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Universality on Spaces Continuously Containing Topological Groups

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Abstract. In 1986 V.V. Uspenskij proved that there exists a universal topological group with a countable base and in 1990 put the problem: does there exist a universal topological group of weight an uncountable cardinal τ? This problem is still open. In 2015 we gave the notion of a continuously containing space for a given collection of topological groups and proved that there exists such a space of weight τ for the collection of all topological groups of weight $\leq \tau$. In the present paper we prove that in the class of all topological spaces of weight $\leq \tau$, which are continuously containing spaces for a collection of topological groups, there are universal elements.

1. Introduction and Preliminaries

The development of Topology is directly connected with the consideration of new classes of objects. One of the main question, which naturally arises in the consideration of any new class of topological spaces, is whether there are universal elements, that is elements of this class, containing topologically all other elements of the class. This fact is emphasized in the paper [1] (see Section 3.3, Problem 7 and Problem 8), where the following two general problems are posed:

"Problem 7. Let $\mathcal{P}$ be a given class of topological spaces. Determine (if that is possible) "standard" topological spaces Y, into which it is possible to imbed each space in $\mathcal{P}$. Find the simplest such space Y."

A variant of Problem 7 is the following *universal object problem*:

"Problem 8. Which classes $\mathcal{P}$ of topological spaces contain a space X into which each space in $\mathcal{P}$ imbeds?"

Here, the spaces Y of the above Problem 7 will be called *containing spaces for* $\mathcal{P}$ and the spaces X of the Problem 8 *universal spaces in* $\mathcal{P}$.

The above problems can be posed not only in classes of topological spaces but also in any category, where the notion of "embedding" is defined. Universal objects for many categories are constructed in [10], where (see also [8]) a method of construction of universal and containing spaces is developed. In the papers [3–7], [9, 11–19] using this method universal objects are also constructed for categories of: topological spaces (with different dimension invariants), separable metric spaces, mappings, topological groups and frames.

Concerning topological groups we recall that a topological group T is *universal* in a class $\mathbb{G}$ of topological groups if $T \in \mathbb{G}$ and each element $X \in \mathbb{G}$ is isomorphic to a subgroup of T. Two universal elements for the

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class of all separable metrizable topological groups are constructed in [21, 22]. These universal elements are the group of all self-homeomorphisms of the Hilbert cube with the topology of uniform convergence and the group of isometries of the Urysohn universal metric space. Another such a space is constructed in [2]. Universal elements in categories of (metrizable) Abelian topological groups are considered in [20]. But the problems of the existence of universal elements for some classes of topological groups remain open. Such classes are the class of all topological groups (see Question 2 of [22]) and the class of all metrizable topological groups (see Problem 4 of [20]) of a given uncountable weight. Other classes of topological groups (connected with dimensions), in which the problems of the existence of universal elements is still open, are given in the paper [15].

In the paper [15], so-called spaces, continuously containing all elements of a given collection of topological groups, were introduced as an alternative of universal topological groups. It was proved that there exists a continuously containing space for the collection of all topological groups of weight $\leq \tau$. In the present paper, using the above mentioned method of [8, 10], we construct universal elements in classes of continuously containing spaces. Below, we recall the definition of these spaces and briefly explain the construction of Containing Spaces, denoted by $\mathrm{T}(\mathbf{B}, \mathrm{R})$, which will be the corresponding universal elements.

Assumptions and notations. In this paper we assume that all considered spaces and topological groups are T_0-spaces of weight $\leq \tau$, where τ is a **fixed** infinite cardinal. The symbol " $\equiv$ " in an expression means that one or both sides of the expression are new notations. The symbol "$\times$" will be used for the product of sets. The operation in each topological group X will be called *product* and the product of two subsets $A \subset X$ and $B \subset X$ will be denoted by AB or by $(AB)|_X$ if we like to indicate the group in which the product is considered. The inverse element of an element $x \in X$ will be denoted by x^{-1} or by $x^{-1}|_X$. We put $A^{-1} \equiv A^{-1}|_X = \{x^{-1}|_X : x \in A\}$. (We note that if A or B is empty set, then $A \times B = AB = \emptyset$.)

