



The Extreme Problem for Orlicz and L_q Torsional Rigidity and their Properties

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Abstract. In this paper, the extreme problem for Orlicz and L_q torsional rigidity is discussed. Moreover, we introduce Orlicz and L_q geominimal torsional rigidity, which is defined as being motivated through Orlicz L_q mixed torsional rigidity. Also, the invariance of Orlicz and L_q geominimal torsional rigidity under orthogonal matrices is proved, and isoperimetric type inequality and circular type inequality for the torsional rigidity are established as well.

1. Introduction

The setting of this paper is in the n -dimensional Euclidean space \mathbb{R}^n . A subset K in \mathbb{R}^n is convex if for all $x, y \in K$ and $a \in [0, 1]$ satisfying $ax + (1 - a)y \in K$. Let \mathcal{K} and \mathcal{K}_0 be the set of convex bodies (compact convex set with nonempty interior) and the set of convex bodies which contain the origin o in their interiors, respectively. Denote $|K|$ as the volume of $K \in \mathcal{K}$. In general, B_2^n is defined as the unit ball of the n -dimensional Euclidean space with the surface area of S^{n-1} , and $|B_2^n| = \omega_n$. The volume radius of $K \in \mathcal{K}$, denoted by $\text{vrad}(K) = (|K|/\omega_n)^{\frac{1}{n}}$. If K is a compact convex set in \mathbb{R}^n , its support function $h(K, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by $h(K, x) = \max\{x \cdot y : y \in K\}$, where $x \cdot y$ denotes the inner product of x and y . We write $h(K, u) = h_K(u)$. Evidently, for $K, L \in \mathcal{K}$ and $a \geq 0$, $h_{aK}(u) = ah_K(u)$ and $h_{K+L}(u) = h_K(u) + h_L(u)$ for any $u \in S^{n-1}$.

For $K, L \in \mathcal{K}$, the mixed volume of K and L , denoted by $V_1(K, L)$ in [20], is defined as

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS(K, u), \quad (1)$$

where $S(K, u)$ is the surface area measure on S^{n-1} of the convex body K (for the definition see section 2). For a convex body $K \in \mathcal{K}$, the geominimal surface area, $G(K)$ of K , raised by Petty [29], could be defined by

$$G(K) = \inf \left\{ \int_{S^{n-1}} h(L, u) dS(K, u) : L \in \mathcal{K}_0, |L^\circ| = \omega_n \right\}, \quad (2)$$

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where L° is the polar body of L , i.e., $L^\circ = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for } y \in L\}$. Combining with (1) and (2), one gets

$$G(K) = \inf\{nV_1(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}. \tag{3}$$

This indicates that the classical geominimal surface area was defined on the basis of the mixed volume. The study of geominimal surface area was first explained by Petty in [29], the classical geominimal surface area naturally connects relative geometry, affine geometry and Minkowski geometry. Therefore, it received a lot of attention (see e.g., [1, 7–9, 13]). Related to this is the classical Brunn-Minkowski theory, which originated from Brunn [3] and Minkowski [27]. It is the core content of convex geometry analysis. Whereafter the L_p Brunn-Minkowski theory originated from the introduction of Firey – p combination of convex bodies into the classical Brunn-Minkowski theory by Lutwak [20]. Due to (3), in [21], Lutwak also introduced L_p geominimal surface area for $p > 1$. And Ye [38] further expanded the $p > 1$ form to $p \in \mathbb{R}$ form. One can also find more resources for L_p geominimal surface area (see e.g., [19, 21, 22, 30, 33–35, 40, 43, 46, 47]). Lately, the L_p Brunn-Minkowski theory is extended to the Orlicz-Brunn-Minkowski theory by Lutwak, Yang and Zhang [23, 24] on the basis of solving asymmetric problems. With the continuous improvement of Orlicz-Brunn-Minkowski theory (see e.g., [4, 13, 15, 37, 45, 48]), this theory has become the mainstream research in the field of convex geometry analysis. Recently, Gardner, Hug and Weil [10] and Xi, Jin and Leng [36] constructed a general framework for Orlicz-Brunn-Minkowski theory, including Orlicz addition and Orlicz mixed volume, respectively. In addition, they established the new Orlicz-Brunn-Minkowski inequality and Orlicz-Minkowski mixed volume inequality. The new theory has gained widespread popularity (see e.g., [16, 17, 23, 24, 36, 41, 49]). At the same time, the geominimal surface area is developed into Orlicz geominimal surface area (see e.g., [39, 41, 42]). Quite recently, the geominimal surface area associated with the capacity is also considered (see e.g., [6, 12, 16, 17, 25, 32, 44]).

In [16], the definition of the nonhomogeneous Orlicz L_φ mixed torsional rigidity was introduced. Consequently, we will define the homogeneous Orlicz L_φ mixed torsional rigidity and the Orlicz geominimal torsional rigidity with respect to \mathcal{E}_0 which is a nonempty subset of S_0 (the set of star bodies about the origin o). For example, suppose $K \in \mathcal{K}_0$ and $\varphi \in \mathcal{I}$ (the definition of \mathcal{I} in section 3), the nonhomogeneous Orlicz geominimal torsional rigidity $H_{1,\varphi}(K, \mathcal{E}_0)$ of K , is defined by the extreme problem as follows:

$$H_{1,\varphi}(K, \mathcal{E}_0) = \inf_{L \in \mathcal{E}_0} \{\tau_{1,\varphi}(K, \text{vrad}(L)L^\circ)\}.$$

Likewise, the homogeneous Orlicz geominimal torsional rigidity $\widehat{H}_{1,\varphi}(K, \mathcal{E}_0)$ of K , is defined by the extreme problem as follows:

$$\widehat{H}_{1,\varphi}(K, \mathcal{E}_0) = \inf_{L \in \mathcal{E}_0} \{\widehat{\tau}_{1,\varphi}(K, \text{vrad}(L)L^\circ)\}.$$

