R}_+^3$ , then for  $\delta P_0 \leq d_3$ ,  $E(P(t)) \geq L_p$ . Here,  $L_p = \min \{ P_0, a - \alpha K_2(1) P_0 - \beta M_0 P_0 - d_1 P_0 - c_1 K_1(2) \}$ ;  $K_1(q)$  and  $K_2(q)$  are as in Theorem 4.1.

*Proof.* From the differential equation of plankton in the system (6), we have

$$dP = \left( a - \alpha FP - \beta MP - d_1 P - c_1 P^2 \right) dt - \sigma_1 P dw_1$$

We consider  $V_4(t) = Pe^t$  and apply Itô formula:

$$\begin{aligned} d(V_4(t)) &= \left[ Pe^t + e^t (a - \alpha FP - \beta MP - d_1 P - c_1 P^2) \right] dt - \sigma_1 P e^t dw_1 \\ &= e^t (a + P - \alpha FP - \beta MP - d_1 P - c_1 P^2) dt - \sigma_1 P e^t dw_1 \end{aligned}$$

Integrating and taking expectation on both sides and using Theorem 4.1, we get

$$\begin{aligned} E(Pe^t) &= P_0 + \int_0^t e^r E(a + P - \alpha FP - \beta MP - d_1 P - c_1 P^2) dr \\ &\geq P_0 + \int_0^t e^r E(a - \alpha K_2(1)P_0 - \beta M_0 P_0 - d_1 P_0 - c_1 K_1(2)) dr \\ &= P_0 + (a - \alpha K_2(1)P_0 - \beta M_0 P_0 - d_1 P_0 - c_1 K_1(2)) (e^t - 1) \end{aligned}$$

Hence,

$$E(P) \geq P_0 e^{-t} + (a - \alpha K_2(1)P_0 - \beta M_0 P_0 - d_1 P_0 - c_1 K_1(2)) (1 - e^{-t})$$

i.e., for  $t = 0$ ,  $E(P) \geq P_0$  and for  $t \rightarrow \infty$ ,

$$E(P) \geq (a - \alpha K_2(1)P_0 - \beta M_0 P_0 - d_1 P_0 - c_1 K_1(2))$$

Therefore,  $E(P(t)) \geq L_P$ , where

$$L_P = \min \{P_0, a - \alpha K_2(1)P_0 - \beta M_0 P_0 - d_1 P_0 - c_1 K_1(2)\}. \text{ Hence the theorem. } \square$$

### 5. Discussion on extinction

One of the most important terms in population dynamics is extinction. Death of last existing member of a population is called extinction of this population, i.e., if there exists no members of a species which can reproduce or create a new generation in the habitat, it is called the extinction of this species. In ecology, extinction usually refers to local extinction, in which a species goes to extinct in the chosen area of study, but may still exist elsewhere. In this article, we have considered a model where both fish ( $F$ ) and mussel ( $M$ ) depend on plankton ( $P$ ) for food. So, density of plankton should play an important role in the condition of extinction of fish and mussel both. Let us define extinction mathematically and verify these facts in the next theorem.

**Definition 5.1.** Population  $x(t)$  is said to be going extinct with probability one if

$$\lim_{t \rightarrow \infty} x(t) = 0 \text{ a.s.}$$

**Theorem 5.1.** Let  $(P(t), F(t), M(t))$  be the solution of system (6). The fish population goes extinct if  $\gamma P_0 < d_2 + \frac{\sigma_1^2}{2}$ , i.e.,

$$\lim_{t \rightarrow \infty} F(t) = 0 \text{ a.s. if } \gamma P_0 < d_2 + \frac{\sigma_1^2}{2}.$$

And mussel population goes extinct if  $\delta P_0 < d_3 + \frac{\sigma_3^2}{2}$ , i.e.,

$$\lim_{t \rightarrow \infty} M(t) = 0 \text{ a.s. if } \delta P_0 < d_3 + \frac{\sigma_3^2}{2}.$$

*Proof.* From the second equation of system (6), we have

$$dF = F(\gamma P - \eta F - d_2 - c_2 F^2) dt - \sigma_2 F dw_2.$$

Now, we apply Itô formula:

$$\begin{aligned} d(\log F(t)) &= \left( \gamma P - \eta F - d_2 - c_2 F^2 - \frac{\sigma_2^2}{2} \right) dt - \sigma_2 dw_2 \\ &\leq \left( \gamma P_0 - d_2 - \frac{\sigma_2^2}{2} \right) dt - \sigma_2 dw_2 \text{ [using Theorem 4.1]} \end{aligned}$$

Integrating both sides and dividing by  $t$ , we get

$$\begin{aligned} \frac{\log F(t)}{t} &\leq \frac{\log F_0}{t} + \left( \gamma P_0 - d_2 - \frac{\sigma_2^2}{2} \right) - \frac{M_2}{t} \\ \Rightarrow \limsup_{t \rightarrow \infty} \frac{\log F(t)}{t} &\leq \left( \gamma P_0 - d_2 - \frac{\sigma_2^2}{2} \right) \end{aligned}$$

Hence, fish population goes extinct if  $\gamma P_0 < d_2 + \frac{\sigma_2^2}{2}$ .