Definition 1.1. [**Continuously containing spaces**; see [15]] Let Q be a topological space, and let $\mathcal{G}$ be a collection of subsets of Q such that each element of $\mathcal{G}$ with the relative topology is a topological group. We say that $\mathcal{G}$ is *continuous* if the following conditions hold:

(1) for any two points x and y of Q, belonging to an element X of $\mathcal{G}$, and for each neighbourhood U of $xy \in X$ in Q, there exist neighbourhoods V and W of x and y, respectively, in Q such that, for each element $Y \in \mathcal{G}$, we have

$$((V \cap Y)(W \cap Y))|_Y \subset U \cap Y;$$

(2) for each point x of Q, belonging to an element X of $\mathcal{G}$, and for each neighbourhood U of $x^{-1} \in X$ in Q there exists a neighbourhood V of x in Q such that for each element $Y \in \mathcal{G}$ we have $(V \cap Y)^{-1}|_Y \subset U \cap Y$;

(3) the union of all elements of $\mathcal{G}$ is Q.

Let $\mathbf{G}$ be an indexed collection of topological groups. We say that a topological space Q is a *continuously containing space for* $\mathbf{G}$ if for each element $X \in \mathbf{G}$ there exists a topological embedding h^X_Q of X into Q (therefore, the subset $h^X_Q(X)$ of Q with the relative topology is a topological group) such that the collection $\{h^X_Q(X) : X \in \mathbf{G}\}$ of subsets of Q is continuous. In this case, we shall say that Q is a continuously containing space for $\mathbf{G}$ *with respect to the collection* $\{h^X_Q : X \in \mathbf{G}\}$ *of embeddings*.

Assumption. In what follows of this note, for convenient of notation, whenever we consider an arbitrary continuously containing space Q for an indexed collection $\mathbf{G}$ of topological groups with respect to a collection $\{h^X_Q : X \in \mathbf{G}\}$ of embeddings, we shall identify each point $x \in X \in \mathbf{G}$ with the point $h^X_Q(x)$ and therefore the group X will be identified with the subset $h^X_Q(X)$ of Q.

On the Containing Spaces $\mathrm{T}(\mathbf{B}, \mathrm{R})$ (see Section 2 of [8] and Chapter 1 of [10]). Let $\mathbf{S}$ be an indexed collection of spaces. Any *Containing Space*, denoted by $\mathrm{T}(\mathbf{B}, \mathrm{R})$, is uniquely determined by a *base* $\mathbf{B}$ for $\mathbf{S}$ (in [8, 10] the base $\mathbf{B}$ is called mark):

$$\mathbf{B} \equiv \{\{U^X_\delta : \delta \in \tau\} : X \in \mathbf{S}\}, \tag{1.1}$$

where $\{U^X_\delta : \delta \in \tau\}$ is an indexed base for the open subsets of $X \in \mathbf{S}$, and by a *family* R *of equivalence relations on* $\mathbf{S}$:

$$R \equiv \{\sim^s : s \in \mathcal{F}\}, \tag{1.2}$$

where $\mathcal{F}$ is the set of all non-empty finite subset of τ. From the family R it is required that the following conditions are satisfied:

(a) the number of equivalence classes of the relation $\sim^s$, $s \in \mathcal{F}$, is finite;

(b) $\sim^s \subset \sim^t$ for each $t \subset s \in \mathcal{F}$;

(c) for each $s \in \mathcal{F}$ and $X, Y \in \mathbf{S}$ the condition $X \sim^s Y$ implies that the algebra A^X_s of subsets of X, generated by the set $\{U^X_\delta : \delta \in s\}$, is isomorphic to the algebra A^Y_s of subsets of Y, generated by the set $\{U^Y_\delta : \delta \in s\}$. Moreover, there is an isomorphism $i : A^X_s \to A^Y_s$ such that $i(U^X_\delta) = U^Y_\delta$ for each $\delta \in s$. A family R of equivalence relations on **S**, satisfying only conditions (a) and (b), is called *admissible* and if, additionally, R satisfies condition (c), then R is called **B**-*admissible*.

We denote by $C(\sim^s)$ the set of all equivalence classes of the relation $\sim^s$ and put $C(R) = \cup\{C(\sim^s) : s \in \mathcal{F}\}$.

