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Multi-Parameter Setting (C,α) Means with Respect to One Dimensional Vilenkin System

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Abstract. We prove that the maximal operator of the (C,α_n)-means of the one dimensional Vilenkin-Fourier series is of weak type(L^1,L^1). Moreover, we prove the almost everywhere convergence of the (C,α_n) means of integrable functions (i.e. $\sigma_n^{\alpha_n} f \longrightarrow f$), where $n \in \mathbb{N}_{\alpha,q}$ and $n \longrightarrow \infty$ for $f \in L^1(G_m)$, G_m is a bounded Vilenkin group, for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$.

1. Introduction

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [12]. In 2007 Akhobadze [3] introduced the notion of (C,α) means of trigonometric Fourier series with variable parameter setting. Fine [6] proved this for Walsh-Paley system for constant sequences. On the rate of convergence of (C,α) means in the constant sequences case see the paper of Fridli [7]. For the two dimensional case see the paper of Goginava [10]. The almost everywhere convergence of this summability method for a constant parameter in the quadraterial partial sums of double Vilenkin-Fourier series was proved by Gát and Goginiva in 2006 [5]. In 2008 Abu Joudeh and Gát [1] proved for variable Parameter setting in the case of Walsh-Paley system. In this paper we proved the almost everywhere convergence of the (C,α) means in a multi-parameter setting with respect to the one dimensional bounded Vilenkin system. The a.e. divergence of Cesàro means with varying parameters of Walsh-Fourier series was investigated by Tetunashvili [15].
First we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced to the theory of Vilenkin systems. These orthonormal systems were introduced by N.Ya. Vilenkin in 1947 (see [16]) as follows.
Denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{P}$ the set of positive integers, respectively. Denote $m := (m_k : k \in \mathbb{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbb{N}$ and Z_{m_k} the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0,1,2,...,m_k-1\}$, with the group operation $\mod m_k$ addition. Since the group is discrete, every subset is open. The normalized Haar measure μ_k on Z_{m_k} is defined by $\mu_k(\{j\}) := \frac{1}{m_k}(j \in \{0,1,...,m_k-1\})$. Let

$$G_m := \bigtimes_{k=0}^{\infty} Z_{m_k}.$$

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Then, every $x \in G_m$ can be represented by a sequence $x = (x_i,\ i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbb{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups, only. The Vilenkin group is metrizable in the following way:

$$d(x,y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods of G_m can be given by the intervals: $I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$ for $x \in G_m$, $n \in \mathbb{P}$. Let $0 = (0,\ i \in \mathbb{N}) \in G_m$ denote the null element of G_m and $I_n(0) := I_n$, $\bar{I}_n = G_m \backslash I_n$.
Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) $(1 \le p \le \infty)$, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ $(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$. We say that an operator $T : L^1 \to L^0$ ($L^0(G_m)$ is the space of measurable functions on G_m) is of type (L^p, L^p) (for $1 \le p \le \infty$) if $\|Tf\|_p \le C_p\|f\|_p$ for all $f \in L^p(G_m)$ and the constant C_p depends only on p. We say that T is of weak type (L^1, L^1) if $\mu(|Tf| > \lambda) \le C\|f\|_1/\lambda$ for all $f \in L^1(G_m)$ and $\lambda > 0$. Let $M_0 := 1$ and $M_{k+1} := m_k M_k$, for $k \in \mathbb{N}$ be the so-called generalized powers. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \le n_k < m_k$, $n_k \in \mathbb{N}$. This allows one to say that the sequence $(n_0, n_1, ...)$ is the expansion of n with respect to m. We often use the following notations. Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \le n < M_{|n|+1}$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$. Next we introduce on G_m an orthonormal system we call Vilenkin system.

For $k \in \mathbb{N}$ and $x \in G_m$ denote by r_k the k-th generalized Rademacher function:

$$r_k(x) := \exp(2\pi\imath \frac{x_k}{m_k}) \quad (x \in G_m,\ \imath := \sqrt{-1},\ k \in \mathbb{N}).$$

The n^{th} Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m and all the characters of G_m are of the this form. Define the m-adic addition as $k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbb{N})$. Then $\psi_{k\oplus n} = \psi_k \psi_n$, $\psi_n(x + y) = \psi_n(x)\psi_n(y)$, $\psi_n(-x) = \bar{\psi}_n(x)$, $|\psi_n| = 1 \quad (k,\ n \in \mathbb{N},\ x,\ y \in G_m)$.
Denote the Dirichlet and the Fejér or (C,1) kernels respectively as,

$$D_n := \sum_{k=0}^{n-1} \psi_k,\ K_n := \frac{1}{n+1} \sum_{k=0}^{n} D_k.$$

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the (C, α) kernels and means with respect to the Vilenkin system ψ as follows:

$$\hat{f}(n) := \int_{G_m} f\bar{\psi}_n d\mu,$$
$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$
$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f,$$
$$K_n^\alpha := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} D_k,$$
$$\sigma_n f := \sigma_n^1 f,\ K_n := K_n^1 \quad (f \in L^1(G_m)).$$