And then we will stress two special cases as follows, let $\mathcal{E}_0 = \mathcal{K}_0$ and $\mathcal{E}_0 = S_0$. For the sake of easy, we replace $H_{1,\varphi}(K, \mathcal{K}_0)$, $\widehat{H}_{1,\varphi}(K, \mathcal{K}_0)$, $H_{1,\varphi}(K, S_0)$, $\widehat{H}_{1,\varphi}(K, S_0)$ with $H_{1,\varphi}(K)$, $\widehat{H}_{1,\varphi}(K)$, $F_{1,\varphi}(K)$, $\widehat{F}_{1,\varphi}(K)$, respectively. And, let's define $B_K = \text{vrad}(K)B_2^n$ to be the origin symmetric ball of radius $\text{vrad}(K)$ for $K \in \mathcal{K}_0$. For definitions of \mathcal{D}_0 , \mathcal{I}_0 and \mathcal{D}_1 , see section 3.

Then we discuss some properties of Orlicz geominimal torsional rigidity, such as establishing some isoperimetric type inequalities, as follows:

Theorem 1.1. *Suppose $L \in \mathcal{K}_0$ and the center or Santaló point of L is the origin o .*

(i) *If $\varphi \in \mathcal{D}_0 \cup \mathcal{I}_0$ and then*

$$\frac{\widehat{F}_{1,\varphi}(L)}{\widehat{F}_{1,\varphi}(B_L)} \leq \frac{\widehat{H}_{1,\varphi}(L)}{\widehat{H}_{1,\varphi}(B_L)} \leq \frac{\tau(L)}{\tau(B_L)}.$$

Equality holds if L is an origin symmetric ball.

(ii) *If $\varphi \in \mathcal{D}_1$, then there exists a constant $d > 0$ such that*

$$\frac{\widehat{F}_{1,\varphi}(L)}{\widehat{F}_{1,\varphi}(B_L)} \geq \frac{\widehat{H}_{1,\varphi}(L)}{\widehat{H}_{1,\varphi}(B_L)} \geq \frac{d \cdot \tau(L)}{\tau(B_L)}.$$

According to the definitions of Orlicz geominimal torsional rigidity in section 3, let $\varphi(t) = t^q$, in section 4, we define the L_q geominimal torsional rigidity of K with respect to \mathcal{K}_0 and S_0 . For instance, for $q \geq 0$, we denote $H_{1,q}(K)$ with respect to \mathcal{K}_0 , by

$$H_{1,q}(K) = \inf_{L \in \mathcal{K}_0} \left\{ \tau_{1,q}(K, L)^{n/n+q} |L^\circ|^{q/n+q} \right\}.$$

And define $F_{1,q}(K)$ with respect to S_0 , by

$$F_{1,q}(K) = \inf_{L \in S_0} \left\{ \tau_{1,q}(K, L^\circ)^{n/n+q} |L|^{q/n+q} \right\}.$$

We also discuss some properties of L_q geominimal torsional rigidity. For example, we establish some isoperimetric type inequalities for $F_{1,q}(K)$ and $H_{1,r}(K)$.

Theorem 1.2. *Let $K \in \mathcal{K}_0$, the center or Santaló point of K is the origin o .*

(i) For $q \geq 0$,

$$\frac{F_{1,q}(K)}{F_{1,q}(B_K)} \leq \frac{H_{1,q}(K)}{H_{1,q}(B_K)} \leq \left(\frac{\tau(K)}{\tau(B_K)} \right)^{n/(n+q)}.$$

(ii) For $-n < q < 0$,

$$\frac{F_{1,q}(K)}{F_{1,q}(B_K)} \geq \frac{H_{1,q}(K)}{H_{1,q}(B_K)} \geq \left(\frac{\tau(K)}{\tau(B_K)} \right)^{n/(n+q)}.$$

(iii) For $q < -n$, there exists a constant $a > 0$ such that

$$\frac{F_{1,q}(K)}{F_{1,q}(B_K)} \geq \frac{H_{1,q}(K)}{H_{1,q}(B_K)} \geq a^{nq/(n+q)} \left(\frac{\tau(K)}{\tau(B_K)} \right)^{n/(n+q)}.$$

Theorem 1.3. *Let $K \in \mathcal{K}_0$.*

(i) If $-n < t < 0 < r < s$, or $-n < s < 0 < r < t$, then

$$H_{1,r}(K) \leq (H_{1,t}(K))^{(r-s)(n+t)/(t-s)(n+r)} (H_{1,s}(K))^{(r-t)(n+s)/(s-t)(n+r)}.$$

(ii) If $-n < t < r < s < 0$, or $-n < s < r < t < 0$, then

$$H_{1,r}(K) \leq (H_{1,t}(K))^{(r-s)(n+t)/(t-s)(n+r)} (H_{1,s}(K))^{(r-t)(n+s)/(s-t)(n+r)}.$$

(iii) If $t < r < -n < s < 0$, or $s < r < -n < t < 0$, then

$$H_{1,r}(K) \geq (H_{1,t}(K))^{(r-s)(n+t)/(t-s)(n+r)} (H_{1,s}(K))^{(r-t)(n+s)/(s-t)(n+r)}.$$

2. Background and Preliminaries

In this section, we will focus on some detailed preliminaries.

If C, D are compact convex sets in \mathbb{R}^n and $a \in \mathbb{R}$, the Minkowski addition $C + D$ is the vector addition, $C + D = \{x + y : x \in C, y \in D\}$, and the scalar multiplication $aC = \{ax : x \in C\}$. Denote $O(n)$ as the class of $n \times n$ orthogonal matrixs. And we use $\det\phi$ to represent the determinant of ϕ . If $\det\phi \neq 0$, then we use ϕ^{-1} to denote the inverse of ϕ , i.e., ϕ is invertible.