Again, from the second equation of system (6), we have

$$dM = M \left( \delta P - d_3 - \frac{d'}{1 + \theta F} \right) dt - \sigma_3 M dw_3$$

Applying Itô formula, we get

$$\begin{aligned} d(\log M(t)) &= \left( \delta P - d_3 - \frac{d'}{1 + \theta F} - \frac{\sigma_3^2}{2} \right) dt - \sigma_3 dw_3 \\ &\leq \left( \delta P_0 - d_3 - \frac{\sigma_3^2}{2} \right) dt - \sigma_3 dw_3 \text{ [using Theorem 4.1]} \end{aligned}$$

Integrating both sides and dividing by  $t$ , we get

$$\begin{aligned} \frac{\log M(t)}{t} &\leq \frac{\log M_0}{t} + \left( \delta P_0 - d_3 - \frac{\sigma_3^2}{2} \right) - \frac{M_3}{t} \\ \Rightarrow \limsup_{t \rightarrow \infty} \frac{\log M(t)}{t} &\leq \left( \delta P_0 - d_3 - \frac{\sigma_3^2}{2} \right) \end{aligned}$$

Therefore, mussel population goes extinct if  $\delta P_0 < d_3 + \frac{\sigma_3^2}{2}$ .

Hence the theorem.  $\square$

From Theorem 5.1, it can be observed that low density of plankton causes extinction of fish and mussel both, which is also biologically significant. In the condition for extinction of mussel, we observe that it goes extinct when transferred energy from highest level of food (plankton) is less than the minimum death rate ( $d_3$ ) of mussel, which occurs for very large value of  $F$  or  $\theta$  ( $\because \lim_{F \rightarrow \infty} \mu(F, \theta) = \lim_{\theta \rightarrow \infty} \mu(F, \theta) = d_3$ ), i.e., huge biomass of fish or maximum contact rate of glochidia larvae (of mussel) with fish cannot make mussel population to survive in a very low density of plankton.

### 6. Discussion on persistence

In this section, we shall discuss the most important properties of a dynamical system called persistence in mean and stability.

**Theorem 6.1.** Let  $(P(t), F(t), M(t))$  be a solution of system (6), then almost every sample path of  $(P(t), F(t), M(t))$  is uniformly continuous on  $t \geq 0$  for any initial value  $(P_0, F_0, M_0) \in \mathbb{R}_+^3$  if  $\delta P_0 + \frac{q-1}{2} \sigma_3^2 \leq d_3$ .

*Proof.* The proof is same as in the research articles [3–7].  $\square$

**Lemma 6.1.** [2] Let  $f : [0, \infty) \rightarrow [0, \infty)$  be an integrable and uniformly continuous function. Then  $\lim_{t \rightarrow \infty} f(t) = 0$ .

**Lemma 6.2.** [4] Suppose  $Z(t) \in \mathbf{C}(\Omega \times [0, \infty), \mathbb{R}_+)$ .

(a) If there exists  $T, \delta, \delta_0 \in \mathbb{R}_+$  such that

$$\log Z(t) \leq \delta t - \delta_0 \int_0^t Z(s) ds + \sum_{i=1}^n \alpha_i W_i(t) \quad \text{a.s. } \forall t \geq T, \text{ where } \alpha_i \text{ are constants for } i = 1, 2, \dots, n, \text{ then}$$

$$\begin{cases} \limsup_{t \rightarrow \infty} \langle Z \rangle_t \leq \frac{\delta}{\delta_0}, \text{ a.s.} & \text{if } \delta > 0, \\ \lim_{t \rightarrow \infty} \langle Z \rangle_t = 0, \text{ a.s.} & \text{if } \delta < 0. \end{cases}$$

(b) If there exists  $T, \delta, \delta_0 \in \mathbb{R}_+$  such that

$$\log Z(t) \geq \delta t - \delta_0 \int_0^t Z(s) ds + \sum_{i=1}^n \alpha_i W_i(t) \quad \text{a.s. } \forall t \geq T,$$

where  $\alpha_i$  are constants for  $i = 1, 2, \dots, n$ , then

$$\liminf_{t \rightarrow \infty} \langle Z \rangle_t \geq \frac{\delta}{\delta_0} \quad \text{a.s.}$$

**Definition 6.1.** If  $\liminf_{t \rightarrow \infty} \langle F \rangle_t > 0$  a.s., then system (6) is said to be persistent in the mean. Here  $\langle F \rangle_t = \frac{1}{t} \int_0^t F(r) dr$ .