We recall the construction of the Containing Space $T \equiv T(\mathbf{B}, R)$. Suppose that (1.1) is a base for a collection **S** of spaces and (1.2) is a **B**-admissible family of equivalence relations on **S**. On the set P of all pairs (x, X), where $x \in X \in \mathbf{S}$, we define an equivalence relation, denoted by $\sim^{\mathbf{B}}_R$, as follows. Two pairs $(x, X), (y, Y) \in P$ are $\sim^{\mathbf{B}}_R$-equivalent if and only if (a) $X \sim^s Y$ for each $s \in \mathcal{F}$ and (b) for each $\delta \in \tau$ we have $x \in U^X_\delta$ if and only if $y \in U^Y_\delta$. The set T is the set of all equivalence classes of the relation $\sim^{\mathbf{B}}_R$. The set

$$B^T \equiv \{U^T_\delta(\mathbf{H}) : \delta \in \tau, \mathbf{H} \in C(R)\},$$

where $U^T_\delta(\mathbf{H})$ is the set, consisting of all point $\mathbf{a} \in T$ such that there exists an element $(x, X) \in \mathbf{a}$ for which $X \in \mathbf{H}$ and $x \in U^X_\delta$ (and, therefore, for each element $(x, X) \in \mathbf{a}$ we have $X \in \mathbf{H}$ and $x \in U^X_\delta$), is a base for a topology on T (see Lemma 2.8 of [8]).

The mapping i^X_T of X into T, defining by relation $i^X_T(x) = \mathbf{a} \in T$, where $x \in X \in \mathbf{S}$ and $\mathbf{a}$ is the point of T, containing the pair (x, X), is an embedding of X into T. This embedding is called *natural* (see Proposition 2.10 of [8]).

We shall use also the notion of an extension of bases for **S** and the notion of final refinement of a family of equivalence relations on **S**. Let

$$\mathbf{B}_1 \equiv \{\{U^X_{1,\delta} : \delta \in \tau\} : X \in \mathbf{S}\} \text{ and } \mathbf{B}_2 \equiv \{\{U^X_{2,\delta} : \delta \in \tau\} : X \in \mathbf{S}\}$$

be two bases for **S**. We say that $\mathbf{B}_2$ is an *extension* of $\mathbf{B}_1$ if there exists an one-to-one mapping $\varphi : \tau \to \tau$ such that $U^X_{1,\delta} = U^Y_{2,\varphi(\delta)}$, $\delta \in \tau$. In this case φ is called the *extension mapping* from $\mathbf{B}_1$ to $\mathbf{B}_2$. Let

$$R_1 \equiv \{\sim^s_1 : s \in \mathcal{F}\} \text{ and } R_2 \equiv \{\sim^s_2 : s \in \mathcal{F}\}$$

be two families of equivalence relations on **S**. We say that R_2 is a *final refinement* of R_1 if for each $s \in \mathcal{F}$ there exists $t \in \mathcal{F}$ such that $\sim^t_2 \subset \sim^s_1$.

Now, we define the notion of a saturated class of spaces (see Section 3 of [8] and Chapter 2 of [10]). Let $\mathbb{S}$ be a class of spaces. We say that $\mathbb{S}$ is *saturated* if for each indexed collection **S** of elements of $\mathbb{S}$, there exists a base $\mathbf{B}_0$ for **S** with the following property: for every extension **B** of $\mathbf{B}_0$, there exists a **B**-admissible family $R^{\mathbf{B}}$ of equivalence relations on **S** such that, for each admissible family R of equivalence relations on **S** being a final refinement of $R^{\mathbf{B}}$, the containing space $T(\mathbf{B}, R)$ belongs to $\mathbb{S}$. The base $\mathbf{B}_0$ is called *initial base* for **S** (*corresponding to the class* $\mathbb{S}$) and the family $R^{\mathbf{B}}$ *initial family of equivalence relations on* **S** *corresponding to* **B** (*and the class* $\mathbb{S}$).

Below, we indicate some saturated classes of spaces.

(1) the class of all completely regular spaces of weight $\leq \tau$ (see Proposition 3.8 of [8]);

(2) the class of all completely regular n-dimensional spaces (in the sense of *ind*) of weight $\leq \tau$ (see Corollary 3.1.6 of [10]);

(3) the class of countable-dimensional spaces (in the sense of *ind*) of weight $\leq \tau$ (see Proposition 4.4.4 of [10]);

(4) the class of strongly countable-dimensional spaces (in the sense of *ind*) of weight $\leq \tau$ (see Proposition 4.4.4 of [10]).

(5) the intersection of any two saturated classes of spaces.

Many other saturated classes are indicated in [10].

The results. Let Q_i, $i = 1, 2$, be a continuously containing space for an indexed collection $\mathbf{G}_i$, $i = 1, 2$, of topological groups. We say that an embedding $f : Q_1 \to Q_2$ is *proper* if there exists a mapping $\varphi : \mathbf{G}_1 \to \mathbf{G}_2$ such that for each $X \in \mathbf{G}_1$, the restriction $f|_X$ of f onto X is an isomorphism of X onto $\varphi(X)$.