The polar body of $K \in \mathcal{K}_0$, denoted by K° , is defined as $K^\circ = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for any } y \in K\}$. By the definition of the polar body, it is easy to get that $K^{\circ\circ} = K$ for $K \in \mathcal{K}_0$ in [31]. Define K^p , the polar body of K with respect to p , by $K^p = (K - p)^\circ + p$, for $K \in \mathcal{K}$ and $p \in \text{int}K$ (where $\text{int}K$ is the interior of K). The Santaló

point of $K \in \mathcal{K}$, denoted by $s(K) \in \text{int}K$, is defined as $|K^{s(K)}| = \inf\{|K^w| : w \in \text{int}K\}$, and the Santaló point of K is unique (see [26]). And there is the distinguished Blaschke-Santaló inequality: For $K \in \mathcal{K}$,

$$|K| \cdot |K^{s(K)}| \leq \omega_n^2 \tag{4}$$

with equality if and only if K is an ellipsoid. In addition, the inverse of this inequality (4) is called the inverse Santaló inequality (see e.g., [2, 14, 28]), for $K \in \mathcal{K}$, there exists a constant $d > 0$ such that

$$|K| \cdot |K^{s(K)}| \geq d^n \omega_n^2. \tag{5}$$

If for any $x \in L \subseteq \mathbb{R}^n$, the line segment from the origin o to x is contained in L , then L is a star-shaped set with respect to the origin o . For a compact star-shaped set L with respect to the origin o , the radial function $\rho_L(u) : S^{n-1} \rightarrow [0, \infty)$ is defined by $\rho_L(u) = \max\{a \geq 0 : au \in L\}$ for $u \in S^{n-1}$. Obviously, the radial function is positive and continuous. We call a star-shaped set M a star body, if the radial function of the star-shaped set is continuous about the variable $x \in M$. Since S_0 is the set of star bodies about the origin o , then $\mathcal{K}_0 \subseteq S_0$. And for $K \in \mathcal{K}_0$, $\rho_{K^\circ}(u) = 1/h_K(u)$ and $h_{K^\circ}(u) = 1/\rho_K(u)$ for $u \in S^{n-1}$ (see [31]). Furthermore, for $K \in S_0$, the following the formula for the volume of L :

$$|L| = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^n du.$$

Combining with each $K \in \mathcal{K}_0$, the surface area measure of K on S^{n-1} is denoted by $S(K, \cdot)$ (see [18]), which is defined as: for any subset measurable $A \subseteq S^{n-1}$,

$$S(K, A) = \int_{\nu_K^{-1}(A)} d\mathcal{H}^{n-1}(u),$$

where $\nu_K^{-1} : S^{n-1} \rightarrow \partial K$ is the inverse Gauss map and \mathcal{H}^{n-1} is the $(n - 1)$ -dimensional Hausdorff measure on ∂K (the boundary of K). Let $C_c^\infty(\text{int}K)$, $C^\infty(\text{int}K)$, $C(K)$, $C^+(S^{n-1})$ be the set of all infinitely differentiable functions on $\text{int}K$ with compact supports, the set of all infinitely differentiable functions on $\text{int}K$, the set of all continuous functions on K and the set of all positive continuous functions on S^{n-1} , respectively. If the surface area measure $S(K, \cdot)$ of $K \in \mathcal{K}$ is absolutely continuous about spherical measure $\sigma(\cdot)$, then K has a curvature function $g_K(\cdot) : S^{n-1} \rightarrow \mathbb{R}$, and $g_K(u) = dS(K, u)/d\sigma(u)$ almost everywhere about $\sigma(\cdot)$. Define \mathcal{G}_0^+ , a subset of \mathcal{K}_0 , by $\mathcal{G}_0^+ = \{K \in \mathcal{K}_0 : g_K(u) \in C^+(S^{n-1})\}$.

For $K \in \mathcal{K}$, the torsional rigidity of K , denoted by $\tau(K)$, is defined as follows [5]:

$$\frac{1}{\tau(K)} = \inf \left\{ \frac{\int_K |\nabla v(x)|^2 dx}{\left(\int_K |v(x)| dx\right)^2} : v \in W_0^{1,2}(\text{int}K) \text{ and } \int_K |v(x)| dx > 0 \right\},$$

where ∇v is the gradient of the function v and $W^{1,2}(\text{int}K)$ is the Sobolev space of the functions in $L^2(\text{int}K)$ whose first-order weak derivatives belong to $L^2(\text{int}K)$, and $W_0^{1,2}(\text{int}K)$ represents the closure of $C_c^\infty(\text{int}K)$ in the Sobolev space $W^{1,2}(\text{int}K)$. By this definition, for $K \in \mathcal{K}_0$ and $b > 0$, one has

$$\tau(bK) = b^{n+2} \tau(K). \tag{6}$$

For any $K \in \mathcal{K}_0$, there exists a unique solution $u_{\tau,K} \in C^\infty(\text{int}K) \cap C(K)$ to the following boundary value problem:

$$\begin{cases} \Delta u = -2 & \text{in } K \\ u = 0 & \text{on } \partial K. \end{cases} \tag{7}$$

In addition, one can easily get the following property by Li and Zhu [16].

Remark 2.1. For any $\phi \in O(n)$ and $K \in \mathcal{K}_0$, the solution $u_{\tau, \phi K}$ to (7) in ϕK has this simple property,

$$u_{\tau, \phi K}(u) = u_{\tau, K}(\phi^t u) \tag{8}$$

for $u \in S^{n-1}$.