It is known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x) \quad (n \in \mathbb{N},\ f \in L^1(G_m)).$$

It is also well-known that(see [4], [5])

$$D_{M_n}(y,x) = \begin{cases} M_n, & \text{if } y \in I_n(x) \\ 0, & \text{if } y \notin I_n(x) \end{cases}$$
$$S_{M_n} f(x) = M_n \int_{I_n(x)} f d\mu = E_n f(x) \quad (f \in L^1(G_m),\ n \in \mathbb{N}), \tag{1}$$
$$D_{sM_n} = D_{M_n} \sum_{k=0}^{s-1} \psi_{kM_n} = D_{M_n} \sum_{k=0}^{s-1} r_n^k.$$

Define the kernel and means of the (C, α_n) summability method as follows

$$K_n^{\alpha_n} = \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n} A_{n-t}^{\alpha_n - 1} D_t$$
$$\sigma_n^{\alpha_n} f(x) := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k(x) = \int_{G_m} f(y) K_n^{\alpha_n}(x-y) d\mu(y) \quad (f \in L^1(G_m))$$

where

$$A_k^{\alpha_n} = \frac{(\alpha_n+1)(\alpha_n+2)...(\alpha_n+n)}{k!} \quad \text{(for all real number } \alpha_n \neq -1, -2, -3, ...).$$

It is known in [18] that,

$$A_n^{\alpha_n} = \sum_{k=0}^{n} A_k^{\alpha_n - 1},\ A_k^{\alpha_n} - A_{k+1}^{\alpha_n} = \frac{-\alpha_n A_k^{\alpha_n}}{k+1}. \tag{2}$$

Introduce the following notations: for a, s, $n \in \mathbb{N}$ let $n_{(s)} := \sum_{j=0}^{s-1} n_j M_j$, that is, $n_{(0)} = 0$, $n_{(1)} = n_0$ and for $M_B \leq n < M_{B+1}$, let $M_B \leq n < M_{B+1}$, $|n| := B$, $n = n_{(B+1)}$.

Next, introduce the following functions and operators for the multi-parameter setting ($n \in \mathbb{N}$, $0 < \alpha_a < 1$).

$$T_n^{\alpha_a} = \frac{1}{A_n^{\alpha_a}} \sum_{k=0}^{n_B M_B - 1} A_{n-k}^{\alpha_a - 1} D_k,$$

$$\tilde{T}_n^{\alpha_a} := \frac{n_B D_{M_B}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1}$$

$$+ \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \left|K_j\right| + \alpha_a \left|K_{n_B M_B - 1}\right|,$$

$$t_n^{\alpha_a} f(y) := \int_{G_m} f(x) T_n^{\alpha_a}(y-x) d\mu(x),$$

$$\tilde{t}_n^{\alpha_a} f(y) := \int_{G_m} f(x) \tilde{T}_n^{\alpha_a}(y-x) d\mu(x).$$

Define two variable function $P(n, \alpha) := \sum_{i=0}^{\infty} n_i M_i^{\alpha}$ for $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$. For example $P(n, 1) = n$. Besides, set for sequences $\alpha = (\alpha_n)$ and positive reals q, the subset of natural numbers

$$\mathbb{N}_{\alpha,q} := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq q \right\}.$$

For sequence α such that $0 < \alpha_0 \leq \alpha_n < 1$ we have $\mathbb{N}_{\alpha,q} = \mathbb{N}$ for some q depending only on α_0. We remark that $M_n \in \mathbb{N}_{\alpha,q}$ for every $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$ and $q \geq 1$.
In this paper, C denotes an absolute constant and C_q another one which may depend only on q. Besides, introduce the following kernel functions and operators for the case where $n \in \mathbb{N}_{\alpha,q}$ and $0 < \alpha_n < 1$.

$$\tilde{K}_n^{\alpha_n} := \left|\tilde{T}_n^{\alpha_n}\right| + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} n_l D_{M_l} + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} |T_{n_{(l-1)}}^{\alpha_n}|,$$

$$\tilde{\sigma}_n^{\alpha_n} f(y) := \int_{G_m} f(x) \tilde{K}_n^{\alpha_n}(y-x) d\mu(x).$$

Lemma 1.1. *[3] If k and n are natural numbers, then*

a). $C_1(1+\alpha_n)(2+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n} < C_2(1+\alpha_n)(2+\alpha_n)k^{\alpha_n}$, $-2 < \alpha_n < -1$;

b). $C_1(1+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n} < C_2(1+\alpha_n)k^{\alpha_n}$, $-1 < \alpha_n < 0$;

c). $C_1(d)k^{\alpha_n} < A_k^{\alpha_n} < C_2(d)k^{\alpha_n}$, $0 < \alpha_n \leq d$.

where C_1, C_2 are positive absolute constants(though in case (c) they depend on d).