Now, we shall find the conditions for which that system (6) is persistent in mean.

**Theorem 6.2.** If  $(P(t), F(t), M(t))$  be a solution of system (6) for any initial value  $(P_0, F_0, M_0) \in \mathbb{R}_+^3$ , then system (6) is persistent in mean if  $\gamma L_P > d_2 + c_2 K_2(2) + \frac{\sigma_2^2}{2}$ .

*Proof.* From the differential equation of fish, we have

$$dF = F(\gamma P - \eta F - d_2 - c_2 F^2) dt - \sigma_2 F d\omega_2$$

We consider the Lyapunov function  $F_1(t) = \log F(t)$  and apply Itô formula:

$$\begin{aligned} d(\log F) &= \left( \gamma P - \eta F - d_2 - c_2 F^2 - \frac{\sigma_2^2}{2} \right) dt - \sigma_2 d\omega_2 \\ &\geq \left[ \left( \gamma L_P - d_2 - c_2 K_2(2) - \frac{\sigma_2^2}{2} \right) - \eta F \right] dt - \sigma_2 d\omega_2 \end{aligned}$$

Integrating both sides and dividing by  $t$ , we get

$$\frac{\log F(t) - \log F_0}{t} \geq \left( \gamma L_P - d_2 - c_2 K_2(2) - \frac{\sigma_2^2}{2} \right) - \frac{\eta}{t} \int_0^t F(r) dr - \frac{M_2}{t}$$

So, applying Lemma 6.2, we get

$$\liminf_{t \rightarrow \infty} \langle F \rangle_t \geq \frac{\gamma L_P - d_2 - c_2 K_2(2) - \frac{\sigma_2^2}{2}}{\eta}$$

Now, if  $\gamma L_P > d_2 + c_2 K_2(2) + \frac{\sigma_2^2}{2}$ , then  $\liminf_{t \rightarrow \infty} \langle F \rangle_t > 0$ .

Hence the theorem.  $\square$

### 7. Globally attractive in mean

In this section, we shall verify whether the solution of the underlying system (6) is globally attractive in mean or not. First, we define when a solution is called globally attractive in mean.

**Definition 7.1.** Let  $(P_1(t), F_1(t), M_1(t))$  and  $(P_2(t), F_2(t), M_2(t))$  be two arbitrary solutions of system (6) with initial values  $(P_1^0, F_1^0, M_1^0) \in \mathbb{R}_+^3$  and  $(P_2^0, F_2^0, M_2^0) \in \mathbb{R}_+^3$ , respectively. Then system (6) is called globally attractive in mean if

$$\lim_{t \rightarrow \infty} [E|P_1(t) - P_2(t)| + E|F_1(t) - F_2(t)| + E|M_1(t) - M_2(t)|] = 0 \text{ a.s.}$$

**Theorem 7.1.** Solutions of system (6) is globally attractive in mean if

$$(i) d_1 > \gamma K'_2(1) + \delta M_1^0, (ii) d_2 > \gamma P_2^0 + d' \theta M_2^0, (iii) d_3 > \max \{ \delta P_1^0, \delta P_2^0 \} \tag{10}$$

where  $K'_2(1) = \max \left\{ F_1^0, \left( \frac{1}{\eta} \right) \left( \frac{1 + \gamma P_1^0 - d_2}{2} \right)^2 \right\}$  (obtained from Theorem 4.1).

*Proof.* Let  $(P_1(t), F_1(t), M_1(t))$  and  $(P_2(t), F_2(t), M_2(t))$  be two arbitrary solutions of system (6) with initial values  $(P_1^0, F_1^0, M_1^0) \in \mathbb{R}_+^3$  and  $(P_2^0, F_2^0, M_2^0) \in \mathbb{R}_+^3$  respectively. Consider,  $U(t) = |P_1(t) - P_2(t)| + |F_1(t) - F_2(t)| + |M_1(t) - M_2(t)|$  and apply generalized Itô formula:

$$\begin{aligned} d(U(t)) &= \text{sgn}(P_1(t) - P_2(t)) \left( -\alpha F_1 P_1 - \beta M_1 P_1 - d_1 P_1 - c_1 P_1^2 + \alpha F_2 P_2 + \beta M_2 P_2 \right. \\ &\quad \left. + d_1 P_2 + c_1 P_2^2 \right) dt + \text{sgn}(F_1(t) - F_2(t)) \left( \gamma P_1 F_1 - \eta F_1^2 - d_2 F_1 - c_2 F_1^3 \right. \\ &\quad \left. - \gamma P_2 F_2 + \eta F_2^2 + d_2 F_2 + c_2 F_2^3 \right) dt + \text{sgn}(M_1(t) - M_2(t)) \left( \delta P_1 M_1 - d_3 M_1 \right. \\ &\quad \left. - \frac{d' M_1}{1 + \theta F_1} - \delta P_2 M_2 + d_3 M_2 + \frac{d' M_2}{1 + \theta F_2} \right) dt \\ &= \left\{ [-\alpha F_1 - \beta M_1 - d_1 - c_1(P_1 + P_2) + \gamma F_1 + \delta M_1] |P_1(t) - P_2(t)| + [-\alpha P_2 \right. \\ &\quad \left. + \gamma P_2 - \eta(F_1 + F_2) - d_2 - c_2(F_1^2 + F_1 F_2 + F_2^2) + \frac{d' \theta M_2}{(1 + \theta F_1)(1 + \theta F_2)} \right] \\ &\quad \times |F_1(t) - F_2(t)| + \left[ -\beta P_2 + \delta P_2 - d_3 - \frac{d'}{(1 + \theta F_1)(1 + \theta F_2)} \right. \\ &\quad \left. - \frac{d' \theta F_2}{(1 + \theta F_1)(1 + \theta F_2)} \right] |M_1(t) - M_2(t)| \Big\} dt \end{aligned}$$

Taking expectation on both sides and using Theorem (4.1), we get

$$\begin{aligned} E(d(U(t))) &\leq E \left\{ [-d_1 + \gamma K'_2(1) + \delta M_1^0] |P_1(t) - P_2(t)| + [\gamma P_2^0 - d_2 + d' \theta M_2^0] \right. \\ &\quad \left. \times |F_1(t) - F_2(t)| + [\delta P_2^0 - d_3] |M_1(t) - M_2(t)| \right\} dt \\ &= E \left\{ [-d_1 + \gamma K'_2(1) + \delta M_1^0] E|P_1(t) - P_2(t)| + [\gamma P_2^0 - d_2 + d' \theta M_2^0] \right. \\ &\quad \left. \times E|F_1(t) - F_2(t)| + [\delta P_2^0 - d_3] E|M_1(t) - M_2(t)| \right\} dt \end{aligned}$$

Integrating both sides, we get

$$E(U(t)) \leq E(U(0)) + [-d_1 + \gamma K_2'(1) + \delta M_1^0] \int_0^t E|P_1(r) - P_2(r)|dr + [\gamma P_2^0 - d_2 + d' \theta M_2^0] \int_0^t E|F_1(r) - F_2(r)|dr + [\delta P_2^0 - d_3] \int_0^t E|M_1(r) - M_2(r)|dr$$

Hence,

$$E(U(t)) + [d_1 - \gamma K_2'(1) - \delta M_1^0] \int_0^t E|P_1(r) - P_2(r)|dr + [d_2 - \gamma P_2^0 - d' \theta M_2^0] \times \int_0^t E|F_1(r) - F_2(r)|dr + [d_3 - \delta P_2^0] \int_0^t E|M_1(r) - M_2(r)|dr \leq U(0) < \infty$$

So, if the given conditions (10) hold then it is easy to observe that  $\int_0^t E|P_1(r) - P_2(r)|dr < \infty$ ,  $\int_0^t E|F_1(r) - F_2(r)|dr < \infty$ ,  $\int_0^t E|M_1(r) - M_2(r)|dr < \infty$ . i.e.,  $E|P_1(t) - P_2(t)|, E|F_1(t) - F_2(t)|, E|M_1(t) - M_2(t)| \in L^1[0, \infty)$ . So,  $\lim_{t \rightarrow \infty} E|P_1(t) - P_2(t)| = \lim_{t \rightarrow \infty} E|F_1(t) - F_2(t)| = \lim_{t \rightarrow \infty} E|M_1(t) - M_2(t)| = 0$ . Hence,  $\lim_{t \rightarrow \infty} \{E|P_1(t) - P_2(t)| + E|F_1(t) - F_2(t)| + E|M_1(t) - M_2(t)|\} = 0$ . Hence the theorem.  $\square$

### 8. Numerical simulation

In this section, we verify our mathematical findings and investigate some effects of fish in reproduction of mussel with numerical simulation using MATLAB. We set some values of environmental parameters in Table 1 with initial point (2, 1.2, 1.5) and we deal with noise intensities  $\sigma_1 = \sigma_2 = \sigma_3 = 0.01$ .

$a$	$\alpha$	$\beta$	$d_1$	$c_1$	$\gamma$	$\eta$	$d_2$	$c_2$	$\delta$	$d_3$	$d'$	$\theta$
1	0.5	0.4	0.1	0.125	0.4	0.1	0.2	0.255	0.3	0.1	0.3	0.6

Table 1: Values of environmental parameters.

Simulating the underlying systems ((4) and (6)), we get the trajectories of the solutions in Figure 2 ((2.a) and (2.b)).