The torsional rigidity measure $\mu_\tau(K, \cdot)$ on S^{n-1} , as follows [6]:

$$\mu_\tau(K, A) = \int_{v_K^{-1}(A)} |\nabla u_{\tau, K}(x)|^2 d\mathcal{H}^{n-1}(x) \text{ for any measurable subset } A \subseteq S^{n-1}.$$

It is easy to calculate that

$$d\mu_\tau(K, u) = |\nabla u_{\tau, K}(v_K^{-1}(u))|^2 dS(K, u) \text{ for } u \in S^{n-1}. \tag{9}$$

Evidently, the torsional rigidity measure $\mu_\tau(K, \cdot)$ is not concentrated on a great subsphere.

According to the above information, the formula was given by Colesanti and Fimiani [6]: If $K \in \mathcal{K}$, then

$$\begin{aligned} \tau(K) &= \frac{1}{n+2} \int_{\partial K} h_K(v(x)) |\nabla u_{\tau, K}(x)|^2 d\mathcal{H}^{n-1}(x) \\ &= \frac{1}{n+2} \int_{S^{n-1}} h_K(v) d\mu_\tau(K, v). \end{aligned} \tag{10}$$

In particular, if $K = B_2^n$, then

$$\tau(B_2^n) = \frac{2}{n-2} \omega_n \tag{11}$$

and

$$d\mu_\tau(B_2^n, u) = \frac{2(n+2)}{n(n-2)} d\sigma(u) \text{ for } u \in S^{n-1}. \tag{12}$$

By (10), one can define the following probability measure $\mu_\tau^*(K, \cdot)$ on S^{n-1} , if $K \in \mathcal{K}_0$,

$$\mu_\tau^*(K, u) = \frac{1}{n+2} \cdot \frac{h_K(u) \mu_\tau(K, u)}{\tau(K)} \text{ for } u \in S^{n-1}.$$

3. The Orlicz geominimal torsional rigidity

In this section, according to the definitions of the nonhomogeneous and homogeneous Orlicz mixed torsional rigidity, we will consider the extreme problems associated with them by studying the properties of the corresponding mixed torsional rigidity.

First of all, let \mathcal{D} be the set of continuous functions $\varphi : (0, \infty) \rightarrow (0, \infty)$, such that φ is strictly decreasing with $\lim_{t \rightarrow 0^+} \varphi(t) = \infty$, $\lim_{t \rightarrow \infty} \varphi(t) = 0$ and $\varphi(1) = 1$. Let \mathcal{I} be the set of continuous functions $\varphi : (0, \infty) \rightarrow (0, \infty)$, such that φ is strictly increasing with $\lim_{t \rightarrow 0^+} \varphi(t) = 0$, $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ and $\varphi(1) = 1$. The following two definitions will be required.

Definition 3.1. (see [16] Definition 4.1) Let $\varphi \in \mathcal{I} \cup \mathcal{D}$ and $K, L \in \mathcal{K}_0$. The nonhomogeneous Orlicz L_φ mixed torsional of K and L is written as $\tau_{1, \varphi}(K, L)$. Define

$$\tau_{1, \varphi}(K, L) = \frac{1}{n+2} \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) d\mu_\tau(K, u).$$

Accordingly, by Definition 3.1, one can easily know that
 (i) If K and L are dilates, i.e., $K = aL$ for some $a > 0$, then

$$\tau_{1,\varphi}(K, L) = \varphi(a)\tau(K);$$

(ii) If $L \in S_0$, one has

$$\tau_{1,\varphi}(K, L^\circ) = \frac{1}{n+2} \int_{S^{n-1}} \varphi\left(\frac{1}{\rho_L(u)h_K(u)}\right)h_K(u)d\mu_\tau(K, u).$$

Definition 3.2. Suppose $\varphi \in \mathcal{I} \cup \mathcal{D}$ and $K, L \in \mathcal{K}_0$. The homogeneous Orlicz L_φ mixed torsional rigidity of K and L , denoted by $\widehat{\tau}_{1,\varphi}(K, L)$, is defined as

$$\int_{S^{n-1}} \varphi\left(\frac{\tau(K)h_L(u)}{\widehat{\tau}_{1,\varphi}(K, L)h_K(u)}\right)d\mu_\tau^*(K, u) = 1. \tag{13}$$

If $s, t > 0$, one can get

$$\widehat{\tau}_{1,\varphi}(sK, tL) = s^{n+1} \cdot t \cdot \widehat{\tau}_{1,\varphi}(K, L), \tag{14}$$

then $\widehat{\tau}_{1,\varphi}(K, L)$ is homogeneous. In particular, if $L \in S_0$, then (13) and (14) can be written as

$$\int_{S^{n-1}} \varphi\left(\frac{\tau(K)}{\widehat{\tau}_{1,\varphi}(K, L^\circ)\rho_L(u)h_K(u)}\right)d\mu_\tau^*(K, u) = 1.$$

$$\widehat{\tau}_{1,\varphi}(sK, (tL)^\circ) = s^{n+1} \cdot t^{-1} \cdot \widehat{\tau}_{1,\varphi}(K, L^\circ). \tag{15}$$

In order to facilitate the study of the Orlicz geominimal torsional rigidity, the following concept will be helpful:

$$\begin{cases} \mathcal{I}_0 = \mathcal{I} \cap \{\varphi : (0, \infty) \rightarrow (0, \infty) | \varphi(t^{-1/n}) \text{ is strictly convex}\}; \\ \mathcal{D}_0 = \mathcal{D} \cap \{\varphi : (0, \infty) \rightarrow (0, \infty) | \varphi(t^{-1/n}) \text{ is strictly concave}\}; \\ \mathcal{D}_1 = \mathcal{D} \cap \{\varphi : (0, \infty) \rightarrow (0, \infty) | \varphi(t^{-1/n}) \text{ is strictly convex}\}. \end{cases}$$

For any $L \in \mathcal{K}_0$, due to $|\text{vrad}(L^\circ)L^\circ| = \omega_n, h_{\text{vrad}(L^\circ)L^\circ} = \text{vrad}(L^\circ)h_L$ and for any $L \in S_0, \rho_{\text{vrad}(L^\circ)L^\circ} = \text{vrad}(L^\circ)\rho_L$, we can define the following Orlicz geominimal torsional rigidity. Suppose \mathcal{E}_0 is a subset of S_0 and is nonempty.