Let $\mathbb{C}$ be a class of continuously containing spaces for indexed collections of topological groups. We say that an element $\mathrm{T} \in \mathbb{C}$ is *properly universal* in this class if for each element $X \in \mathbb{C}$ there exists a proper embedding of X into T.

The main results of this paper are stated in the following theorem.

Theorem 1.2. *Let $\mathbb{S}$ be a saturated class of spaces and* $\mathbf{G}$ *a collection of topological groups. Then:*

(1) *in the subclass $\mathbb{C}(\mathbf{G})$ of $\mathbb{S}$ of all continuously containing spaces for the indexations of* $\mathbf{G}$ *there are properly universal elements;*

(2) *in the subclass $\mathbb{C}$ of $\mathbb{S}$ of all continuously containing spaces for indexed collections of topological groups, there are properly universal elements.*

Considering in the above theorem $\mathbb{S}$ as concrete saturated classes we obtain different results, which are independent each of other. For example, since the classes of all completely regular spaces and all completely regular n-dimensional spaces (in the sense of *ind*) are saturated we have the following corollaries.

Corollary 1.3. (1) *In the class of all completely regular spaces, which are continuously containing spaces for indexations of a fixed collection of topological groups, there are properly universal elements.*

(2) *In the class of all completely regular spaces, which are continuously containing spaces for indexed collections of topological groups, there are properly universal elements.*

Corollary 1.4. (1) *In the class of all completely regular n-dimensional spaces (in the sense of ind), which are continuously containing spaces for indexations of a fixed collection of topological groups, there are properly universal elements.*

(2) *In the class of all completely regular n-dimensional spaces (in the sense of ind), which are continuously containing spaces for indexed collections of topological groups, there are properly universal elements.*

2. Proof of Theorem 1.2

Let Q be a continuously containing space for an indexation $\mathbf{G}_{in}$ of the collection $\mathbf{G}$ of topological groups. We define a subset A_Q of $Q \times Q$, a mapping $p_Q : A_Q \to Q$, and a mapping $i_Q : Q \to Q$ as follows: a point $(x, y) \in Q \times Q$ belongs to A_Q if and only if there exists an element $X \in \mathbf{G}_{in}$ such that $x, y \in X$. In this case we put $p_Q(x, y) = (xy)|_X$ and $i_Q(x) = x^{-1}|_X$.

Lemma 2.1. *Let Q be a continuously containing space for an indexation $\mathbf{G}_{in}$ of the collection $\mathbf{G}$ of topological groups. Then, the mappings p_Q and i_Q are well-defined (that is, they are independent of the topological group $X \in \mathbf{G}$ that is considered in their definitions) and continuous.*

Proof. Let $x, y \in X \cap Y$, where $X, Y \in \mathbf{G}_{in}$. To prove that p_Q is well-defined it suffices to prove that for each points $x, y \in X \cap Y$ we have $(xy)|_X = (xy)|_Y$. Suppose that $(xy)|_X \neq (xy)|_Y$. Consider, for example, a neighbourhood U of $(xy)|_X$ in Q such that $(xy)|_Y \notin U$. Since Q is a continuously containing space for $\mathbf{G}_{in}$ there exist neighbourhoods V and W of x and y in Q, respectively, such that

$$((V \cap Z)(W \cap Z))|_Z \subset U \tag{2.1}$$

for neach $Z \in \mathbf{G}_{in}$. Then, since $x \in V \cap Y$ and $y \in W \cap Y$ for $Z = Y$ we must have $(xy)|_Y \in U$, which is a contradiction. Similarly we prove that i_Q is well-defined.

