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The Cesàro Operator on Weighted Bergman Fréchet and (LB)-Spaces of Analytic Functions

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Abstract. The spectrum of the Cesàro operator C is determined on the spaces which arises as intersections $A^p_{\alpha+}$ (resp. unions $A^p_{\alpha-}$) of Bergman spaces A^p_α of order $1 < p < \infty$ induced by standard radial weights $(1-|z|)^\alpha$, for $0 < \alpha < \infty$. We treat them as reduced projective limits (resp. inductive limits) of weighted Bergman spaces A^p_α, with respect to α. Proving that these spaces admit the monomials as a Schauder basis paves the way for using Grothendieck-Pietsch criterion to deduce that we end up with a non-nuclear Fréchet-Schwartz space (resp. a non-nuclear (DFS)-space). We show that C is always continuous, while it fails to be compact or to have bounded inverse on $A^p_{\alpha+}$ and $A^p_{\alpha-}$.

1. Introduction

Let $H(\mathbb{D})$ denote the Fréchet space of all analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ equipped with the topology of uniform convergence on the compact subsets of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The classical Cesàro operator C is given by

$$f \mapsto C(f) : z \mapsto \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta}\mathrm{d}\zeta, \quad z \in \mathbb{D}\setminus\{0\},\ C(f)(0) := f(0), \tag{1.1}$$

for $f \in H(\mathbb{D})$. The Cesàro operator C is an isomorphism of $H(\mathbb{D})$ onto itself. From (1.1) one may obtain that

$$f(z) = (1-z)(zC(f)(z))', \quad f \in H(\mathbb{D}). \tag{1.2}$$

In the sense of Taylor coefficients

$$\hat{f}(j) := \frac{f^{(j)}(0)}{j!}, \quad n \in \mathbb{N}_0 \tag{1.3}$$

of the function $f \in H(\mathbb{D})$ given by

$$f(z) = \sum_{j=0}^{\infty} \hat{f}(j) z^j, \tag{1.4}$$

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one has the expression

$$Cf(z) = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \sum_{j=0}^{k} \hat{f}(j) \right) z^k, \quad z \in \mathbb{D}. \tag{1.5}$$

Let $L(E)$ denote the space of all continuous operators on a topological vector space E. For $\varphi \in H^\infty$, we denote M_φ the operator in $H(\mathbb{D})$ of multiplication by φ so that $M_\varphi \in L(H(\mathbb{D}))$. The differentiation operator $D\colon f \mapsto f'$ for $f \in H(\mathbb{D})$ also belongs to $L(H(\mathbb{D}))$. Then it follows from (1.1) that $C^{-1} \in L(H(\mathbb{D}))$ given by $C^{-1} = M_{1-z} D M_z$ (see [28, p. 1185]). That is, for all $h \in H(\mathbb{D})$ we have

$$C^{-1}(h)(z) = (1-z)(h(z) + zh'(z)), \quad z \in \mathbb{D}. \tag{1.6}$$

The continuity, compactness and spectrum of generalized Cesàro operators on Banach spaces of analytic functions on the unit disc have been studied by many authors [6–13, 16, 17, 20, 28, 30, 31]. Continuity of the Cesàro operator on the Hilbert space $H^2(\mathbb{D})$ was studied by Hardy, Littlewood, and Pólya [20]. Continuity of C on the general Hardy spaces and unweighted Bergman spaces A^p is due to Siskakis [30, 31]. Andersen [13] proved that the Cesàro operator is bounded on a class of spaces of analytic functions on the unit disc, including the weighted Bergman space. Boundedness and compactness of the class of a certain type of integral operators (also containing the Cesàro operator) acting on spaces of analytic functions on the unit disc have been studied by Aleman and Cima [7], Aleman and Siskakis [12]. Persson [28], Aleman and Constantin [8], Aleman and Persson [10] investigated the spectrum of (generalized) Cesàro operators on various spaces of analytic functions such as Hardy spaces, weighted Bergman spaces and Dirichlet spaces in detail. We refer the reader to the introduction of [10] for a comprehensive information on the development of the research in this area. The *Bergman space* $A^p_\alpha = A^p_\alpha(\mathbb{D})$ of order $1 < p < \infty$ induced by standard radial weight $(1-|z|)^\alpha$ for $0 < \alpha < \infty$ is given by

$$A^p_\alpha := \{f \in H(\mathbb{D}) : \|f\|_{p,\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p \, ds_\alpha(z) \right)^{1/p} < \infty\}, \tag{1.7}$$

where $ds_\alpha(z) = (1-|z|)^\alpha ds(z)$, and $ds(z) = \frac{1}{\pi} dx dy$. Some authors prefer to define the space A^p_α with the weight $(1-|z|^2)^\alpha$ instead of $(1-|z|)^\alpha$. Since we have $1-|z| \le 1-|z|^2 \le 2(1-|z|)$, these spaces coincide and the norms are equivalent. Each A^p_α is a closed subspace of $L^p(\mathbb{D}, ds_\alpha(z))$ in which the polynomials are dense [21, Section 1.1]. The weighted Bergman space A^p_α is a Banach space with the norm $\|\cdot\|_{p,\alpha}$. Classical Bergman space $A^p(\mathbb{D})$ corresponds to the case $\alpha = 0$. Contrary to $H(\mathbb{D})$, the Cesàro operator C does not have a bounded inverse on the Banach space A^p_α (see e.g. [28]). The aim of this paper is to investigate the Cesàro operator C on spaces that arise as intersections and unions of Bergman spaces of order $1 < p < \infty$ induced by the standard weights $(1-|z|)^\alpha$ for $0 < \alpha < \infty$:

$$\begin{aligned} A^p_{\alpha+} &:= \{f \in H(\mathbb{D}) : \left(\int_{\mathbb{D}} |f(z)|^p \, ds_\mu(z) \right)^{1/p} < \infty, \ \forall \mu > \alpha\} \\ &= \bigcap_{\mu > \alpha} A^p_\mu = \bigcap_{n \in \mathbb{N}} A^p_{(\alpha + \frac{1}{n})} = \operatorname*{proj}_{n \in \mathbb{N}} A^p_{(\alpha+\frac{1}{n})}, \end{aligned} \tag{1.8}$$

$$\begin{aligned} A^p_{\alpha-} &:= \{f \in H(\mathbb{D}) : \left(\int_{\mathbb{D}} |f(z)|^p \, ds_\mu(z) \right)^{1/p} < \infty, \text{ for some } \mu < \alpha\} \\ &= \bigcup_{\mu < \alpha} A^p_\mu = \bigcup_{n \in \mathbb{N}} A^p_{(\alpha - \frac{1}{n})} = \operatorname*{ind}_{n \in \mathbb{N}} A^p_{(\alpha-\frac{1}{n})}, \end{aligned} \tag{1.9}$$