Definition 3.3. Suppose $K \in \mathcal{K}_0$, the nonhomogeneous Orlicz geominimal torsional rigidity of K with respect to \mathcal{E}_0 , denoted by $H_{1,\varphi}(K, \mathcal{E}_0)$, is defined as

$$H_{1,\varphi}(K, \mathcal{E}_0) = \inf_{L \in \mathcal{E}_0} \{\tau_{1,\varphi}(K, \text{vrad}(L)L^\circ)\} \text{ for } \varphi \in \mathcal{I} \cup \mathcal{D}_1,$$

$$H_{1,\varphi}(K, \mathcal{E}_0) = \sup_{L \in \mathcal{E}_0} \{\tau_{1,\varphi}(K, \text{vrad}(L)L^\circ)\} \text{ for } \varphi \in \mathcal{D}_0.$$

Definition 3.4. Suppose $K \in \mathcal{K}_0$, the homogeneous Orlicz geominimal torsional rigidity of K with respect to \mathcal{E}_0 , denoted by $\widehat{H}_{1,\varphi}(K, \mathcal{E}_0)$, is defined as

$$\widehat{H}_{1,\varphi}(K, \mathcal{E}_0) = \inf_{L \in \mathcal{E}_0} \{\widehat{\tau}_{1,\varphi}(K, \text{vrad}(L)L^\circ)\} \text{ for } \varphi \in \mathcal{I} \cup \mathcal{D}_0,$$

$$\widehat{H}_{1,\varphi}(K, \mathcal{E}_0) = \sup_{L \in \mathcal{E}_0} \{\widehat{\tau}_{1,\varphi}(K, \text{vrad}(L)L^\circ)\} \text{ for } \varphi \in \mathcal{D}_1.$$

Next we have the following two special cases according to the previous definitions and will discuss their related properties.

Remark 3.5. (i) If $\mathcal{E}_0 = \mathcal{K}_0$, one gets

$$\begin{aligned} H_{1,\varphi}(K) &= H_{1,\varphi}(K, \mathcal{K}_0); \\ \widehat{H}_{1,\varphi}(K) &= \widehat{H}_{1,\varphi}(K, \mathcal{K}_0). \end{aligned}$$

(ii) If $\mathcal{E}_0 = S_0$, one gets

$$\begin{aligned} F_{1,\varphi}(K) &= H_{1,\varphi}(K, S_0); \\ \widehat{F}_{1,\varphi}(K) &= \widehat{H}_{1,\varphi}(K, S_0). \end{aligned}$$

As a result of $\mathcal{K}_0 \subseteq S_0$, one has

$$\begin{aligned} F_{1,\varphi}(K) &\leq H_{1,\varphi}(K) \text{ for } \varphi \in \mathcal{I} \cap \mathcal{D}_1, \\ F_{1,\varphi}(K) &\geq H_{1,\varphi}(K) \text{ for } \varphi \in \mathcal{D}_0 \end{aligned}$$

and

$$\begin{aligned} \widehat{F}_{1,\varphi}(K) &\leq \widehat{H}_{1,\varphi}(K) \text{ for } \varphi \in \mathcal{I} \cap \mathcal{D}_0, \\ \widehat{F}_{1,\varphi}(K) &\geq \widehat{H}_{1,\varphi}(K) \text{ for } \varphi \in \mathcal{D}_1. \end{aligned}$$

From (14) and (15), if $a > 0$, one has

$$\begin{aligned} \widehat{H}_{1,\varphi}(aK) &= a^{n+1}\widehat{H}_{1,\varphi}(K); \\ \widehat{F}_{1,\varphi}(aK) &= a^{n+1}\widehat{F}_{1,\varphi}(K). \end{aligned}$$

Proposition 3.6. Suppose $\varphi \in \mathcal{I} \cup \mathcal{D}_0 \cup \mathcal{D}_1$ and $K \in \mathcal{K}_0$, then for any $\phi \in O(n)$, one has

$$H_{1,\varphi}(\phi K) = H_{1,\varphi}(K), \widehat{H}_{1,\varphi}(\phi K) = \widehat{H}_{1,\varphi}(K)$$

and

$$F_{1,\varphi}(\phi K) = F_{1,\varphi}(K), \widehat{F}_{1,\varphi}(\phi K) = \widehat{F}_{1,\varphi}(K).$$

Proof. We only prove $H_{1,\varphi}(\phi K) = H_{1,\varphi}(K)$, the other conclusions follow along the same argument. Let $\phi \in O(n)$ and $L \in \mathcal{K}_0$, then $|\phi L| = |L|$, $\text{vrad}(\phi L) = \text{vrad}(L)$, by (8) and (9) for $u \in S^{n-1}$, one has

$$\begin{aligned} d\mu_\tau(\phi K, u) &= |\nabla u_{\tau, \phi K}(v_{\phi K}^{-1}(u))|^2 dS(\phi K, u) \\ &= |\nabla u_{\tau, K}(v_K^{-1}(\phi^t u))|^2 dS(K, \phi^t u) \\ &= d\mu_\tau(K, \phi^t u). \end{aligned} \tag{16}$$