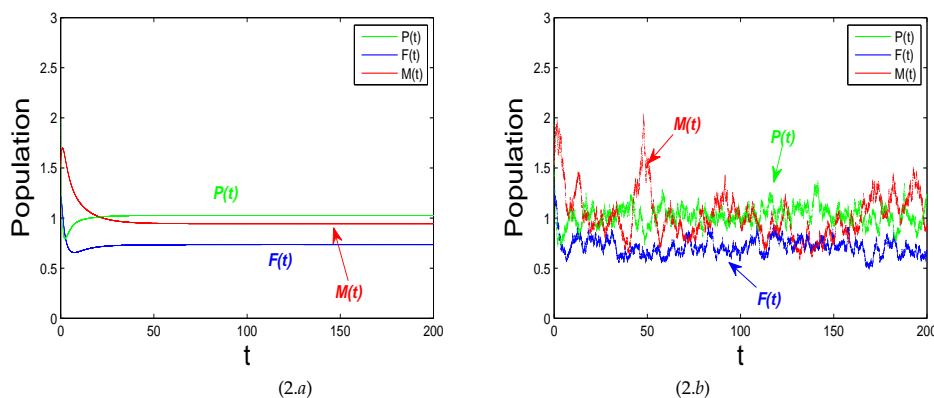


Figure 2: Trajectories for both the deterministic (2.a) and the stochastic (2.b) systems for the values in Table 1.

In this section, we handle both deterministic (4) and stochastic (6) systems. In each figure, we depict both images side by side to understand the effect of environmental noise on these systems.

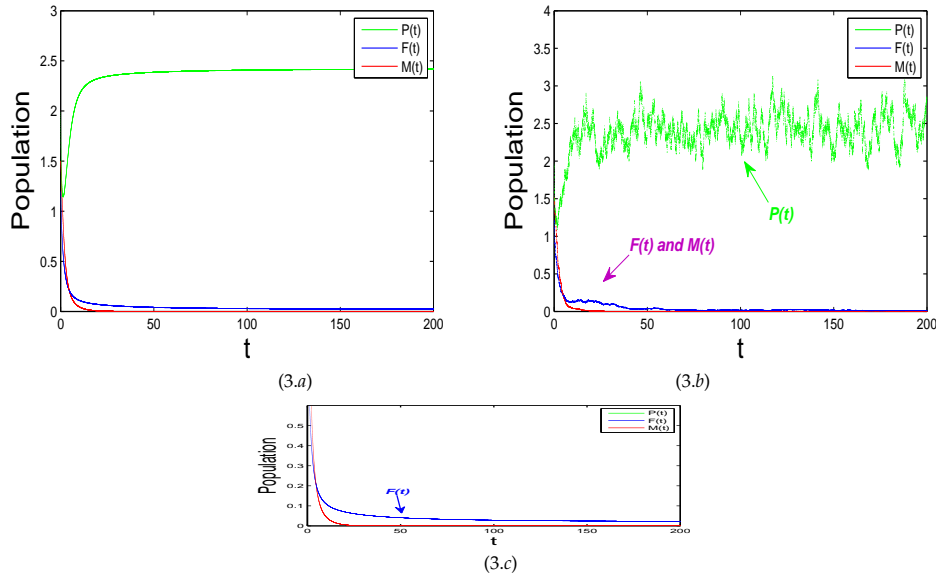


Figure 3: Fish does not go extinct for the deterministic (3.a) system but both the species go extinct for the stochastic (3.b) system, a specific region of (3.a) is in (3.c).

Now, we consider  $\gamma = 0.2, d_2 = 0.48$  and  $d_3 = 0.6$  to satisfy the conditions of extinction of fish and mussel, which are proved mathematically in Theorem 5.1. Keeping all other parametric values same as in Table 1 other than  $\gamma, d_2$  and  $d_3$ , we simulate both the systems and observe in Figure 3, that both the species (fish and mussel) go extinct (see Figure (3.b)) in stochastic system (6) but fish does not go to extinct in deterministic system (4)(see Figures (3.a) and (3.c)). For a clear view and better understanding, a specific portion of Figure (3.a) is displayed in Figure (3.c).

In our model,  $\theta$  is described as the contact rate of glochidia larvae (of mussel) with fish. Now, we shall verify the effect of this consideration by changing the value of  $\theta$ .