The monograph [21] presents an investigation of Bergman type spaces (of infinite order) in that fashion with relevance to interpolation and sampling of analytic functions. The paper [24] gives a description of intersections and unions of weighted Bergman spaces of order $0 < p < \infty$. Unlike those, we treat the space

$A^p_{\alpha+}$ as a Fréchet space when equipped with the locally convex topology generated by the increasing system of norms

$$\|\!|f|\!\|_{p,\alpha,n} := \left(\int_{\mathbb{D}} \left| f(z) \right|^p \mathrm{d}s_{(\alpha+\frac{1}{n})}(z) \right)^{1/p}, \tag{1.10}$$

for $f \in A^p_{\alpha+}$ and each $n \in \mathbb{N}$. The space $A^p_{\alpha-}$ is an (LB)-space endowed with the finest locally convex topology such that each natural inclusion map from A^p_μ into A^p_γ, for $0 < \mu < \gamma$ is continuous. It is also regular, since every bounded set $B \subseteq A^p_{\alpha-}$ is contained and bounded in the Banach space A^p_μ, for some $0 < \mu < \alpha$. We also mention that for $0 < \beta < \alpha < \infty$, we have $A^p_\beta \subset A^p_{\alpha-} \subset A^p_\alpha \subset A^p_{\alpha+}$.

We address the inspiration and motivation of this research to three sources. Aleman and Constantin [8] investigated the spectrum of the Cesàro operator on weighted Bergman spaces. Albanese, Bonet and Ricker [5] studied continuity, compactness and spectrum of the Cesàro operator in growth Banach spaces. In [3] they conducted the same investigation for Cesàro operator within the context of intersections and unions of growth spaces. We keep their setup in present paper, and use it for weighted Bergman spaces. Retaining similar techniques, the patterns of our proofs are very close to the those of [3] concerning the spectrum of the Cesàro operator.

The paper is organized as follows. In Section 2, we focus on the properties of the Cesàro operator C defined on $A^p_{\alpha+}$ and $A^p_{\alpha-}$. We immediately reach that C is always continuous on $A^p_{\alpha+}$ and $A^p_{\alpha-}$ since it is continuous on each step. To show that C is not an isomorphism on $A^p_{\alpha+}$ and $A^p_{\alpha-}$, we construct specific functions. Then, we determine its spectrum on $A^p_{\alpha+}$ and $A^p_{\alpha-}$. Spectral properties of C reveals that it is non-compact on $A^p_{\alpha+}$ and $A^p_{\alpha-}$. In Section 3 we concentrate on the structural properties of $A^p_{\alpha+}$ and $A^p_{\alpha-}$. By proving that for each pair $0 < \mu < \gamma < \infty$, each inclusion map from A^p_μ to A^p_γ is compact, we establish that $A^p_{\alpha+}$ is a Fréchet-Schwartz space and $A^p_{\alpha-}$ is a (DFS)-space. Using the approach in [14, Section 2], we show that $A^p_{\alpha+}$ and $A^p_{\alpha-}$ are non-nuclear spaces both admitting the monomials $\{z^j\}_{j=1}^\infty$ as a Schauder basis.

2. The Cesàro operator C on $A^p_{\alpha+}$ and $A^p_{\alpha-}$

2.1. C is continuous on $A^p_{\alpha+}$ and $A^p_{\alpha-}$

Boundedness of the Cesàro operator on the Banach space A^p_α is due to Andersen [13, Corollary 1.2], as it is proved for a more general class of spaces of analytic functions on the unit disc. This implies C is continuous at every step $A^p_{(\alpha+\frac{1}{n})}$ or $A^p_{(\alpha-\frac{1}{n})}$, and hence C is also continuous on $A^p_{\alpha+}$ and $A^p_{\alpha-}$ by means of the properties of Fréchet and inductive limit topologies, respectively. Both $A^p_{\alpha+}$ and $A^p_{\alpha-}$ contain polynomials. Hence $A^p_{\alpha+}$ is dense in A^p_μ, for every $\mu > \alpha$ and $A^p_{\alpha-}$ is dense in A^p_α, for every $0 < \alpha < \infty$. Let us denote $\iota_n \colon A^p_{\alpha+} \hookrightarrow A^p_{(\alpha+\frac{1}{n})}$ and $\iota_{n,n+1} \colon A^p_{(\alpha+\frac{1}{n+1})} \hookrightarrow A^p_{(\alpha+\frac{1}{n})}$ the canonical maps with dense range. We denote the Cesàro operator $C_n \colon A^p_{(\alpha+\frac{1}{n})} \to A^p_{(\alpha+\frac{1}{n})}$, for each $n \in \mathbb{N}$. Observe that $\iota C = C_n \iota_n$ and $\iota_{n,n+1} C_{n+1} = C_n \iota_{n,n+1}$, for every $n \in \mathbb{N}$.

2.2. Inverse of C on $A^p_{\alpha+}$ and $A^p_{\alpha-}$

Proposition 2.1. *Let $1 < p < \infty$, and $0 < \alpha < \infty$. Then,*

(1) *The Cesàro operator C fails to be an isomorphism on $A^p_{\alpha+}$.*

(2) *The Cesàro operator C fails to be an isomorphism on $A^p_{\alpha-}$.*

Proof. Let $p > 1$. Then, there exists $\varepsilon \in (0, 1)$ such that $p \geq 1 + 2\varepsilon$.