Let $L \in \mathcal{K}_0$ and $\phi \in O(n)$, by $(\phi L)^\circ = \phi L^\circ$ and (16), one has

$$\begin{aligned} \tau_{1,\varphi}(\phi K, \text{vrad}(\phi L)(\phi L)^\circ) &= \tau_{1,\varphi}(\phi K, \text{vrad}(L)\phi L^\circ) \\ &= \frac{1}{n+2} \int_{S^{n-1}} \varphi \left(\frac{h_{\text{vrad}(L)\phi L^\circ}(u)}{h_{\phi K}(u)} \right) h_{\phi K}(u) d\mu_\tau(\phi K, u) \\ &= \frac{1}{n+2} \int_{S^{n-1}} \varphi \left(\frac{h_{\text{vrad}(L)L^\circ}(\phi^t u)}{h_K(\phi^t u)} \right) h_K(\phi^t u) d\mu_\tau(K, \phi^t u) \\ &= \tau_{1,\varphi}(K, \text{vrad}(L)L^\circ). \end{aligned}$$

Together with Definition 3.3, then if $\varphi \in \mathcal{I} \cup \mathcal{D}_1$,

$$\begin{aligned} H_{1,\varphi}(\phi K) &= \inf_{\phi L \in \mathcal{K}_0} \{ \tau_{1,\varphi}(\phi K, \text{vrad}(\phi L)(\phi L)^\circ) \} \\ &= \inf_{L \in \mathcal{K}_0} \{ \tau_{1,\varphi}(K, \text{vrad}(L)L^\circ) \} \\ &= H_{1,\varphi}(K). \quad \square \end{aligned}$$

In addition, it is difficult to calculate $H_{1,\varphi}(K)$, $\widehat{H}_{1,\varphi}(K)$, $F_{1,\varphi}(K)$ and $\widehat{F}_{1,\varphi}(K)$. But in some special cases, one can calculate their exact values, i.e., let $K = \lambda B_2^n$ for some $\lambda > 0$.

Theorem 3.7. Suppose $\varphi \in \mathcal{I}_0 \cup \mathcal{D}_0 \cup \mathcal{D}_1$ and $\lambda > 0$. Then

$$\widehat{F}_{1,\varphi}(B_2^n) = \widehat{H}_{1,\varphi}(B_2^n) = \tau(B_2^n), \tag{17}$$

$$F_{1,\varphi}(\lambda B_2^n) = H_{1,\varphi}(\lambda B_2^n) = \varphi(1/\lambda)\tau(\lambda B_2^n). \tag{18}$$

Proof. Since the proof process of $\varphi \in \mathcal{D}_0 \cup \mathcal{D}_1$ and $\varphi \in \mathcal{I}_0$ of (17) and (18) are similar, so we just prove that $\varphi \in \mathcal{I}_0$. First, we prove (17). Let $L_0 = L/\text{vrad}(L)$ for any $L \in \mathcal{S}_0$. Then $\text{vrad}(L_0) = 1$ and $|L_0| = \omega_n$. By $\varphi \in \mathcal{I}_0$, combining with Jensen’s inequality, (11) and (12), one has

$$\begin{aligned} 1 &= \int_{S^{n-1}} \varphi \left(\frac{\tau(B_2^n)}{\widehat{\tau}_{1,\varphi}(B_2^n, L_0^\circ) \rho_{L_0}(u) h_{B_2^n}(u)} \right) d\mu_\tau^*(B_2^n, u) \\ &= \int_{S^{n-1}} \varphi \left(\frac{\tau(B_2^n)}{\widehat{\tau}_{1,\varphi}(B_2^n, L_0^\circ) \rho_{L_0}(u)} \right) \frac{d\sigma(u)}{n\omega_n} \\ &\geq \varphi \left(\left(\int_{S^{n-1}} \left(\frac{\tau(B_2^n)}{\widehat{\tau}_{1,\varphi}(B_2^n, L_0^\circ) \rho_{L_0}(u)} \right)^{-n} \frac{d\sigma(u)}{n\omega_n} \right)^{-1/n} \right) \\ &= \varphi \left(\frac{\tau(B_2^n)}{\widehat{\tau}_{1,\varphi}(B_2^n, L_0^\circ)} \right). \end{aligned}$$

Due to $\varphi \in \mathcal{I}_0$ and $\varphi(1) = 1$, then

$$\tau(B_2^n) \leq \widehat{\tau}_{1,\varphi}(B_2^n, L_0^\circ) = \widehat{\tau}_{1,\varphi}(B_2^n, \text{vrad}(L)L^\circ).$$

According to the definition of $\widehat{H}_{1,\varphi}(K, \mathcal{E}_0)$ in Definition 3.4, one has

$$\tau(B_2^n) \leq \widehat{F}_{1,\varphi}(B_2^n) \leq \widehat{H}_{1,\varphi}(B_2^n) = \inf_{L \in \mathcal{K}_0} \{ \widehat{\tau}_{1,\varphi}(B_2^n, \text{vrad}(L)L^\circ) \} \leq \tau(B_2^n).$$

Therefore, $\tau(B_2^n) = \widehat{F}_{1,\varphi}(B_2^n) = \widehat{H}_{1,\varphi}(B_2^n)$. Along the same line, one can prove (18). \square

Recall that $B_K = \text{vrad}(K)B_2^n$ be the origin symmetric ball of radius $\text{vrad}(K)$ for $K \in \mathcal{K}_0$. The following theorems are the isoperimetric type inequalities of $F_{1,\varphi}(\cdot)$, $H_{1,\varphi}(\cdot)$, $\widehat{F}_{1,\varphi}(\cdot)$, and $\widehat{H}_{1,\varphi}(\cdot)$.

Theorem 3.8. Suppose $L \in \mathcal{K}_0$ and the center or Santaló point of L is the origin o .

(i) If $\varphi \in \mathcal{D}_0 \cup \mathcal{I}_0$ and then

$$\frac{\widehat{F}_{1,\varphi}(L)}{\widehat{F}_{1,\varphi}(B_L)} \leq \frac{\widehat{H}_{1,\varphi}(L)}{\widehat{H}_{1,\varphi}(B_L)} \leq \frac{\tau(L)}{\tau(B_L)}.$$

Equality holds if L is an origin symmetric ball.