Lemma 1.2. *[5] Let $0 \leq j < n_t M_t$ and $0 \leq n_t < m_t$. Then, $D_{n_t M_t - j} = D_{n_t M_t} - \psi_{n_t M_t - 1} \bar{D}_j$.*

Proof. We know that this result is not a new one, but in order to give some introduction to the methods of Vilenkin system we give here the proof of [5].
It is clear that

$$D_{n_t M_t} = D_{n_t M_t - j} + \sum_{k=n_t M_t - j}^{n_t M_t - 1} \psi_k = D_{n_t M_t - j} + \sum_{k=0}^{j-1} \psi_{n_t M_t - k - 1}.$$

Consequently,

$$\psi_{n_t M_t - k - 1}(x) = \psi_{(n_t-1)M_t + (m_{t-1}-1)M_{t-1} + \ldots + (m_0-1)M_0 - k}(x)$$
$$= \psi_{(n_t-k_t-1)M_t + (m_{t-1}-k_{t-1}-1)M_{t-1} + \ldots + (m_0-k_0-1)M_0}(x)$$
$$= \psi_{(n_t-1)M_t + (m_{t-1}-1)M_{t-1} + \ldots + (m_0-1)M_0}(x) \bar{\psi}_k(x)$$
$$= \psi_{n_t M_t - 1}(x) \bar{\psi}_k(x).$$

Hence, the Lemma follows. □

2. Main Results

Lemma 2.1. *For* $n, a \in \mathbb{N}$, $M_B \le n < M_{B+1}$, $|n| = B$, $\alpha_a \in (0,1)$. *Then,* $|T_n^{\alpha_a}| \le \tilde{T}_n^{\alpha_a}$.

Proof. Since $|n| = B$. Then,

$$A_n^{\alpha_a} T_n^{\alpha_a} = \sum_{j=0}^{n_B M_B - 1} A_{n-j}^{\alpha_a - 1} D_j = \sum_{j=0}^{n_B M_B - 1} A_{n_B M_B + n_{(B)} - j}^{\alpha_a - 1} D_j$$
$$= \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1} D_{n_B M_B - j}.$$

By Lemma 1.2 and (1) we have

$$T_n^{\alpha_a} = \frac{D_{n_B M_B}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1} - \frac{\psi_{n_B M_B - 1}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a} \bar{D}_j$$
$$= \frac{D_{M_B}}{A_n^{\alpha_a}} \sum_{k=0}^{n_B - 1} r_n^k \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1} - \frac{\psi_{n_B M_B - 1}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a} \bar{D}_j$$
$$= \frac{D_{M_B}}{A_n^{\alpha_a}} \sum_{k=0}^{n_B - 1} r_n^k \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1} + I.$$

This implies that

$$\left|T_n^{\alpha_a}\right| \le \left| \frac{D_{M_B}}{A_n^{\alpha_a}} \sum_{k=0}^{n_B - 1} r_n^k \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1} \right| + \left|I\right|$$
$$\le \frac{D_{M_B}}{A_n^{\alpha_a}} \sum_{k=0}^{n_B - 1} \left|r_n^k\right| \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1} + \left|I\right|$$
$$= \frac{n_B D_{M_B}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n^{(B)}+j}^{\alpha_a - 1} + \left|I\right|.$$

By the help of Abel's transformation and (2) we get

$$|I| = \left| - \frac{\psi_{n_B M_B - 1}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1} \bar{D}_j \right|$$
$$= \frac{1}{A_n^{\alpha_a}} \left| \sum_{j=0}^{n_B M_B - 2} \left[A_{n_{(B)}+j}^{\alpha_a - 1} - A_{n_{(B)}+j+1}^{\alpha_a - 1} \right] \sum_{i=0}^{j} \bar{D}_i + A_{n_{(B)}+n_B M_B}^{\alpha_a - 1} \sum_{i=0}^{n_B M_B - 1} \bar{D}_i \right|$$
$$\le \sum_{j=0}^{n_B M_B - 2} \frac{(1-\alpha_a) A_{n_{(B)}+j}^{\alpha_a - 1}}{A_n^{\alpha_a}} \frac{j+1}{n_{(B)} + j + 1} \left|K_j\right| + \frac{A_n^{\alpha_a - 1}}{A_n^{\alpha_a}} \left| \sum_{i=0}^{n_B M_B - 1} D_i \right| =: h_1 + h_2.$$