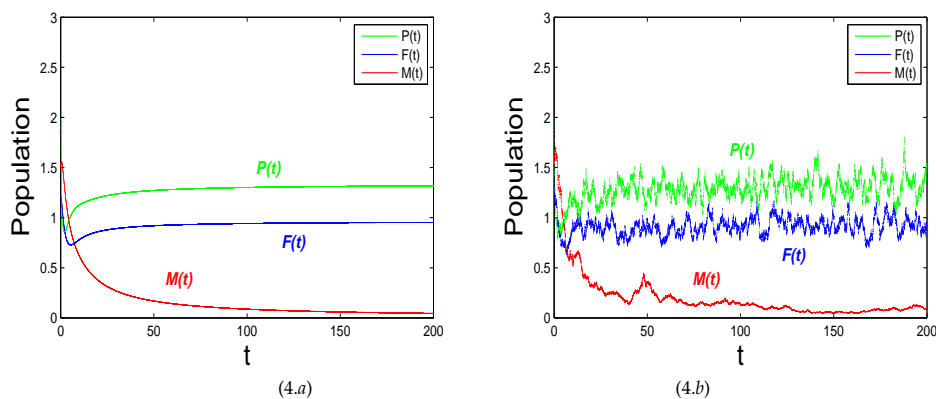


Figure 4: Effect of zero contact rate of glochidia larvae (of mussel) with fish ( $\theta = 0$ ) on the systems (4) and (6).

First, we consider the case when glochidia larvae does not find any suitable host, i.e.,  $\theta = 0$ . In that case, mussel reaches to their maximum death rate ( $d$ ). Taking  $\theta = 0$ , we simulate the systems and the output is depicted in Figure 4.



Second, we consider the case when a suitable host is easily available and most of the microscopic larvae (glochidia) find a host fish to infect with glochidia to complete the reproductive process. We take  $\theta = 10$  (a high value of  $\theta$ ) and simulate the system to observe the effect (see Figure 5).

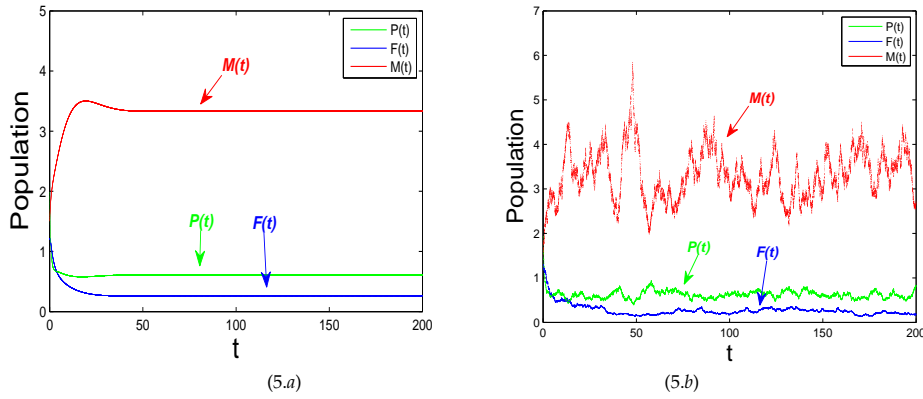


Figure 5: Effect of respectively high contact rate of glochidia larvae (of mussel) with fish ( $\theta = 10$ ) on the systems (4) and (6).

In the third case, we consider  $\theta \rightarrow \infty$ , i.e.,  $d' = 0$ , in that case mussel survives with minimum death rate ( $d_3$ ). We simulate the systems to verify the consideration of fish dependent death rate of mussel (see Figure 6).

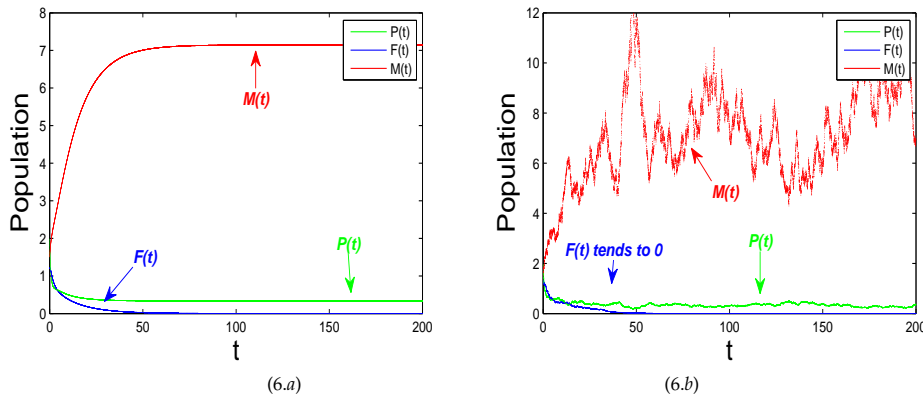


Figure 6: Effect of maximum contact rate of glochidia larvae (of mussel) with fish ( $\theta \rightarrow \infty$ ) on systems (4) and (6).

We verify the effect of toxicity by considering the values of  $c_1$  and  $c_2$  very high. We consider  $c_1 = 1.2$  and  $c_2 = 1.5$  and the results are depicted in Figure 7.