(1) Given $0 < \alpha < \infty$, define $f_\varepsilon(z) := \frac{1}{(1+z)^{\frac{\alpha+1-\varepsilon}{p}}}$ for $z \in \mathbb{D}$. Since $1 = |1 + z - z| \leq |1 + z| + |z|$, straightforward calculation shows $f_\varepsilon \in A^p_\alpha$, and so $f_\varepsilon \in A^p_{\alpha+}$. Suppose that $C^{-1} f_\varepsilon \in A^p_{\alpha+}$. Since clearly $(1 - z) f_\varepsilon \in A^p_{\alpha+}$, this is equivalent to assume that $z(1 - z) f'_\varepsilon \in A^p_{\alpha+}$. Let us define the region $A \subset \mathbb{D}$ by the intersection of $\mathrm{Re}(z) \leq -\frac{1}{2}$,

and the Stolz angle with vertex $(-1,0)$ in which the inequality $|1+z| \leq 2(1-|z|)$ is satisfied (see e.g. [25, Lemma 6.20]). It is easy to verify that $|1+z|<1$ and $|1-z|>|z|>\frac{1}{2}$ whenever $z \in A$. Let us pick $n_0 \in \mathbb{N}$ such that $\frac{1}{n}<\varepsilon$, for all $n \geq n_0$. Hence, $p+1-\varepsilon-\frac{1}{n} \geq 2$ for every $n \geq n_0$. Then, by the fact that $z(1-z)f_\varepsilon' \in A^p_{(\alpha+\frac{1}{n})'}$ for a constant $M>0$ and for all $n \geq n_0$ we have

$$\begin{aligned} M &\geq \left\|\!\left| z(1-z)f_\varepsilon' \right|\!\right\|^p_{p,\alpha,n} = \int_{\mathbb{D}} \left|z(1-z)f_\varepsilon'(z)\right|^p \mathrm{d}s_{(\alpha+\frac{1}{n})}(z) \\ &\gtrsim \int_A \left| \frac{z(1-z)}{(1+z)^{\frac{\alpha+1-\varepsilon}{p}+1}} \right|^p \mathrm{d}s_{(\alpha+\frac{1}{n})}(z) \gtrsim \int_A \left| \frac{1}{(1+z)^{\frac{\alpha+1-\varepsilon}{p}+1}} \right|^p \mathrm{d}s_{(\alpha+\frac{1}{n})}(z) \\ &\gtrsim \int_A \frac{\mathrm{d}s(z)}{|1+z|^2} = \int_{-1}^{-\frac{1}{2}} \frac{1}{1+x} \arctan\left(\frac{y_0}{1+x}\right) \mathrm{d}x, \end{aligned}$$

where $y_0 := \frac{1}{3}\sqrt{-9x^2+6x-8\sqrt{6x+7}+23}$. An integration by parts shows that the right hand side fails to be convergent. This is a contradiction.

(2) Define $g_\varepsilon := \frac{1}{(1+z)^{\frac{\alpha+1-2\varepsilon}{p}}}$, for $z \in \mathbb{D}$. Note that there exists $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{n}<\varepsilon$, for all $n \geq n_\varepsilon$. Then, similar to part (1) we obtain $g_\varepsilon \in A^p_{(\alpha-\frac{1}{n_\varepsilon})'}$, and so $g_\varepsilon \in A^p_{\alpha-}$. Now suppose that C^{-1} is continuous on $A^p_{\alpha-}$. Then, there exists $m > n_\varepsilon$ such that the restriction $C^{-1}\colon A^p_{(\alpha-\frac{1}{n_\varepsilon})} \to A^p_{(\alpha-\frac{1}{m})}$ is continuous. Then, for a constant $M>0$, and the region $A \subset \mathbb{D}$ defined in part (1)

$$\begin{aligned} M &\geq \left\|z(1-z)g_\varepsilon'\right\|^p_{p,(\alpha-\frac{1}{m})} = \int_{\mathbb{D}} \left|z(1-z)g_\varepsilon'(z)\right|^p \mathrm{d}s_{(\alpha-\frac{1}{m})}(z) \\ &\gtrsim \int_A \left| \frac{z(1-z)}{(1+z)^{\frac{\alpha+1-2\varepsilon}{p}+1}} \right|^p \mathrm{d}s_{(\alpha-\frac{1}{m})}(z) \gtrsim \int_A \frac{\mathrm{d}s_{(\alpha-\frac{1}{m})}(z)}{|1+z|^{\alpha+1-2\varepsilon+p}} \\ &\gtrsim \int_A \frac{\mathrm{d}s(z)}{|1+z|^{p+1-2\varepsilon+\frac{1}{m}}} \geq \int_A \frac{\mathrm{d}s(z)}{|1+z|^2}. \end{aligned}$$

Similar to part (1), the right hand side is not convergent. This is a contradiction. □

Corollary 2.2. *Let $1<p<\infty$, and $0<\alpha<\infty$. Then,*

(1) *The differentiation operator D does not act on on $A^p_{\alpha+}$.*

(2) *The differentiation operator D does not act on on $A^p_{\alpha-}$.*

Proof. Suppose that $D \in L(A^p_{\alpha+})$. Since clearly both M_{1-z} and M_z are continuous in $A^p_{\alpha+}$, then by (1.6), $C^{-1} = M_{1-z}DM_z$ is continuous in $A^p_{\alpha+}$. However, this contradicts Proposition 2.1(1). Part (2) is similar. □

2.3. Spectrum of C on $A^p_{\alpha+}$

Let X be a locally convex Hausdorff space, and Γ_X a system of continuous seminorms determining the topology of X. Let X' denote the space of all continuous linear functionals on X. Denote the identity operator on X by I. Let $L(X)$ denote the space of all continuous linear operators from X into itself. For $T \in L(X)$, the *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda,T) := (\lambda I - T)^{-1}$ exists in $L(X)$. The set $\sigma(T;X) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{pt}(T;X)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. Contrary to Banach spaces, concerning the spectrum of an operator T

on the Fréchet space X, one may encounter that $\rho(T) = \varnothing$ or $\rho(T)$ fails to be an open set in $\mathbb{C}$. For this reason, some authors prefer to consider the subset $\rho^*(T)$ of $\rho(T)$ consisting of $\lambda \in \mathbb{C}$ such that there exists $\delta > 0$ such that $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\} \subseteq \rho(T)$ and $\{R(\mu, T) : \mu \in B(\lambda, \delta)\}$ is equicontinuous in $L(X)$. Define the *Waelbroeck spectrum* $\sigma^*(T; X) := \mathbb{C} \setminus \rho^*(T; X)$, which is a closed set containing $\sigma(T; X)$. If $T \in L(X)$ with X a Banach space, then $\sigma^*(T; X) = \sigma(T; X)$. For every $r \geq 1$ we denote the open disk by $D_r := \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2r}\right| < \frac{1}{2r}\right\}$. Let us write $\overline{D}_r := \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2r}\right| \leq \frac{1}{2r}\right\}$.