It is Known from Lemma 1.1 that $\frac{A^{\alpha_a-1}_{n_{(B)}+j}}{A^{\alpha_a}_n} \leq \frac{\alpha_a(n_{(B)}+j)^{\alpha_a-1}}{n^{\alpha_a}}$.
So, the situation for h_1 becomes

$$\sum_{j=0}^{n_BM_B-2}\left|\frac{(1-\alpha_a)A^{\alpha_a-1}_{n_{(B)}+j}}{A^{\alpha_a}_n}\frac{j+1}{n_{(B)}+j+1}K_j\right|$$
$$\leq \sum_{j=0}^{n_BM_B-2}\left|\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}(n_{(B)}+j)^{1-\alpha_a}}\frac{j+1}{n_{(B)}+j+1}K_j\right|$$
$$\leq \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}}\sum_{j=0}^{n_BM_B-2}\frac{j+1}{(n_{(B)}+j)^{1-\alpha_a}(n_{(B)}+j+1)}\left|K_j\right|$$
$$\leq \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}}\sum_{j=0}^{n_BM_B-2}\frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}}\left|K_j\right|.$$

The case for h_2 becomes

$$h_2 = \frac{A^{\alpha_a-1}_n}{A^{\alpha_a}_n}\left|\sum_{i=0}^{n_BM_B-1}D_i\right| = \frac{A^{\alpha_a-1}_n}{A^{\alpha_a}_n}(n_BM_B)\left|K_{n_BM_B-1}\right|$$
$$\leq \frac{\alpha_a(n_BM_B)}{n}\left|K_{n_BM_B-1}\right| \leq \alpha_a\left|K_{n_BM_B-1}\right|.$$

Thus,

$$|I| \leq \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}}\sum_{j=0}^{n_BM_B-2}\frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}}\left|K_j\right| + \alpha_a\left|K_{n_BM_B-1}\right|.$$

The proof completed. □

Now, we need to prove the next Lemma which means that the maximal operator $\tilde{t}^{\alpha_a}_* := \sup_{n,a\in\mathbb{N}}|\tilde{t}^{\alpha_a}_n|$ is quasi-local. This Lemma together with the next one are the most important tools in the proof of the main results of this paper.

Lemma 2.2. *Let* $1 > \alpha_a > 0$, $f \in L^1(G_m)$ *such that* $\mathrm{supp} f \subset I_k(u)$, $\int_{I_k(u)} f d\mu(x) = 0$ *for some m-adic interval* $I_k(u)$. *Then, we have* $\int_{\bar{I}_k(u)} \sup_{n,a\in N}|\tilde{t}^{\alpha_a}_n f|d\mu(x) \leq C\|f\|_1$.

Proof. We can easily show that for $n < M_k$ and $x \in I_k(u)$, $y \in \bar{I}_k(u)$ we have

$$\tilde{T}^{\alpha_a}_n(y-x) = \tilde{T}^{\alpha_a}_n(y-u),$$
$$\int_{I_k(u)} f(x)\tilde{T}^{\alpha_a}_n(y-x)d\mu(x) = \tilde{T}^{\alpha_a}_n(y-u)\int_{I_k(u)} f(x)d\mu(x) = 0.$$

Consequently,

$$\int_{\bar{I}_k(u)}\sup_{n,a\in N}\left|\tilde{t}^{\alpha_a}_n f\right|d\mu = \int_{\bar{I}_k(u)}\sup_{n\geq M_k,a\in N}\left|\tilde{t}^{\alpha_a}_n f\right|d\mu.$$

By the shift invariance of the Haar measure it can be supposed that $u = 0$. That is, $I_k(u) = I_k$. Thus,

$$\int_{\bar{I}_k(u)}\sup_{n\geq M_k,a\in N}|\tilde{t}^{\alpha_a}_n f|d\mu = \int_{\bar{I}_k}\sup_{n\geq M_k,a\in N}\left|\int_{I_k}\tilde{T}^{\alpha_a}_n(y-x)f(x)d\mu(x)\right|d\mu(y).$$

By Lemma 2.1 we have,

$$\int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} \tilde{T}_n^{\alpha_a}(y-x)f(x)d\mu(x) \right| d\mu(y)$$

$$= \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) \left[\frac{n_B D_{M_B}(y-x)}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n-j}^{\alpha_a - 1} \right.\right.$$

$$+ \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \frac{j+1}{(n^{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right|$$

$$\left.\left. + \alpha_a \left| K_{n_B M_B - 1}(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$= \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) \left[\frac{n_B D_{M_B}(y-x)}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n-j}^{\alpha_a - 1} \right] d\mu(x) \right| d\mu(y)$$

$$+ \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \frac{j+1}{(n^{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$+ \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) \left[\alpha_a \left| K_{n_B M_B - 1}(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$:= \phi_1 + \phi_2 + \phi_3.$$

It is simple to find out that

$$\frac{n_B D_{M_B}(y-x)}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n-j}^{\alpha_a - 1} = 0,$$

for any $y - x \in \bar{I}_k$. This holds because $D_{M_B}(y-x) = 0$ for $B = |n| \geq k$ and $y - x \in \bar{I}_k$. Hence, $\phi_1 = 0$. Besides,

$$\phi_2 = \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \frac{j+1}{(n^{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$= \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{M_k - 1} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right.\right.$$

$$\left.\left. + \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=M_k}^{n_B M_B - 2} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$\leq \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{M_k - 1} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$+ \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=M_k}^{n_B M_B - 2} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$:= \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) H_1(y-x) d\mu(x) \right| d\mu(y)$$

$$+ \int_{\bar{I}_k} \sup_{n\geq M_k, a\in N} \left| \int_{I_k} f(x) H_2(y-x) d\mu(x) \right| d\mu(y).$$