Now, we prove that p_Q is continuous. Let $(x, y) \in A_Q$ and let U be a neighbourhood of $p_Q(x, y)$ in Q. Then, there exists $X \in \mathbf{G}_{in}$ such that $x, y \in X$ and therefore $p_Q(x, y) = (xy)|_X = xy$. Consider the neighbourhoods V and W of x and y in Q respectively, which satisfy relation (2.1) and put $H = A_Q \cap (V \times W)$. Then, H is a neighbourhood of the point (x, y) in A_Q. We prove that $p_Q(H) \subset U$. Indeed, let $(x', y') \in H$. Then, there exists $Z \in \mathbf{G}_{in}$ such that $x', y' \in Z$ and therefore $x' \in V \cap Z$ and $y' \in W \cap Z$. By relation (2.1) we have $x'y' = p_Q(x', y') \in U$, proving that p_Q is continuous. Similarly we prove that i_Q is continuous. □

2.2 Proof of the case (1) **of Theorem 1.2.** By set-theoretical reasons, we can assume that there exists a collection $\mathbf{Q}(\mathbf{G})$ of elements of $\mathbb{C}(\mathbf{G})$ having the following property: for each continuously containing space Q for an indexation of $\mathbf{G}$ with respect to some collection of embeddings there exists a proper embedding of Q onto an element of $\mathbf{Q}(\mathbf{G})$. Moreover, we can suppose that $\mathbf{Q}(\mathbf{G})$ is indexed by a set $M_{\mathbf{G}}$, that is

$$\mathbf{Q}(\mathbf{G}) = \{Q_\mu : \mu \in M_{\mathbf{G}}\}.$$

In the rest of the proof of the case (1) we put $M = M_{\mathbf{G}}$ and $\mathbf{Q} = \mathbf{Q}(\mathbf{G})$. Therefore, each element $Q_\mu \in \mathbf{Q}$, $\mu \in M$, is a continuously containing space for an indexation $\mathbf{G}_\mu$ of $\mathbf{G}$ by a set Λ_μ:

$$\mathbf{G}_\mu \equiv \{X_{(\mu,\lambda)} : \lambda \in \Lambda_\mu\}, \tag{2.2}$$

with respect to an indexed collection $\{h^{X_{(\mu,\lambda)}}_{Q_\mu} : \lambda \in \Lambda_\mu\}$ of embeddings. We shall prove that there is an object $\mathrm{T} \in \mathbb{C}(\mathbf{G})$ such that for each element $Q \in \mathbf{Q}$ there is a proper embedding of Q into T. In this case, since the composition of proper embeddings is a proper embedding, for each space $Q \in \mathbb{C}(\mathbf{G})$ there will be a proper embedding of Q into T, that is T will be the required proper universal element. This will prove the case (1).

The space T will be constructed as the Containing Space $\mathrm{T}(\mathbf{B}, \mathrm{R})$ for some indexed base $\mathbf{B}$ for $\mathbf{Q}$ and suitable $\mathbf{B}$-admissible family R of equivalence relations on $\mathbf{Q}$. Below, we give the construction of $\mathbf{B}$ and R.

We define the base $\mathbf{B}$ to be an arbitrary initial base for $\mathbf{Q}$ corresponding to the saturated class $\mathbb{S}$. We will assume that

$$\mathbf{B} = \{\{W^{Q_\mu}_\eta : \eta \in \tau\} : \mu \in M\},$$

where $\{W^{Q_\mu}_\eta : \eta \in \tau\}$ is an indexed base for the open subsets of the space Q_μ.

We define the family

$$\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$$

to be an initial family of equivalence relations on $\mathbf{Q}$, corresponding to the base $\mathbf{B}$ and the class $\mathbb{S}$, satisfying the following two conditions:

(A) for each $s \in \mathcal{F}$ and for each two $\sim^s$-equivalent elements Q_{μ_0} and Q_{μ_1} of $\mathbf{Q}$, $\mu_0, \mu_1 \in M$, the relation

$$(W^{Q_{\mu_0}}_{\eta_1} \times W^{Q_{\mu_0}}_{\eta_2}) \cap A_{Q_{\mu_0}} \subset (p_{Q_{\mu_0}})^{-1}(W^{Q_{\mu_0}}_{\eta_0}) \tag{2.3}$$

for some $\eta_0, \eta_1, \eta_2 \in s$, implies the relation

$$(W^{Q_{\mu_1}}_{\eta_1} \times W^{Q_{\mu_1}}_{\eta_2}) \cap A_{Q_{\mu_1}} \subset (p_{Q_{\mu_1}})^{-1}(W^{Q_{\mu_1}}_{\eta_0}); \tag{2.4}$$

(B) for each $s \in \mathcal{F}$ and for each two $\sim^s$-equivalent elements Q_{μ_0} and Q_{μ_1} of $\mathbf{Q}$, $\mu_0, \mu_1 \in M$, the relation

$$W^{Q_{\mu_0}}_{\eta_1} \subset i^{-1}_{Q_{\mu_0}}(W^{Q_{\mu_0}}_{\eta_0}) \tag{2.5}$$

for some $\eta_0, \eta_1 \in s$, implies the relation

$$W^{Q_{\mu_1}}_{\eta_1} \subset i^{-1}_{Q_{\mu_1}}(W^{Q_{\mu_1}}_{\eta_0}). \tag{2.6}$$

The existence of the family R with the above properties can easily be proved.