Proposition 2.3. [28, Theorem A; B] *For the weighted Bergman space $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha \geq 0$, the following statements hold:*

(i) *For each λ in the interior of $\sigma(C; A^p_\alpha)$, the set $\mathrm{Im}(\lambda I - C)$ is a closed one codimensional subspace of A^p_α.*

(ii) $\sigma_{pt}(C; A^p_\alpha) = \left\{\frac{1}{m} : m \in \mathbb{N},\ m < \frac{2+\alpha}{p}\right\}$.

(iii) $\sigma(C; A^p_\alpha) = \overline{D}_{\frac{2+\alpha}{p}} \cup \sigma_{pt}(C; A^p_\alpha)$.

To prove one of our main theorems, we need the following abstract spectral result.

Lemma 2.4. [2, Lemma 2.1] *Let $E = \bigcap_{n\in\mathbb{N}} E_n$ be a Fréchet space which is the intersection of a sequence of Banach spaces $((E_n, \|\cdot\|_n))_{n\in\mathbb{N}}$ satisfying $E_{n+1} \subseteq E_n$ with $\|x\|_n \leq \|x\|_{n+1}$, for all $n \in \mathbb{N}$ and $x \in E_{n+1}$. Let $T \in L(E)$ satisfy:*

(A) *For all $n \in \mathbb{N}$, there exists $T_n \in L(E_n)$ such that the restriction of T_n to E (resp. of T_n to E_{n+1}) coincides with T (resp. T_{n+1}).*

Then, the following statements hold:

(i) *$\sigma(T; E) \subseteq \cup_{n\in\mathbb{N}}\sigma(T_n; E_n)$ and $R(\lambda, T)$ coincides with the restriction of $R(\lambda, T_n)$ to E, for all $n \in \mathbb{N}$ and $\lambda \in \bigcap_{n\in\mathbb{N}} \rho(T_n; E_n)$.*

(ii) *If $\cup_{n\in\mathbb{N}}\sigma(T_n; E_n) \subseteq \overline{\sigma(T; E)}$, then*

$$\sigma^*(T; E) = \overline{\sigma(T; E)}.$$

For $n \in \mathbb{N}$, let us denote by C_n the Cesàro operator acting on the Banach space $A^p_{(\alpha+\frac{1}{n})}$.

Theorem 2.5. *Let $1 < p < \infty$. Then, for $0 < \alpha < \infty$, the following statements hold:*

(1) *We have the inclusion*

$$\left\{\frac{1}{m} : m \in \mathbb{N},\ m < \frac{2+\alpha}{p}\right\} \subset \sigma_{pt}(C; A^p_{\alpha+}) \subset \left\{\frac{1}{m} : m \in \mathbb{N},\ m \leq \frac{2+\alpha}{p}\right\}$$

(2) $\sigma(C; A^p_{\alpha+}) = \{0\} \cup D_{\frac{2+\alpha}{p}} \cup \sigma_{pt}(C; A^p_{\alpha+})$.

(3) $\sigma^*(C; A^p_{\alpha+}) = \overline{\sigma(C; A^p_{\alpha+})}$.

Proof. (1) One inclusion follows from Proposition 2.3. For the other inclusion, take any $\lambda \in \sigma_{pt}(C; A^p_{\alpha+})$. Then, there exists $f \in A^p_{\alpha+}$ such that $Cf = \lambda f$. Since $f \in A^p_\mu$ for every $\mu > \alpha$, we have $Cf = \lambda f$ in A^p_μ as well. Then $\lambda \in \sigma_{pt}(C; A^p_\mu)$, for all $\mu > \alpha$. Hence $\lambda \in \cap_{\mu>\alpha}\sigma_{pt}(C; A^p_\mu)$. By Proposition 2.3(ii) we obtain

$$\begin{aligned}\sigma_{pt}(C; A^p_{\alpha+}) \subseteq \bigcap_{\mu>\alpha} \sigma_{pt}(C; A^p_\mu) &= \bigcap_{\mu>\alpha} \left\{\frac{1}{m} : m \in \mathbb{N}, m < \frac{2+\mu}{p}\right\} \\ &= \left\{\frac{1}{m} : m \in \mathbb{N}, m \leq \frac{2+\alpha}{p}\right\}.\end{aligned}$$

(2) By part (1) and Lemma 2.4(i), we already know that

$$\sigma_{pt}(C;A^p_{\alpha+}) \subseteq \sigma(C;A^p_{\alpha+}) \subseteq \bigcup_{n\in\mathbb{N}} \sigma(C_n;A^p_{(\alpha+\frac{1}{n})}).$$

Proposition 2.1 implies that $0 \in \sigma(C;A^p_{\alpha+})$. By Proposition 2.3, we obtain

$$\sigma(C;A^p_{\alpha+}) \subseteq \{0\} \cup \bigcup_{n\in\mathbb{N}} \left(\left\{ \frac{1}{m} : m \in \mathbb{N}, m < \frac{2+(\alpha+\frac{1}{n})}{p} \right\} \cup \overline{D}_{\frac{2+(\alpha+\frac{1}{n})}{p}} \right).$$

Then we clearly have

$$\sigma(C;A^p_{\alpha+}) \subseteq \{0\} \cup \left\{ \frac{1}{m} : m \in \mathbb{N}, m \leq \frac{2+\alpha}{p} \right\} \cup D_{\frac{2+\alpha}{p}}.$$