However, since for any $j < M_k$ we have that the Fejér kernel $K_j(y-x)$depends with respect to x only on coordinates $x_0 = 0, ..., x_{k-1} = 0,$ then

$$\int_{I_k} f(x)\left|K_j(y-x)\right|d\mu(x) = |K_j(y)| \int_{I_k} f(x) = 0$$

gives $\int_{I_k} f(x)H_1(y-x)d\mu(x) = H_1(y)\int_{I_k} f(x)d\mu(x) = 0.$

On the other hand,

$$\begin{aligned}
&\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=M_k}^{n_B M_B - 1} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}}\left|K_j\right| \\
&\leq \sup_{j\geq M_k}\left|K_j\right| \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=1}^{n} \frac{j+1}{j^{2-\alpha_a}} \\
&\leq \sup_{j\geq M_k}\left|K_j\right| \frac{2\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=1}^{n} j^{\alpha_a - 1} \\
&\leq 2(1-\alpha_a)\sup_{j\geq M_k}\left|K_j\right|.
\end{aligned}$$

By Lemma 2.1 in [9], this implies

$$\begin{aligned}
&\int_{\bar{I}_k} \sup_{n\geq M_k,\, a\in N}\left|\int_{I_k} f(x)H_2(y-x)d\mu(x)\right|d\mu(y) \\
&\leq \int_{I_k}|f(x)|\Big(\int_{\bar{I}_k}\sup_{n\geq M_k,\, a\in N}\left|H_2(y-x)\right|d\mu(y)\Big)d\mu(x) \\
&\leq C\int_{I_k}|f(x)|\Big(\int_{\bar{I}_k}\sup_{j\geq M_k}\left|K_j(y-x)\right|d\mu(y)\Big)d\mu(x) \\
&\leq C\int_{I_k}|f(x)|d\mu(x) = C\left\|f\right\|_1.
\end{aligned}$$

Thus, $\phi_2 \leq C\left\|f\right\|_1.$

Similarly, for the case ϕ_3 we apply Lemma 2.1 in [9]

$$\begin{aligned}
\phi_3 &= \int_{\bar{I}_k} \sup_{n\geq M_k,\, a\in N}\left|\int_{I_k} f(x)\Big[\left|K_{n_B M_B - 1}(y-x)\right|\Big]d\mu(x)\right|d\mu(y) \\
&\leq \int_{I_k}|f(x)|\Big(\int_{\bar{I}_k}\sup_{n\geq M_k}\left|K_{n_B M_B - 1}(y-x)\right|d\mu(y)\Big)d\mu(x) \\
&\leq C\int_{I_k}|f(x)|d\mu(x) = C\left\|f\right\|_1.
\end{aligned}$$

Hence, the Lemma follows. □

Corollary 2.3. *Let* $1 > \alpha_a > 0$*.Then, we have*

$$||T_n^{\alpha_a}||_1 \leq ||\tilde{T}_n^{\alpha_a}||_1 \leq C;$$
$$||t_n^{\alpha_a} f||_1,\ ||\tilde{t}_n^{\alpha_a} f||_1 \leq C||f||_1$$

and

$$||t_n^{\alpha_a} g||_\infty,\ ||\tilde{t}_n^{\alpha_a} g||_\infty \leq C||g||_\infty$$

for all natural numbers a, n where C is some absolute constant and $f \in L^1$, $g \in L^\infty$. That is, operator $t_n^{\alpha_a}$, $\tilde{t}_n^{\alpha_a}$ are of type (L^1, L^1) and (L^∞, L^∞) and Uniformly in n.

Proof. The proof is direct consequence of Lemma 2.2. Then

$$\|\tilde{T}_n^{\alpha_a}\|_1 \leq C\frac{n_B\|D_{M_B}\|_1}{A_n^{\alpha_a}}\sum_{j=0}^{n_BM_B-1}A_{n-j}^{\alpha_a-1}$$
$$+\frac{(1-\alpha_a)}{n^{\alpha_a}}\sum_{j=0}^{n_BM_B-2}\frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}}\left\|K_j\right\|_1+\|K_{n_BM_B-1}\|_1.$$

Consequently, by $\|D_{M_B}\|_1$, $\|K_j\|_1 \leq C$, the proof of Corollary 5 follows. □

In the sequel we prove that maximal operator $\tilde{\sigma}_{*,q}^{\alpha} := \sup_{n\in\mathbb{N}_{\alpha,q}}|\tilde{\sigma}_n^{\alpha_n}|$ is quasi-local. The way we get this is by the investigation of kernel functions, its maximal function on the Vilenkin group by making a hole around zero and some quasi-locality issues (for the notion of quasi-locality see[13]). This is the very base of the proof of the main results of this paper. That is, Theorem 2.7.