Now, we prove that the Containing Space $\mathrm{T} \equiv \mathrm{T}(\mathbf{B}, \mathrm{R})$ is the required element of $\mathbb{C}(\mathbf{G})$. First, we note that by the choice of $\mathbf{B}$ and R, the Containing Space T is an element of $\mathbb{S}$. Moreover, if $i^{Q_\mu}_{\mathrm{T}}$, $\mu \in M$, is the natural embedding of Q_μ into T, then for each $\lambda \in \Lambda_\mu$ the mapping

$$h^{X_{(\mu,\lambda)}}_{\mathrm{T}} \equiv i^{Q_\mu}_{\mathrm{T}} \circ h^{X_{(\mu,\lambda)}}_{Q_\mu}$$

is an embedding of $X_{(\mu,\lambda)}$ into T and

$$\mathrm{T} = \cup\{i^{Q_\mu}_{\mathrm{T}}(Q_\mu) : \mu \in M\}.$$

Since

$$Q_\mu = \cup\{h_{Q_\mu}^{X_{(\mu,\lambda)}}(X_{(\mu,\lambda)}) : \lambda \in \Lambda_\mu\}$$

we have

$$i_{\mathrm{T}}^{Q_\mu}(Q_\mu) = \cup\{i_{\mathrm{T}}^{Q_\mu}(h_{\mathrm{T}}^{X_{(\mu,\lambda)}})(X_{(\mu,\lambda)})) : \lambda \in \Lambda_\mu\}$$

and, therefore (using Assumption of Section 1),

$$\mathrm{T} = \cup\{h_{\mathrm{T}}^{X_{(\mu,\lambda)}}(X_{(\mu,\lambda)}) : (\mu,\lambda) \in N\}, \tag{2.7}$$

where

$$N = \{(\mu,\lambda) : \mu \in M, \lambda \in \Lambda_\mu\}.$$

To complete the proof of the case (1) it is enough to prove that T is a continuously containing space for the indication

$$\mathbf{G}_{in} \equiv \{X_{(\mu,\lambda)} : (\mu,\lambda) \in N\}$$

of $\mathbf{G}$ with respect to the indexed collection

$$\{h_{\mathrm{T}}^{X_{(\mu,\lambda)}} : (\mu,\lambda) \in M \times \Lambda_\mu\}$$

of embeddings, that is to prove that conditions (1) − (3) of Definition 1.1 are satisfied. We note that in this case the natural embedding $i_{\mathrm{T}}^{Q_\mu}$, $\mu \in M$, of Q_μ into T will be a proper embedding.

Condition (3) of Definition 1.1 is the relation (2.7). We prove condition (1). Let $x, y \in X \equiv X_{(\mu_0,\lambda_0)} \in \mathbf{G}_{in}$, where $\mu_0 \in M$ and $\lambda_0 \in \Lambda_{\mu_0}$, and let U be an open neighbourhood of $(xy)|_X$ in T. Without loss of generality, we can suppose that U is an element of the standard base of T, that is $U = U_{\eta_0}^{\mathrm{T}}(\mathbf{H}_0)$, where $\eta_0 \in \tau$ and $\mathbf{H}_0 \in \mathrm{C}(\sim^{s_0})$ for some $s_0 \in \mathcal{F}$. Since $(xy)|_X \in Q_{\mu_0}$, by the definition of the elements of the standard base of T, we have $Q_{\mu_0} \in \mathbf{H}_0$ and $W_{\eta_0}^{Q_{\mu_0}}$ is an open neighbourhood of $(xy)|_X$ in Q_{μ_0}. Since Q_{μ_0} is a continuously containing space for the indexation $\mathbf{G}_{\mu_0}$ of $\mathbf{G}$, there are open neighbourhoods V and W of x and y in Q_{μ_0}, respectively, such that

$$((V \cap X_{(\mu_0,\lambda)})(W \cap X_{(\mu_0,\lambda)}))|_{X_{(\mu_0,\lambda)}} \subset W_{\eta_0}^{Q^{\mu_0}}$$