(ii) If $\varphi \in \mathcal{D}_1$, then there exists a constant $d > 0$ such that

$$\frac{\widehat{F}_{1,\varphi}(L)}{\widehat{F}_{1,\varphi}(B_L)} \geq \frac{\widehat{H}_{1,\varphi}(L)}{\widehat{H}_{1,\varphi}(B_L)} \geq \frac{d \cdot \tau(L)}{\tau(B_L)}.$$

Proof. (i) If $\varphi \in \mathcal{D}_0 \cup \mathcal{I}_0$. By Theorem 3.7, the homogeneity of $\widehat{H}_{1,\varphi}(\cdot)$, $\widehat{F}_{1,\varphi}(\cdot)$ and $\tau(\cdot)$, one has

$$\widehat{F}_{1,\varphi}(B_L) = \widehat{H}_{1,\varphi}(B_L) = \frac{\tau(B_L)}{\text{vrad}(L)}. \tag{19}$$

Due to (14) and Definition 3.4, one gets

$$\widehat{F}_{1,\varphi}(L) \leq \widehat{H}_{1,\varphi}(L) \leq \widehat{\tau}_{1,\varphi}(L, \text{vrad}(L^\circ)L) = \text{vrad}(L^\circ)\tau(L).$$

Combining with (4) and (19), one gets

$$\frac{\widehat{F}_{1,\varphi}(L)}{\widehat{F}_{1,\varphi}(B_L)} \leq \frac{\widehat{H}_{1,\varphi}(L)}{\widehat{H}_{1,\varphi}(B_L)} \leq \frac{\tau(L)}{\tau(B_L)}.$$

Let $L = cB_2^n$ for some $c > 0$, one can easily get $L = B_L$ and then the equality holds.

(ii) According to the same argument above, if $\varphi \in \mathcal{D}_1$, by (5), one has

$$\frac{\widehat{F}_{1,\varphi}(L)}{\widehat{F}_{1,\varphi}(B_L)} \geq \frac{\widehat{H}_{1,\varphi}(L)}{\widehat{H}_{1,\varphi}(B_L)} \geq \frac{\text{vrad}(L) \cdot \text{vrad}(L^\circ) \cdot \tau(L)}{\tau(B_L)} \geq \frac{d \cdot \tau(L)}{\tau(B_L)}. \quad \square$$

In the same manner, one can check the similar results for $F_{1,\varphi}(L)$ and $H_{1,\varphi}(L)$.

Theorem 3.9. *Suppose $L \in \mathcal{K}_0$ and the center or Santaló point of L is the origin o .*

(i) *If $\varphi \in \mathcal{D}_1 \cup \mathcal{I}_0$ and then*

$$\frac{F_{1,\varphi}(L)}{F_{1,\varphi}((B_{L^\circ})^\circ)} \leq \frac{H_{1,\varphi}(L)}{H_{1,\varphi}((B_{L^\circ})^\circ)} \leq \frac{\tau(L)}{\tau((B_{L^\circ})^\circ)}.$$

In addition, if $\varphi \in \mathcal{I}_0$, then

$$\frac{F_{1,\varphi}(L)}{F_{1,\varphi}(B_L)} \leq \frac{H_{1,\varphi}(L)}{H_{1,\varphi}(B_L)} \leq \frac{\tau(L)}{\tau(B_L)}.$$

Equality holds if L is an origin symmetric ball.

(ii) *If $\varphi \in \mathcal{D}_0$, then*

$$\frac{F_{1,\varphi}(L)}{F_{1,\varphi}(B_L)} \geq \frac{H_{1,\varphi}(L)}{H_{1,\varphi}(B_L)} \geq \frac{\tau(L)}{\tau(B_L)}.$$

Equality holds if L is an origin symmetric ball.

Proof. (i) If $\varphi \in \mathcal{D}_1 \cup \mathcal{I}_0$, together with Definition 3.3, one has

$$F_{1,\varphi}(L) \leq H_{1,\varphi}(L) \leq \tau_{1,\varphi}(L, \text{vrad}(L^\circ)L) = \varphi(\text{vrad}(L^\circ))\tau(L). \tag{20}$$

Since (18) and $(B_{L^\circ})^\circ = (\text{vrad}(L^\circ)B_2^n)^\circ = (1/\text{vrad}(L^\circ))B_2^n$, then

$$F_{1,\varphi}(B_L) = H_{1,\varphi}(B_L) = \varphi(1/\text{vrad}(L))\tau(B_L); \tag{21}$$

$$F_{1,\varphi}((B_{L^\circ})^\circ) = H_{1,\varphi}((B_{L^\circ})^\circ) = \varphi(\text{vrad}(L^\circ))\tau((B_{L^\circ})^\circ). \tag{22}$$

Together with (20) and (22), one has

$$\frac{F_{1,\varphi}(L)}{F_{1,\varphi}((B_{L^\circ})^\circ)} \leq \frac{H_{1,\varphi}(L)}{H_{1,\varphi}((B_{L^\circ})^\circ)} \leq \frac{\tau(L)}{\tau((B_{L^\circ})^\circ)}.$$

If $\varphi \in \mathcal{I}_0$, by (4) and (20), one has

$$F_{1,\varphi}(L) \leq H_{1,\varphi}(L) \leq \varphi(\text{vrad}(L^\circ))\tau(L) \leq \varphi(1/\text{vrad}(L))\tau(L). \tag{23}$$

The desired inequality follows from (21) and (23). If L is an origin symmetric ball, the equality obviously holds. In the case of $\varphi \in \mathcal{D}_0$, we can prove that (ii) is true by the same way. \square