Lemma 2.4. *Let $0 < \alpha_n < 1$, $f \in L^1(G_m)$ such that $suppf \subset I_k(u)$, $\int_{I_k(u)} fd\mu = 0$ for some m-adic interval $I_k(u)$. Then we have $\int_{G_m\setminus I_k(u)}\tilde{\sigma}_{*,q}^{\alpha}fd\mu \leq C_q\|f\|_1$. Where constants C_q can depend only on q.*

Proof. From the formula of the kernel function $\tilde{K}_n^{\alpha_n}$ we have

$$\tilde{K}_n^{\alpha_n} = \left|T_n^{\alpha_n}\right| + \sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}n_lD_{M_l} + \sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}|T_{n_{(l-1)}}^{\alpha_n}| =: N_1+N_2+N_3.$$

The integral,

$$\int_{G_m\setminus I_k(u)}\sup_{n\in N}\left|\int_{I_k(u)}f(x)\Big(N_2(y-x)\Big)d\mu(x)\right|d\mu(y) = 0$$

since $f * D_{M_l} = 0$ for $l < s \leq k$ because of the $\mathcal{A}_k$ measurablity of D_{M_l} and $\int f = 0$. Besides, $D_{M_l}(y-x) = 0$; for $s > k$, $y - x \notin I_k$.
Since from Lemma [3] we have

$$\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} \leq \frac{(n_{(l-1)})^{\alpha_n}}{n^{\alpha_n}} \leq C\frac{M_l^{\alpha_n}}{n^{\alpha_n}}.$$

Besides, by the help of Lemma 2.2 and by the fact that $n \in \mathbb{N}_{\alpha,q}$ implies $\sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} \leq C\sum_{l=0}^{B}\frac{M_l^{\alpha_n}}{n^{\alpha_n}} \leq C_q$ we get

$$\int_{G_m\setminus I_k(u)}\sup_{n\in\mathbb{N}_{\alpha,q}}\left|\int_{I_k(u)}f(x)\Big(N_1(y-x)+N_3(y-x)\Big)d\mu(x)\right|d\mu(y)$$
$$\leq \int_{G_m\setminus I_k(u)}\sup_{n\in\mathbb{N}_{\alpha,q}}\left|\int_{I_k(u)}f(x)\left(\left|\tilde{T}_n^{\alpha_n}(y-x)\right| + \sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}\left|\tilde{T}_{n_{(l-1)}}^{\alpha_a}(y-x)\right|\right)d\mu(x)\right|d\mu(y)$$
$$\leq C_q\int_{G_m\setminus I_k(u)}\sup_{n\in\mathbb{N}_{\alpha,q}}\left|\int_{I_k(u)}f(x)\left|\tilde{T}_n^{\alpha_n}(y-x)\right|d\mu(x)\right|d\mu(y)$$
$$\leq C_q\|f\|_1.$$

Hence, the Lemma follows. □

Lemma 2.5. *Let $0 < \alpha_n < 1$, $n \in \mathbb{N}$, $M_B \le n < M_{B+1}$, $|n| = B$. Then,*

$$|K_n^{\alpha_n}| \le \tilde{K}_n^{\alpha_n}.$$

Proof. By definition, we have

$$\begin{aligned} K_n^{\alpha_n} &= \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n-1} A_{n-j}^{\alpha_n-1} D_j \\ &= \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n_B M_B-1} A_{n-j}^{\alpha_n-1} D_j + \frac{1}{A_n^{\alpha_n}} \sum_{j=n_B M_B}^{n-1} A_{n-j}^{\alpha_n-1} D_j \\ &= T_n^{\alpha_n} + \frac{1}{A_n^{\alpha_n}} \sum_{j=n_B M_B}^{n_B M_B + n_{(B)}-1} A_{n_{(B)}+n_B M_B - j}^{\alpha_n-1} D_j. \end{aligned}$$

By Lemma 1.2 the situation for

$$\begin{aligned} &\frac{1}{A_n^{\alpha_n}} \sum_{j=n_B M_B}^{n_B M_B + n_{(B)}-1} A_{n_{(B)}+n_B M_B - j}^{\alpha_n-1} D_j \\ &= \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n-1} A_{n-t}^{\alpha_n-1} D_{t+n_B M_B} \\ &= \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n_{(B)}-1} A_{n-t}^{\alpha_n-1} \left(D_{n_B M_B} + \psi_{n_B M_B -1} \overline{D_t} \right) \\ &= \frac{D_{n_B M_B}}{A_n^{\alpha_n}} \sum_{t=0}^{n_{(B)}-1} A_{n-t}^{\alpha_n-1} + \frac{\psi_{n_B M_B-1}}{A_n^{\alpha_n}} \sum_{t=0}^{n_{(B)}-1} A_{n-t}^{\alpha_n-1} \overline{D_t}. \\ &= \frac{A_{n_{(B)}}^{\alpha_n}}{A_n^{\alpha_n}} \left(D_{n_B M_B} + \psi_{n_B M_B-1} \overline{K_{n_{(B)}}^{\alpha_n}} \right). \end{aligned}$$