So it remains to show that $D_{\frac{2+\alpha}{p}} \subseteq \sigma(C;A^p_{\alpha+})$. Let us fix $\alpha > 0$, and let $\lambda \in D_{\frac{2+\alpha}{p}}$. Let us set $m_\alpha := \max\{m \in \mathbb{N} : \frac{1}{m} \geq \frac{p}{2+\alpha}\}$. Then find $n_0 \in \mathbb{N}$ such that $\frac{p}{2+(\alpha+\frac{1}{n})} \geq \frac{1}{m_\alpha+1}$ and since $\lambda \in D_{m_\alpha+1}$ we have

$$\left|\lambda - \frac{p}{2(2+(\alpha+\frac{1}{n}))}\right| \leq \left|\lambda - \frac{1}{2(m_\alpha+1)}\right| < \frac{1}{2(m_\alpha+1)} \leq \frac{p}{2(2+(\alpha+\frac{1}{n}))},$$

for every $n \geq n_0$. Hence, $\lambda \in D_{\frac{2+(\alpha+\frac{1}{n})}{p}}$, for all $n \geq n_0$. So by Proposition 2.3(ii), λ belongs to the interior of $\sigma(C_n;A^p_{(\alpha+\frac{1}{n})})$, for each $n \geq n_0$. We next show that the set $A^p_{\alpha+} \setminus \mathrm{Im}(\lambda I - C)$ is a non-empty open set. Due to Proposition 2.3(i), the argument is as in the proofs of [2, Theorem 2.2] and [3, Proposition 2.3]. We first take any sequence $(g_j)_{j\in\mathbb{N}} \subseteq \mathrm{Im}(\lambda I - C)$ such that $g_j \underset{j}{\to} g \in A^p_{\alpha+}$. For every $j \in \mathbb{N}$ let us select $f_j \in A^p_{\alpha+}$ such that $(\lambda I - C)f_j = g_j$. In particular, $f_j \subseteq A^p_{(\alpha+\frac{1}{n})}$, for every $n \in \mathbb{N}$. Then we have $g_j \underset{j}{\to} g \in A^p_{(\alpha+\frac{1}{n})}$. Since $\mathrm{Im}(\lambda I - C_n)$ is closed in $A^p_{(\alpha+\frac{1}{n})}$, $g \in \mathrm{Im}(\lambda I - C_n)$, for all $n \geq n_0$. Then there exists $h_n \in A^p_{(\alpha+\frac{1}{n})}$ such that $\mathrm{Im}(\lambda I - C_n)h_n = g$. Moreover, for $n \geq n_0$ we have $\mathrm{Im}(\lambda I - C_n)h_n = g = (\lambda I - C_{n+1})h_{n+1}$. Since the restriction of C_n to $A^p_{(\alpha+\frac{1}{n+1})}$ coincides with C_{n+1} and $\lambda I - C_n$ is injective, we have $h_n = h_{n+1}$, for all $n \geq n_0$. So $g \in \mathrm{Im}(\lambda I - C)$, and hence $\mathrm{Im}(\lambda I - C)$ is closed. Now it remains to show that $\mathrm{Im}(\lambda I - C)$ is a proper subspace. Assume not, that is, suppose that $\mathrm{Im}(\lambda I - C) = A^p_{\alpha+}$. Since $A^p_{\alpha+}$ is dense in $A^p_{(\alpha+\frac{1}{n})}$, for all $n \in \mathbb{N}$,

$$A^p_{(\alpha+\frac{1}{n})} = \overline{A^p_{\alpha+}} = \overline{(\lambda I - C)(A^p_{\alpha+})} \subseteq (\lambda I - C_n)(A^p_{(\alpha+\frac{1}{n})}), \tag{2.1}$$

where all the closures are taken in $A^p_{(\alpha+\frac{1}{n})}$. However, this contradicts the fact that $\mathrm{Im}(\lambda I - C_n)$ is a closed subspace of $A^p_{(\alpha+\frac{1}{n})}$. Hence $\lambda I - C$ is not surjective, so $\lambda \in \sigma(C;A^p_{\alpha+})$.

(3) By part (2), we observe that

$$\overline{\sigma(C;A^p_{\alpha+})} = \{0\} \cup \overline{D}_{\frac{2+\alpha}{p}} \cup \left\{ \frac{1}{m} : m \in \mathbb{N}, m \leq \frac{2+\alpha}{p} \right\}.$$

From Proposition 2.3(ii), we deduce that

$$\bigcup_{n\in\mathbb{N}} \sigma(C_n;A^p_{(\alpha+\frac{1}{n})}) \subseteq \{0\} \cup \overline{D}_{\frac{2+\alpha}{p}} \cup \left\{ \frac{1}{m} : m \in \mathbb{N}, m \leq \frac{2+\alpha}{p} \right\}.$$

So by Lemma 2.4(ii), we get the desired result.

□

2.4. Spectrum of C on $A^p_{\alpha-}$

Let us state an abstract spectral lemma which is needed for our next result.

Lemma 2.6. [3, Lemma 5.2] *Let $E = \text{ind}_{n\in\mathbb{N}}(E_n, \|\cdot\|_n)$ be a Hausdorff inductive limit of Banach spaces. Let $T \in L(E)$ satisfy the following condition:*

(A) *For each $n \in \mathbb{N}$, the restriction T_n of T to E_n maps E_n into itself and $T_n \in L(E_n)$.*

Then, the following properties are satisfied:

(i) $\sigma_{pt}(T; E) = \bigcup_{n\in\mathbb{N}} \sigma_{pt}(T_n; E_n)$.

(ii) $\sigma(T; E) \subseteq \bigcap_{m\in\mathbb{N}}(\bigcup_{n=m}^{\infty} \sigma(T_n; E_n)$. *Moreover, if $\lambda \in \bigcap_{n=m}^{\infty} \rho(T_n; E_n)$ for some $m \in \mathbb{N}$, then $R(\lambda, T_n)$ coincides with the restriction of $R(\lambda, T)$ to E_n, for every $n \geq m$.*

(iii) *If $\bigcup_{n=m}^{\infty} \sigma(T_n; E_n) \subseteq \overline{\sigma(T; E)}$, for some $m \in \mathbb{N}$, then*

$$\sigma^*(T; E) = \overline{\sigma(T; E)}.$$

In the light of Lemma 2.6, the following result will follow by using the arguments in [3, Propositions 2.5 - 2.9], adapted to our setting.

Theorem 2.7. *Let $1 < p < \infty$ be fixed, and let $0 < \alpha < \infty$. Then, the following statements hold:*

(1) $\sigma_{pt}(C; A^p_{\alpha-}) = \{\frac{1}{m} : m \in \mathbb{N}, m < \frac{2+\alpha}{p}\}$.

(2) $\sigma(C; A^p_{\alpha-}) = \sigma_{pt}(C; A^p_{\alpha-}) \cup \overline{D}_{\frac{2+\alpha}{p}}$.

(3) $\sigma^*(C; A^p_{\alpha-}) = \sigma(C; A^p_{\alpha-})$.