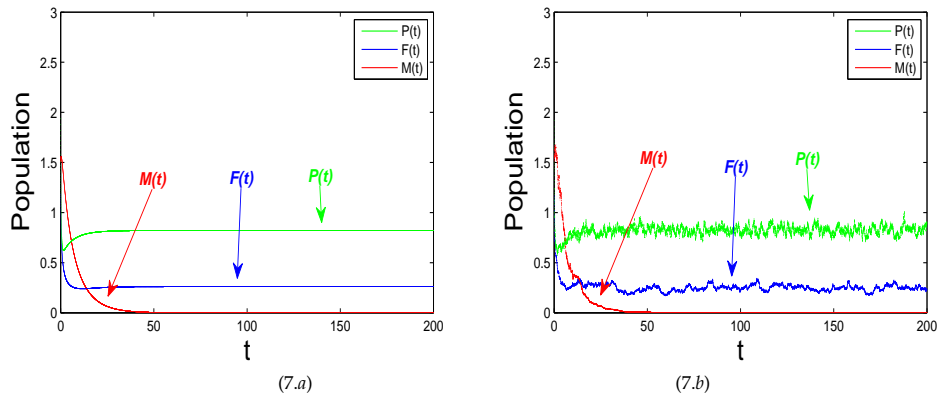


Figure 7: Effect of high toxicity on the systems (4) and (6).

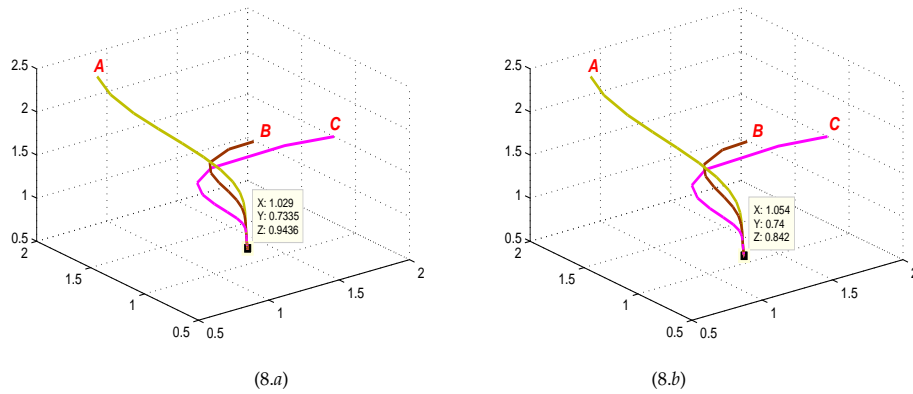


Figure 8: Phase portrait of deterministic (8.a) and stochastic (8.b) system, starting from different initial points  $A(0.7, 1.7, 2.5)$ ,  $B(1.2, 0.9, 2)$  and  $C(2, 1.2, 1.5)$ , where  $(X, Y, Z) \equiv (P, F, M)$ .

In Figure 8 ((8.b) and (8.a)), we see in a phase portrait that starting from different initial points  $A(0.7, 1.7, 2.5)$ ,  $B(1.2, 0.9, 2)$  and  $C(2, 1.2, 1.5)$ , the solutions converge to the point  $(1.054, 0.74, 0.842)$  for the underlying system (6) (stochastic system) and to the point  $(1.029, 0.7335, 0.9436)$  for the underlying system (4) (deterministic system), i.e. both the systems are asymptotically stable.

Now, we consider a high input rate of plankton into the system and investigate the effects. Taking  $a = 10$  and keeping all other parameters same as Table 1, we simulate the systems, depicted in Figure 9.

In this section, we find effects of various factors on the systems by changing numerical values of the parameters. In the next section, we shall discuss the effects and shall try to reach at some conclusions which should be biologically meaningful.

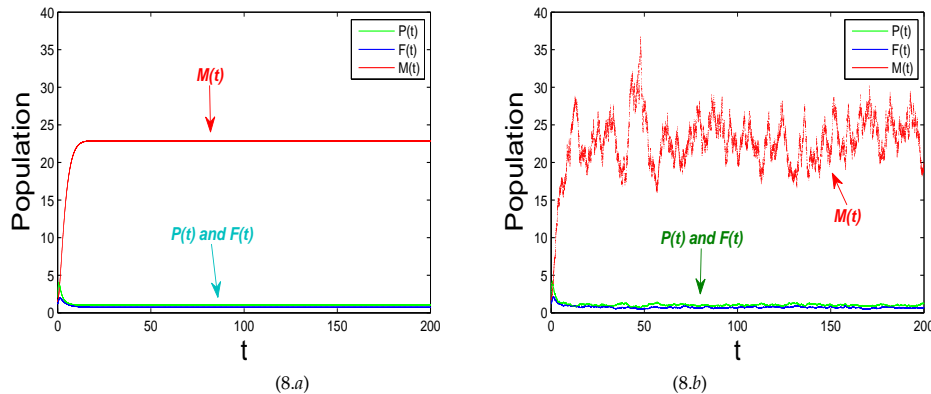


Figure 9: Effect of high constant input rate ( $a = 10$ ) of plankton on the systems (4) and (6).