Then,

$$K_n^{\alpha_n} = T_n^{\alpha_n} + \frac{A_{n_{(B)}}^{\alpha_n}}{A_n^{\alpha_n}} \left(D_{n_B M_B} + \psi_{n_B M_B-1} \overline{K_{n_{(B)}}^{\alpha_n}} \right).$$

In general, for $j = 1, ..., B+1,$ we get

$$K_{n_{(j)}}^{\alpha_n} = T_{n_{(j)}}^{\alpha_n} + \frac{A_{n_{(j-1)}}^{\alpha_n}}{A_{n_{(j)}}^{\alpha_n}} \left(D_{n_{(j-1)} M_{(j-1)}} + \psi_{n_{(j-1)} M_{(j-1)}-1} \overline{K_{n_{(j-1)}}^{\alpha_n}} \right).$$

Recursively applying this formula and Considering that $n_{(-1)} = 0$, $T_0^{\alpha_n} = K_0^{\alpha_n} = 0$, $A_0^{\alpha_n} = 1$, we get

$$\begin{aligned} |K_n^{\alpha_n}| &\le |T_n^{\alpha_n}| + \sum_{l=0}^{B} \left(\prod_{j=l}^{B} \frac{A_{n_{(j-1)}}^{\alpha_n}}{A_{n_{(j)}}^{\alpha_n}} D_{M_l} \sum_{k=0}^{n_l-1} |r_n^k| + \prod_{j=l}^{B} \frac{A_{n_{(j-1)}}^{\alpha_n}}{A_{n_{(j)}}^{\alpha_n}} |T_{n_{(l-1)}}^{\alpha_n}| \right) \\ &= \left| \tilde{T}_n^{\alpha_n} \right| + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} n_l D_{M_l} + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} |T_{n_{(l-1)}}^{\alpha_n}| = \tilde{K}_n^{\alpha_n}. \end{aligned}$$

Hence, the Lemma follows. □

Now, we plug into the main tool for the proof of Theorem 2.7. Define operators as follows

$$\sigma^{\alpha}_{*,q} f := \sup_{n \in \mathbb{N}_{\alpha,q}} |\sigma^{\alpha_n}_n f|,\ \tilde{\sigma}^{\alpha}_{*,q} f := \sup_{n \in \mathbb{N}_{\alpha,q}} |\tilde{\sigma}^{\alpha_n}_n f|.$$

Lemma 2.6. *The operator $\tilde{\sigma}^{\alpha}_{*}$ is of type (L^∞, L^∞) and Weak type (L^1, L^1); σ^{α}_{*} is of Weak type (L^1, L^1).*

Proof. By the help of the method of Lemma 2.2 and Corollary 2.3 we get that

$$\|\tilde{K}^{\alpha_n}_n\|_1 \le \|T^{\alpha_n}_n\|_1 + \sum_{l=0}^{B} \frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n} n_l \|D_{M_l}\|_1 + \sum_{l=0}^{B} \frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n} \|T^{\alpha_n}_{n_{(l-1)}}\|_1$$
$$\le C + C \sum_{l=0}^{B} \frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n} \le C_q$$

since $n \in \mathbb{N}_{\alpha,q}$. Thus, $\tilde{\sigma}^{\alpha}_{*,q}$ is of type (L^∞, L^∞).
To proof the weak type(L^1, L^1) case we apply Calderon-Zygmund decomposition Lemma [9].
Let $f \in L^1$ and $\|f\|_1 < \delta$. Then there is a decomposition:

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that

$$\|f_0\|_\infty \le C\delta,\ \|f_0\|_1 \le C\|f\|_1,\ G^j_m = I_{k_j}(u^j)$$

are disjoint m-adic intervals for which

$$supp f_j \subset G^j_m,\ \int_{G^j_m} f_j d\mu = 0,\ |F| \le \frac{C\|f\|_1}{\delta}$$

$(u^j \in G_m, k_j \in \mathbb{N}, j \in \mathbb{P})$, where $F = \bigcup_{i=1}^{\infty} G^j_m$.
By the σ-sublinearity of the maximal operator with an appropriate constant C_q we have

$$\mu(\tilde{\sigma}^{\alpha}_{*,q} f > 2C_q\delta) \le \mu(\tilde{\sigma}^{\alpha}_{*,q} f_0 > C_q\delta) + \mu(\tilde{\sigma}^{\alpha}_{*,q} \sum_{j=1}^{\infty} f_j > C_q\delta) := W + M.$$