Proof. (1) By Lemma 2.6(i), we know that

$$\sigma_{pt}(C; A^p_{\alpha-}) = \bigcup_{n\in\mathbb{N}} \sigma_{pt}(C_n; A^p_{(\alpha-\frac{1}{n})}) = \bigcup_{n\in\mathbb{N}} \left\{\frac{1}{m} : m \in \mathbb{N}, m < \frac{2 + (\alpha - \frac{1}{n})}{p}\right\}$$
$$= \left\{\frac{1}{m} : m \in \mathbb{N}, m < \frac{2+\alpha}{p}\right\}.$$

(2) If we apply Lemma 2.6(ii) we obtain

$$\sigma(C; A^p_{\alpha-}) \subseteq \bigcap_{m\in\mathbb{N}, m>\frac{1}{\alpha}} \left(\bigcup_{n=m}^{\infty} \sigma(C_n; A^p_{(\alpha-\frac{1}{n})})\right).$$

On the other hand, by Proposition 2.3(ii) we know that for every $n \in \mathbb{N}$ satisfying $n > \frac{1}{\alpha}$

$$\sigma(C_n; A^p_{(\alpha-\frac{1}{n})}) = \left\{\frac{1}{m} : m \in \mathbb{N}, m < \frac{2 + (\alpha - \frac{1}{n})}{p}\right\} \cup D_{\frac{2+(\alpha-\frac{1}{n})}{p}}$$

Since for each $n, m \in \mathbb{N}$ with $n \geq m > \frac{1}{\alpha}$ one has

$$D_{\frac{2+(\alpha-\frac{1}{n})}{p}} \subseteq D_{\frac{2+(\alpha-\frac{1}{m})}{p}},$$

it follows

$$\sigma(C; A^p_{\alpha-}) \subseteq \left\{\frac{1}{m} : m \in \mathbb{N}, m < \frac{2+\alpha}{p}\right\} \cup \overline{D}_{\frac{2+\alpha}{p}}.$$

For the other inclusion, first let us notice that by part (1),

$$\left\{\frac{1}{m} : m \in \mathbb{N}, m < \frac{2+\alpha}{p}\right\} \subseteq \sigma(C; A^p_{\alpha-}).$$

Now let us assume that there exists $\lambda \in \mathbb{C}$ satisfying $\left|\lambda - \frac{p}{2(2+\alpha)}\right| < \frac{p}{2(2+\alpha)}$, but $\lambda \notin \sigma(C; A^p_{\alpha-})$. Then, $(\lambda I - C)(A^p_{\alpha-}) = A^p_{\alpha-}$. However, Proposition 2.3(i) implies that $\mathrm{Im}(\lambda I - C)$ is a one-dimensional closed subspace of A^p_α. Since $A^p_{\alpha-}$ is dense in A^p_α, we have

$$A^p_\alpha = \overline{A^p_{\alpha-}} = \overline{(\lambda I - C)(A^p_{\alpha-})} \subseteq (\lambda I - C)(A^p_\alpha),$$

closures taken in A^p_α. Then, $\mathrm{Im}(\lambda I - C)$ is dense in A^p_α. Contradiction. Therefore $\lambda \in \sigma(C; A^p_{\alpha-})$. Now it remains to show that

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p}{2(2+\alpha)}\right| = \frac{p}{2(2+\alpha)}\right\} \subseteq \sigma(C; A^p_{\alpha-}).$$

Fix $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{p}{2(2+\alpha)}\right| = \frac{p}{2(2+\alpha)}$, that is, $\mathrm{Re}(\frac{1}{\lambda}) = \frac{2+\alpha}{p}$. Then, by [28, Proposition 4], constant functions do not belong to $(\lambda I - C)(A^p_\alpha)$. For instance, take $\mathbf{1} \in A^p_{\alpha-} \subseteq A^p_\alpha$ so that $\mathbf{1} \notin (\lambda I - C)$ and hence $(\lambda I - C)\colon A^p_{\alpha-} \to A^p_{\alpha-}$ fails to be surjective. Therefore, $\lambda \in \sigma(C; A^p_{\alpha-})$, and this completes the proof.

(3) By definition, it is always true to say $\sigma(C; A^p_{\alpha-}) \subseteq \sigma^*(C; A^p_{\alpha-})$. For the reverse inclusion, let us take $\lambda \notin \sigma(C; A^p_{\alpha-})$ and show $\lambda \notin \sigma^*(C; A^p_{\alpha-})$. By part (2), the set $\sigma(C; A^p_{\alpha-})$ is compact. So there exist $r > 0$ and $n_0 \in \mathbb{N}$ with $n_0 > \frac{1}{\alpha}$ such that $\overline{B(\lambda, r)} \cap \sigma(C; A^p_{(\alpha-\frac{1}{n_0})}) = \emptyset$. By Proposition 2.3(iii), we have $\overline{B(\lambda, r)} \cap \sigma(C_n; A^p_{(\alpha-\frac{1}{n})}) = \emptyset$, for every $n \geq n_0$. That is, $\overline{B(\lambda, r)} \subseteq \rho(C_n; A^p_{(\alpha-\frac{1}{n})})$, for every $n \geq n_0$. Hence the set $\{(\mu I - C_n)^{-1}\colon A^p_{(\alpha-\frac{1}{n})} \to A^p_{(\alpha-\frac{1}{n})} | \, \mu \in \overline{B(\lambda, r)}\}$ is equicontinuous, for every $n \geq n_0$. Now we show that the set $\{(\mu I - C)^{-1}\colon A^p_{\alpha-} \to A^p_{\alpha-} | \, \mu \in \overline{B(\lambda, r)}\}$ is equicontinuous in $L(A^p_{\alpha-})$. As we shall show that $A^p_{\alpha-}$ is a (DFS)-space (see Corollary 3.2), it is barrelled. Hence, by means of Banach-Steinhaus Theorem, it suffices to show that the set

$$\{(\mu I - C)^{-1} f : \mu \in \overline{B(\lambda, r)}\}$$

is bounded in $A^p_{\alpha-}$. This will guarantee $\lambda \notin \sigma^*(C; A^p_{\alpha-})$. Let us fix $f \in A^p_{(\alpha-\frac{1}{n})}$, for some $n \geq n_0$. So $\{(\mu I - C_n)^{-1} f\colon \mu \in \overline{B(\lambda, r)}\}$ is a bounded set in $A^p_{(\alpha-\frac{1}{n})}$ and hence also in $A^p_{\alpha-}$. Since $(\mu I - C)^{-1}|_{A^p_{(\alpha-\frac{1}{n})}} = (\mu I - C_n)^{-1}$ for $\mu \in \overline{B(\lambda, r)}$, we are done.