## 9. Discussion and conclusion

We have considered a three species model where one is a primary link of a food chain and other two species depend on it for survival. It is taken into consideration the reproduction procedure of a parasite on other species by introducing host dependent death rate. Effect of external toxic substance is also considered to formulate the model. The dependency of growth rate, death rate, reproduction procedure on various natural factors such as temperature, humidity, environmental pollution etc. lead us to study the dynamics of the system in a randomly fluctuating environment. This model may be used to study the dynamics of an ecosystem which includes parasite. This model can also be used in fish farm by considering  $P$  as an external food source and the input rate can be controlled as per requirement.

In mathematical studies, we have proved that global positive solution exists uniquely for both the deterministic (4) and stochastic (6) systems. Boundedness of solutions are also proved. Since plankton has a constant input rate in the system, it can not extinct and it should have a lower bound which is also found mathematically. Extinction criterion for fish and mussel are derived and it is observed that the conditions involve effective transferred energy rate, initial value of plankton (food source), death rate and intensity of noise. These conditions clearly indicate that species will go extinct when (i) food is not available, (ii) zooplankton is unavailable to transfer the energy from phytoplankton, (iii) death rate becomes very high for any external reasons (environment or artificial reasons) or (iv) random change of climate highly affects on the death rate of these species. Numerically, we have justified the result in Figure 3. In Figure 3, it is also found that fish does not go extinct (3.a) in deterministic system although it goes extinct (3.b) in stochastic system for the same parametric values. This fact leads us to realize the effect of environmental noise on the system, i.e., environmental factors should not be ignored.

Mathematically, we have derived the conditions for persistence in mean of the system and numerically have shown that the system is stochastically asymptotically stable under some conditions. Through trajectories of solutions in Figure 1 we have justified our finding of persistence. We have also proved that the solutions of the underlying system (6) are globally attractive in mean.

The effects of consideration of dependent death rate of mussel on host species (fish) are investigated through numerical simulation using MATLAB. The function of death rate involves contact rate ( $\theta$ ) of microscopic larvae with fish. We have considered  $\theta = 0, 0.6, 10$  and  $\theta \rightarrow \infty$  to realize the behaviour. Comparing Figures 4, 1, 5 and 6 (respectively for corresponding values of  $\theta$ ), it can be concluded that our consideration meets the biological expectation that high contact rate can increase mussel population. It is also found in Figure 6 that if each and every microscopic larvae is able to find a host fish for their reproduction, then huge mussel population causes extinction of fish from this habitat.

Effect of toxic substance in the models ((4) and (6)) is also interesting. From Figure 7, it can be concluded that consideration of high value toxic coefficient causes extinction of mussel population although toxic does not affect directly on mussel. Since plankton and fish are directly affected by environmental toxic substance,

high toxicity causes low density of both these species, and, since mussel is totally dependent on plankton for food and fish for reproduction, low density of both these species causes extinction of mussel.

In Figure 9, we observe that high constant input rate of plankton mostly benefits mussel population, i.e., more availability of food highly increases the mussel but fish does not go to extinct. So, it is observed that high input rate of food does not affect the coexistence of species.

The model we have studied can be made more interesting if we take the functional responses as Holling Type II [13]. Consumption rates can be considered as  $\frac{\alpha P}{h_1 + P}$  and  $\frac{\beta P}{h_2 + P}$  towards plankton by fish and mussel respectively, where  $h_1$  and  $h_2$  are half saturation constants for fish and mussel respectively. The following model can be used for future studies:

$$\begin{aligned} dP &= \left( a - \frac{\alpha P}{h_1 + P} F - \frac{\beta P}{h_2 + P} M - d_1 P - c_1 P^2 \right) dt - \sigma_1 P dw_1 \\ dF &= F \left( \frac{\gamma P}{h_1 + P} - \eta F - d_2 - c_2 F^2 \right) dt - \sigma_2 F dw_2 \\ dM &= M \left( \frac{\delta P}{h_2 + P} - d_3 - \frac{d'}{1 + \theta F} \right) dt - \sigma_3 M dw_3 \end{aligned} \quad (11)$$

with initial conditions  $P_0 > 0$ ,  $F_0 > 0$  and  $M_0 > 0$ . Here, all the parameters are same as in the model (6) and  $h_1, h_2$  are half saturation constants.

Harvesting can also be introduced in these models ((6) and (11)) and may be used to study the fish farm ecosystem without neglecting environmental factors.

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