Since $\tilde{\sigma}^{\alpha}_{*,q}$ is of type (L^∞, L^∞), we have that

$$\|\tilde{\sigma}_{*,q} f_0\|_\infty \le C_q \|f_0\|_\infty \le C_q\delta$$

then we have $W = 0$. The situation for M becomes,

$$M = \mu(\tilde{\sigma}^{\alpha}_{*,q} \sum_{j=1}^{\infty} f_j > C_q\delta) \le |F| + \mu(\bar{F} \cap [\tilde{\sigma}^{\alpha}_{*,q} \sum_{j=1}^{\infty} f_j > C_q\delta])$$
$$\le \frac{C\|f\|_1}{\delta} + \frac{C_q}{\delta} \sum_{i=1}^{\infty} \int_{G_m \setminus G^j_m} \sigma^{\alpha}_{*,q} f_j d\mu =: \frac{C\|f\|_1}{\delta} + \frac{C_q}{\delta} \sum_{i=1}^{\infty} N_j,$$

in which

$$N_j = \int_{G_m \setminus G^j_m} \sigma^{\alpha}_{*,q} f_j d\mu \le \int_{G_m \setminus I_{k_j}(u^j)} \sup_{n \in \mathbb{N}_{\alpha,q}} \left| \int_{I_{k_j}(u^j)} f_j(x) \tilde{K}^{\alpha_n}_n (y - x) d\mu(x) \right| d\mu(y).$$

The next estimation for N_j is given by Lemma 2.4. Then,

$$N_j \leq C_q \|f_j\|_1.$$

That is, operator $\tilde{\sigma}^{\alpha}_{*,q}$ is of weak type (L^1, L^1).
By Lemma 2.5 and since

$$\mu(\sigma^{\alpha}_{*,q} f > 2C_q\delta) \leq \mu(\tilde{\sigma}^{\alpha}_{*,q}|f| > 2C_q\delta) \leq C_q \frac{\|f\|_1}{\delta}.$$

We concluded that the maximal operator $\sigma^{\alpha}_{*,q}$ is of weak type (L^1, L^1).
Hence, the Lemma follows. □

Theorem 2.7. *Let* $0 < \alpha_n < 1$. *Let* $f \in L^1(G_m)$. *Then* $\sigma^{\alpha_n}_n f \longrightarrow f$ *if* $n \longrightarrow \infty$, $n \in \mathbb{N}_{\alpha,q}$.

Proof. Let us consider a Vilenkin Polynomial P such that $P(x) = \sum_{i=0}^{M_k-1} c_i\psi_i$. Then for all natural number $n \geq M_k$, $n \in \mathbb{N}_{\alpha,q}$ we have that $S_n P \equiv P$. Thus, the statement $\sigma^{\alpha_n}_n P \longrightarrow P$ holds everywhere which is not only for $n \in \mathbb{N}_{\alpha,q}$. Now, let ϵ, $\delta > 0$, $f \in L^1$. Let P be a Vilenkin polynomial such that $\|f - P\|_1 < \delta$. Then,

$$\begin{aligned}
&\mu(\overline{\lim_{n\in\mathbb{N}_{\alpha,q}}} |\sigma^{\alpha_n}_n f - f| > \epsilon) \\
&\leq \mu(\overline{\lim_{n\in\mathbb{N}_{\alpha,q}}} |\sigma^{\alpha_n}_n (f - P)| > \frac{\epsilon}{3}) + \mu(\overline{\lim_{n\in\mathbb{N}_{\alpha,q}}} |\sigma^{\alpha_n}_n P - P| > \frac{\epsilon}{3}) \\
&+ \mu(\overline{\lim_{n\in\mathbb{N}_{\alpha,q}}} |\sigma^{\alpha_n}_n P - f| > \frac{\epsilon}{3}) \\
&\leq \mu(\overline{\lim_{n\in\mathbb{N}_{\alpha,q}}} |\sigma^{\alpha_n}_n (f - P)| > \frac{\epsilon}{3}) + 0 + \frac{3}{\epsilon}\|P - f\|_1 \\
&\leq C_q\|P - f\|_1 \frac{3}{\epsilon} \leq \frac{C_q}{\epsilon}\delta
\end{aligned}$$

since (from Lemma 2.6) $\sigma^{\alpha}_{*,q}$ is of weak type(L^1, L^1) with any fixed $q > 0$. This holds for all $\delta > 0$.
That is, for an arbitrary $\epsilon > 0$

$$\mu(\overline{\lim_{n\in\mathbb{N}_{\alpha,q}}} |\sigma^{\alpha_n}_n f - f| > \epsilon) = 0$$

and as a result we also have

$$\mu(\overline{\lim_{n\in\mathbb{N}_{\alpha,q}}} |\sigma^{\alpha_n}_n f - f| > 0) = 0.$$

This finally gives $\overline{\lim}_{n\in\mathbb{N}_{\alpha,q}} |\sigma^{\alpha_n}_n f - f| = 0$ a.e. Consequently, $\sigma^{\alpha_n}_n f \longrightarrow f$ a.e provided that $n \longrightarrow \infty$, $n \in \mathbb{N}_{\alpha,q}$.
Hence, the Theorem follows. □

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