for each $\lambda \in \Lambda_{\mu_0}$ (see condition (1) of Definition 1.1). Without loss of generality, we can suppose that $V = W_{\eta_1}^{Q_{\mu_0}}$ and $W = W_{\eta_2}^{Q_{\mu_0}}$ for some $\eta_1, \eta_2 \in \tau$. In this case, the above relation takes the form

$$((W_{\eta_1}^{Q_{\mu_0}} \cap X_{(\mu_0,\lambda)})(W_{\eta_2}^{Q_{\mu_0}} \cap X_{(\mu_0,\lambda)}))|_{X_{(\mu_0,\lambda)}} \subset W_{\eta_0}^{Q_{\mu_0}}$$

or, equivalently,

$$(W_{\eta_1}^{Q_{\mu_0}} \times W_{\eta_2}^{Q_{\mu_0}}) \cap A_{Q_{\mu_0}} \subset (p_{Q_{\mu_0}})^{-1}(W_{\eta_0}^{Q_{\mu_0}}). \tag{2.8}$$

Let s be an element of $\mathcal{F}$ such that $s_0 \cup \{\eta_0, \eta_1, \eta_2\} \subset s$. Denote by $\mathbf{H}$ the equivalence class of $\sim^s$ containing the space Q_{μ_0}. Then, $W_{\eta_1}^{\mathrm{T}}(\mathbf{H})$ and $W_{\eta_2}^{\mathrm{T}}(\mathbf{H})$ are open sets of T (they belong to the standard base of T). Since $x \in W_{\eta_1}^{Q_{\mu_0}}$ and $y \in W_{\eta_2}^{Q_{\mu_0}}$, by the definition of the elements of the standard base of a containing space, we have (using Assumption of Section 1) $x \in W_{\eta_1}^{\mathrm{T}}(\mathbf{H})$ and $y \in W_{\eta_2}^{\mathrm{T}}(\mathbf{H})$. Let $x', y' \in X_{(\mu_1,\lambda_1)} \in \mathbf{G}$, where $\mu_1 \in M$, $\lambda_1 \in \Lambda_{\mu_1}$, $x' \in W_{\eta_1}^{\mathrm{T}}(\mathbf{H})$ and $y' \in W_{\eta_2}^{\mathrm{T}}(\mathbf{H})$. Then, $x' \in W_{\eta_1}^{Q_{\mu_1}}$, $y' \in W_{\eta_2}^{Q_{\mu_1}}$ and $Q_{\mu_1} \in \mathbf{H}$. Since $Q_{\mu_0}, Q_{\mu_1} \in \mathbf{H}$ we have $Q_{\mu_0} \sim^s Q_{\mu_1}$. By the choice of s and the condition (a) of the definition of the family R, relation (2.8) implies the relation

$$(W_{\eta_1}^{Q_{\mu_1}} \times W_{\eta_2}^{Q_{\mu_1}}) \cap A_{Q_{\mu_1}} \subset (p_{Q_{\mu_1}})^{-1}(W_{\eta_0}^{Q_{\mu_1}})$$

or, equivalently,

$$((W_{\eta_1}^{Q_{\mu_1}} \cap X_{(\mu_1,\lambda)})(W_{\eta_2}^{Q_{\mu_1}} \cap X_{(\mu_1,\lambda)}))|_{X_{(\mu_1,\lambda)}} \subset W_{\eta_0}^{Q_{\mu_1}}$$

for each $\lambda \in \Lambda_{\mu_1}$. In particular, for $\lambda = \lambda_1$ we have

$$(x'y')|_{X_{(\mu_1,\lambda_1)}} \in W_{\eta_0}^{Q_{\mu_1}} \subset U_{\eta_0}^{\mathrm{T}}(\mathbf{H}),$$

proving condition (1) of the Definition 1.1.

We prove condition (2) of Definition 1.1. Let $x \in X_{(\mu_0,\lambda_0)} \in \mathbf{G}$, $\mu_0 \in M$, and let $U \equiv U^{\mathrm{T}}_{\eta_0}(\mathbf{H}_0)$ be an open neighbourhood of $x^{-1}|_{X_{(\mu_0,\lambda_0)}}$ in T, where $\eta_0 \in \tau$ and $\mathbf{H}_0 \in \mathrm{C}(\sim^{s_0})$ for some $s_0 \in \mathcal{F}$. Therefore, $Q_{\mu_0} \in \mathbf{H}_0$ and $x^{-1} = x^{-1}|_{X_{(\mu_0,\lambda_0)}} \in W^{Q_{\mu_0}}_{\eta_0}$. Since Q_{μ_0} is a continuously containing space for $\mathbf{G}_{\mu_0}$ there exists a neighbourhood $W^{Q_{\mu_0}}_{\eta_1}$ of x in Q_{μ_0}, $\eta_1 \in \tau$, such that