□

2.5. Other properties of C on $A^p_{\alpha+}$ and $A^p_{\alpha-}$

The following result follows from Section 2.3 and Section 2.4.

Proposition 2.8. *Let $1 < p < \infty$ be fixed, and let $0 < \alpha < \infty$. Then, the Cesàro operator C fails to be compact on $A^p_{\alpha+}$ and $A^p_{\alpha-}$.*

Proof. We make use of the results in [18, Theorem 9.10.2(4)] and [19, Theorem 2.4] stating that a compact operator $T\colon X \to X$ on a Hausdorff locally convex space X necessarily has a spectrum $\sigma(T; X)$ which is compact as a set in $\mathbb{C}$ and every element of $\sigma(T; X)$ except for the origin is isolated. We see in Theorem 2.5 C has a non-compact spectrum on $A^p_{\alpha+}$. Although it has a compact spectrum on $A^p_{\alpha-}$ as shown in Theorem 2.7, the points in the spectrum are not isolated. □

We conclude this section with some remarks on the dynamical properties of the Cesàro operator C on $A^p_{\alpha+}$. A Fréchet space operator $T \in L(X)$, where X is separable, is called *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^k x : k \in \mathbb{N}_0\}$ is dense in X. If, for some $z \in X$, the projective orbit $\{\lambda T^k z : \lambda \in \mathbb{C}, k \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Clearly, if T is hypercyclic then T is supercyclic.

Proposition 2.9. *Let $1 < p < \infty$, and $0 < \alpha < \infty$. Then, both $C\colon A^p_{\alpha+} \to A^p_{\alpha+}$ and $C\colon A^p_{\alpha-} \to A^p_{\alpha-}$ fail to be supercyclic. In particular, they are not hypercyclic.*

Proof. It is proved in [4, Proposition 2.20] that the Cesàro operator C acting on $H(\mathbb{D})$ fails to be supercyclic. So if $C\colon A^p_{\alpha-} \to A^p_{\alpha-}$ or $C\colon A^p_{\alpha+} \to A^p_{\alpha+}$ were supercyclic, by the fact that $A^p_{\alpha+}$ and $A^p_{\alpha-}$ are dense in $H(\mathbb{D})$ since it contains the polynomials, C would also be supercyclic on $H(\mathbb{D})$. This is a contradiction. □

3. Further discussion on the structures of $A^p_{\alpha+}$ and $A^p_{\alpha-}$

The well-known *Korenblum space* [23] is defined by

$$A^{-\infty} := \bigcup_{0<\alpha<\infty} A^{-\alpha} = \bigcup_{n\in\mathbb{N}} A^{-n},$$

where

$$A^{-\alpha} = \{f \in H(\mathbb{D})\colon \sup_{z\in\mathbb{D}}(1-|z|)^{\alpha}\left|f(z)\right| < \infty\}.$$

The classical Korenblum space $A^{-\infty}$ is a regular (LB)-space when endowed with the finest locally convex topology which makes each natural inclusion map $A^{-n} \subseteq A^{-\infty}$ continuous. So, $A^{-\infty} = \operatorname{ind}_{n\in\mathbb{N}} A^{-n}$. Let $f \in A^p_\alpha$. Then, by means of [28, Lemma 3.1], we obtain $f \in A_0^{-\left(\frac{2+\alpha}{p}\right)}$, where

$$A_0^{-\alpha} = \{f \in H(\mathbb{D})\colon \lim_{|z|\to 1^-}(1-|z|)^{\alpha}\left|f(z)\right| = 0\} \subseteq A^{-\alpha}.$$

This means $\cup_{\alpha\in\mathbb{N}} A^p_\alpha \subset A^{-\infty}$. For the converse, let us take any $f \in A^{-\infty}$. Then there exists $n \in \mathbb{N}$ and a constant $M > 0$ such that $\sup_{z\in\mathbb{D}}(1-|z|)^n\left|f(z)\right| < M$. Then,

$$\int_{\mathbb{D}} \left|f(z)\right|^p (1-|z|)^{np}\mathrm{d}s(z) \le M^p \int_{\mathbb{D}} \mathrm{d}s(z) = M^p,$$

which implies that $f \in A^p_{np}$. This shows $A^{-\infty} \subset \cup_{\alpha>0} A^p_\alpha$. See also [21, p. 111]. Continuity, compactness and the spectrum of the Cesàro operator acting on the classical Korenblum space have been studied completely in [3]. This was the reason we avoided any study of C acting on the (LB)-space $\cup_{\alpha>0} A^p_\alpha$ although it seems quite tempting concerning the nature of the spaces we deal with.

3.1. Schwartz property in $A^p_{\alpha+}$ and $A^p_{\alpha-}$

For each $0 < \mu < \gamma$ it is easy to observe that it holds for the pair of weighted Bergman spaces $A^p_\mu \subseteq A^p_\gamma$.

Lemma 3.1. *For each pair $0 < \mu < \gamma < \infty$, the canonical inclusion map $\iota : A^p_\mu \hookrightarrow A^p_\gamma$ is compact.*

Proof. Fix $0 < \mu < \gamma < \infty$, $M > 0$. By the fact that $\lim_{|z|\to 1^-}(1-|z|)^{\gamma-\mu} = 0$, for any $\varepsilon > 0$ given, we find $R \in (0,1)$ such that $(1-|z|)^{\gamma-\mu} < \frac{\varepsilon}{2M^p}$, for all $|z| > R$. Let us take $f = (f_j) \in A^p_\mu$ with $\left\|f\right\|_{p,\mu} \le M$, which converges to 0 in the topology of uniform convergence on compact subsets of $\mathbb{D}$. Then, we have

$$\int_{|z|>R} \left|f_j(z)\right|^p \mathrm{d}s_\gamma(z) = \int_{|z|>R} \left|f_j(z)\right|^p (1-|z|)^{\gamma-\mu}\mathrm{d}s_\mu(z)$$

$$\leq \frac{\varepsilon}{2M^p} \int\limits_{|z|>R} \left|f_j(z)\right|^p \mathrm{d}s_\mu(z) \leq \frac{\varepsilon}{2M^p} \left\|f\right\|_{p,\mu}^p \leq \frac{\varepsilon}{2}.$$