$$(W^{Q_{\mu_0}}_{\eta_1} \cap X_{(\mu_0,\lambda)})^{-1}|_{X_{(\mu_0,\lambda)}} \subset W^{Q_{\mu_0}}_{\eta_0} \cap X_{(\mu_0,\lambda)}$$

for each $\lambda \in \Lambda_{\mu_0}$ (see condition (2) of Definition 1.1), or, equivalently,

$$W^{Q_{\mu_0}}_{\eta_1} \subset i^{-1}_{Q_{\mu_0}}(W^{Q_{\mu_0}}_{\eta_0}). \tag{2.9}$$

Let $s_1 = s_0 \cup \{\eta_0, \eta_1\}$ and let $\mathbf{H}_1$ be the element of $\mathrm{C}(\sim^{s_1})$ containing the space Q_{μ_0}. Since $x \in W^{Q_{\mu_0}}_{\eta_1}$, by the definition of the elements of the standard base of T, $W^{\mathrm{T}}_{\eta_1}(\mathbf{H}_1)$ is a neighbourhood of x in T. Let $y \in X_{(\mu_1,\lambda_1)} \cap W^{\mathrm{T}}_{\eta_1}(\mathbf{H}_1)$, where $\mu_1 \in M$ and $\lambda_1 \in \Lambda_{\mu_1}$. Then, $Q_{\mu_1} \in \mathbf{H}_1$. Since Q_{μ_0} and Q_{μ_1} belong to the same equivalence class of the equivalence relation $\sim^{s_1}$, relation (2.9) implies that

$$W^{Q_{\mu_1}}_{\eta_1} \subset i^{-1}_{Q_{\mu_1}}(W^{Q_{\mu_1}}_{\eta_0}),$$

or, equivalently,

$$(W^{Q_{\mu_1}}_{\eta_1} \cap X_{(\mu_1,\lambda)}))^{-1}|_{X_{(\mu_1,\lambda)}} \subset W^{Q_{\mu_1}}_{\eta_0} \cap X_{(\mu_1,\lambda)},$$

proving condition (2) of Definition 1.1 and completing the proof of the case (1) of Theorem 1.2.

Proof of the case (2) **of Theorem 1.2.** By set-theoretical reasons, we can suppose that there is a collection $\mathcal{G}$ of elements $\mathbb{S}$, which are topological groups, such that each element of $\mathbb{S}$, which is a topological group, is isomorphic to an element of $\mathcal{G}$. Then, any continuously containing space Q for an indexation of a collection of topological groups from $\mathbb{S}$, that is any element of $\mathbb{C}$, will be a continuously containing space for an indexation of a subset of $\mathcal{G}$. For each non-empty subset $\mathbf{G}$ of $\mathcal{G}$ we consider the indexed collection

$$\mathbf{Q}(\mathbf{G}) = \{Q_\mu : \mu \in M_{\mathbf{G}}\}$$

of elements of $\mathbb{C}(\mathbf{G})$, constructed in the proof of the case (1) of Theorem 1.2, and put

$$\mathbf{Q}(\mathcal{G}) = \cup\{\mathbf{Q}(\mathbf{G}) : \mathbf{G} \text{ is a non-empty subset of } \mathcal{G}\}.$$

Considering that $M_{\mathbf{G}_1} \cap M_{\mathbf{G}_2} = \emptyset$ if $\mathbf{G}_1$ and $\mathbf{G}_2$ are distinct subsets of $\mathcal{G}$ we can suppose that $\mathbf{Q}(\mathcal{G})$ is indexed by the set

$$M_{\mathcal{G}} \equiv \cup\{M_{\mathbf{G}} : \mathbf{G} \text{ is non-empty subset of } \mathcal{G}\},$$

that is

$$\mathbf{Q}(\mathcal{G}) = \{Q_\mu : \mu \in M_{\mathcal{G}}\}.$$

The rest part of the proof is the same as in the case (1) replacing only the indexed collection $\mathbf{Q}(\mathbf{G})$ by the indexed collection $\mathbf{Q}(\mathcal{G})$ and the set $M_{\mathbf{G}}$ by the set $M_{\mathcal{G}}$.□

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