Now let us find $j_0 \in \mathbb{N}$ such that we have $\left|f_j(z)\right| < (\frac{\varepsilon}{2(\gamma+1)})^{1/p}$, for all $j \geq j_0$. So, for every $j \geq j_0$ we obtain

$$\int\limits_{|z|\leq R} \left|f_j(z)\right|^p \mathrm{d}s_\gamma(z) \leq \frac{\varepsilon}{2(\gamma+1)} \int\limits_{|z|\leq R} \mathrm{d}s_\gamma(z) = \frac{\varepsilon}{2}.$$

Therefore, combining the arguments above, we obtain $\left\|\iota(f_j)\right\|_{p,\gamma}^p \leq \varepsilon$. This means, f_j converges to 0 in norm topology as well. Then, ι is a compact operator. □

Corollary 3.2. *For a fixed $1 < p < \infty$ and for all $0 < \alpha < \infty$. Then,*

(1) *$A^p_{\alpha+}$ is a Fréchet-Schwartz space.*

(2) *$A^p_{\alpha-}$ is a (DFS)-space.*

Proof. (1) By (1.8), this result follows directly by the combination of arguments in Lemma 3.1, and [22, §21.1, Example 1(b)].

(2) By (1.9), it follows by Lemma 3.1 and [27, Proposition 25.20].
□

3.2. On nuclearity of $A^p_{\alpha+}$ and $A^p_{\alpha-}$

A sequence $(x_j)_{j=0}^\infty$ in a locally convex space E is said to be a *Schauder basis* if each element $y \in E$ can be written uniquely as $y = \sum_{j=1}^\infty f_j(y)x_j$, where $f_j \colon E \to \mathbb{K}$, $j \in \mathbb{N}$ are continuous linear forms. See [22] for further information on Schauder basis in Fréchet spaces. The work of Lusky [26, Theorem 2.2] tells us that the monomials $\Lambda = \{z^j\}_{j=0}^\infty$ is a Schauder basis for A^p_α, since it is proved for a more general setup (see also [15]). In the light of that, and with the help of Fréchet and inductive limit topologies, it is straightforward to assert that

Theorem 3.3. *Let $1 < p < \infty$ be fixed, and let $0 < \alpha < \infty$. Then,*

(1) *Λ is a Schauder basis for $A^p_{\alpha+}$.*

(2) *Λ is a Schauder basis for $A^p_{\alpha-}$.*

Proof. (1) We need to prove that the Taylor series of any $f \in A^p_{\alpha+}$, $\alpha \geq 0$, converges in $A^p_{\alpha+}$ to f. Let us fix $\mu > \alpha$ and pick μ_1 with $\alpha < \mu_1 < \mu$. Since $f \in A^p_{\mu_1}$ we may apply [26, Theorem 2.2] to deduce that the Taylor series of f converges to f in $A^p_{\alpha+}$. This implies Λ is a Schauder basis for $A^p_{\alpha+}$.

(2) A direct consequence of [26, Theorem 2.2] and the properties of inductive limits.
□

Thanks to Theorem 3.3, we are now allowed to make use of Grothendieck-Pietsch criterion to determine whether $A^p_{\alpha+}$ and $A^p_{\alpha-}$ are nuclear or not. We adapt this approach from [14]. First we need the following estimate, which is essentially known (cf. [29, Lemma 4]).

Lemma 3.4. *Let $1 < p < \infty$, and $0 < \alpha < \infty$. For $j \in \mathbb{N}$,*

$$\left\|z^j\right\|_{p,\alpha} \asymp \left(\frac{1}{j^{\alpha+1}}\right)^{1/p}.$$

Proof. For any $f \in A^p_\alpha$ we have $\|f\|^p_{p,\alpha} = \int_0^1 r(1-r)^\alpha \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^p \mathrm{d}t\right)\mathrm{d}r$. Let $\beta(\cdot,\cdot)$ denote the usual Beta function, and $\Gamma(\cdot)$ denote the usual Gamma function. Then we have the estimate

$$\begin{aligned}\left\|z^j\right\|^p_{p,\alpha} &\asymp \int_0^1 (1-r)^\alpha \left(\frac{1}{2\pi}\int_0^{2\pi} (r^j)^p \mathrm{d}t\right)\mathrm{d}r = \int_0^1 (1-r)^\alpha r^{jp}\mathrm{d}r \\ &= \beta(jp+1,\alpha+1) = \frac{\Gamma(jp+1)\Gamma(\alpha+1)}{\Gamma(jp+\alpha+2)} \\ &\asymp \frac{(jp)^{jp+\frac{1}{2}}\alpha^{(\alpha+\frac{1}{2})}e^{-(\alpha+jp)}}{(jp+\alpha+1)^{jp+\alpha+\frac{3}{2}}e^{-(jp+\alpha+1)}} \\ &= \left(\frac{jp}{jp+\alpha+1}\right)^{jp+\frac{1}{2}} \frac{\alpha^{(\alpha+\frac{1}{2})}}{e(jp+\alpha+1)} \asymp \frac{1}{j^{\alpha+1}},\end{aligned}$$

where the first estimate is due to an extension of Stirling's formula (see e.g. [1, p. 257]). □

Theorem 3.5. *Let $1 < p < \infty$ be fixed and let $0 < \alpha < \infty$. Then,*

(1) *The Fréchet-Schwartz space $A^p_{\alpha+}$ fails to be nuclear.*

(2) *The (DFS)-space $A^p_{\alpha-}$ fails to be nuclear.*

Proof. We give the argument for part (1). Suppose that $A^p_{\alpha+}$ is nuclear. Then, by Grothendieck-Pietsch criterion (see e.g. [27, Proposition 28.15]), given $n = 1$ we find $m > 1$ such that

$$\sum_{j=1}^\infty \frac{\left\|z^j\right\|_{p,\alpha+1}}{\left\|z^j\right\|_{p,\alpha+\frac{1}{m}}} < \infty.$$

On the other hand, by Lemma 3.4 we have

$$\sum_{j=1}^\infty \frac{\left\|z^j\right\|_{p,\alpha+1}}{\left\|z^j\right\|_{p,\alpha+\frac{1}{m}}} \asymp \left(\sum_{j=1}^\infty \frac{j^{\alpha+\frac{1}{m}+1}}{j^{\alpha+2}}\right)^{1/p} = \left(\sum_{j=1}^\infty \frac{1}{j^{1-\frac{1}{m}}}\right)^{1/p} = \infty.$$

This is a contradiction. □

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