4. The L_q geominimal torsional rigidity

In Definition 3.1, $\varphi \in \mathcal{D} \cup \mathcal{I}$, here we let $\varphi(t) = t^q$ and think about the L_q geominimal torsional rigidity of K with respect to \mathcal{K}_0 and S_0 . Namely, let

$$\tau_{1,q}(K, L) = \frac{1}{n+2} \int_{S^{n-1}} \left(\frac{h_L(u)}{h_K(u)} \right)^q h_K(u) d\mu_\tau(K, u) \text{ for } L \in \mathcal{K}_0;$$

$$\tau_{1,q}(K, L^\circ) = \frac{1}{n+2} \int_{S^{n-1}} \left(\frac{1}{\rho_L(u) h_K(u)} \right)^q h_K(u) d\mu_\tau(K, u) \text{ for } L \in S_0.$$

Definition 4.1. Suppose $K \in \mathcal{K}_0$ and $-n \neq q \in \mathbb{R}$. Define the L_q geominimal torsional rigidity $H_{1,q}(K)$ on \mathcal{K}_0 , by

$$H_{1,q}(K) = \inf_{L \in \mathcal{K}_0} \left\{ \tau_{1,q}(K, L)^{n/(n+q)} |L^\circ|^{q/(n+q)} \right\}, q \geq 0; \tag{24}$$

$$H_{1,q}(K) = \sup_{L \in \mathcal{K}_0} \left\{ \tau_{1,q}(K, L)^{n/(n+q)} |L^\circ|^{q/(n+q)} \right\}, -n \neq q < 0. \tag{25}$$

Define the L_q geominimal torsional rigidity $F_{1,q}(K)$ about S_0 , by

$$F_{1,q}(K) = \inf_{L \in S_0} \left\{ \tau_{1,q}(K, L^\circ)^{n/(n+q)} |L|^{q/(n+q)} \right\}, q \geq 0; \tag{26}$$

$$F_{1,q}(K) = \sup_{L \in S_0} \left\{ \tau_{1,q}(K, L^\circ)^{n/(n+q)} |L|^{q/(n+q)} \right\}, -n \neq q < 0. \tag{27}$$

Obviously, $H_{1,0}(K) = F_{1,0}(K) = \tau(K)$ for $K \in \mathcal{K}_0$. And, for $\varphi(t) = t^q (q \neq -n)$,

$$H_{1,q}(aK) = a^{(n+2-q)n/(n+q)} H_{1,q}(K),$$

$$F_{1,q}(aK) = a^{(n+2-q)n/(n+q)} F_{1,q}(K)$$

for any $a > 0$;

$$H_{1,q}(\phi K) = H_{1,q}(K),$$

$$F_{1,q}(\phi K) = F_{1,q}(K)$$

for $\phi \in O(n)$. In addition, if $q \neq 0, -n$, by $\varphi(t) = t^q$, then

$$\widehat{H}_{1,\varphi}(K) = \frac{\tau(K)^{1-(1/q)}}{\omega_n^{1/n}} H_{1,q}(K)^{(n+q)/nq}; \tag{28}$$

$$\widehat{F}_{1,\varphi}(K) = \frac{\tau(K)^{1-(1/q)}}{\omega_n^{1/n}} F_{1,q}(K)^{(n+q)/nq}. \tag{29}$$

According to (17) and (24), one can get the following corollary.

Corollary 4.2. If $-n \neq q \in \mathbb{R}$, then

$$\begin{aligned} H_{1,q}(B_2^n) &= F_{1,q}(B_2^n) \\ &= (\tau(B_2^n))^{n/(n+q)} |B_2^n|^{q/(n+q)} \\ &= (\tau_{1,q}(B_2^n, B_2^n))^{n/(n+q)} |B_2^n|^{q/(n+q)}. \end{aligned}$$

Using the following theorem we will introduce a convenient way to calculate $F_{1,q}(K)$. Let

$$g_{\mu,q}(K, u) = h_K^{1-q}(u) |\nabla u_{\tau,K}(v_K^{-1}(u))|^2 g_K(u) \text{ for } K \in \mathcal{G}_0^+, q \neq -n,$$

where $g_K(\cdot)$ is the curvature function of K in S^{n-1} , $u_{\tau,K}$ is the solution of (7). For $-n \neq q \in \mathbb{R}$, let

$$\zeta_{\mu,q} = \left\{ K \in \mathcal{G}_0^+ : \exists D \in S_0, \text{ s.t. } g_{\mu,q}(K, u) = \rho_D(u)^{n+q}, u \in S^{n-1} \right\}.$$

Thus,

$$g_{\mu,q}(B_2^n, u) = \frac{2(n+2)}{n(n-2)} = \rho_{D_0}(u)^{n+q} \text{ for } u \in S^{n-1},$$

where $D_0 = \left(\frac{2(n+2)}{n(n-2)}\right)^{1/(n+q)} B_2^n \in S_0$.

Theorem 4.3. *If $K \in \zeta_{\mu,q}$, then for $-n \neq q \in \mathbb{R}$,*

$$F_{1,q}(K) = \left(\frac{1}{n+q}\right)^{n/(n+q)} \left(\frac{1}{n}\right)^{q/(n+q)} \int_{S^{n-1}} g_{\mu,q}(K, u)^{n/(n+q)} d\sigma(u).$$

Proof. Let $L \in S_0$, if $q = 0$,

$$F_{1,0}(K) = \frac{1}{n+2} \int_{S^{n-1}} h_K(u) d\mu_\tau(K, u) = \tau(K),$$

the conclusion is clearly true.

If $q > 0$, by the integral form of the Hölder inequality [11], one has

$$\begin{aligned} & \left(\frac{1}{n+2}\right)^{n/(n+q)} \left(\frac{1}{n}\right)^{q/(n+q)} \int_{S^{n-1}} g_{\mu,q}(K, u)^{n/(n+q)} d\sigma(u) \\ &= \left(\frac{1}{n+2}\right)^{n/(n+q)} \left(\frac{1}{n}\right)^{q/(n+q)} \int_{S^{n-1}} [g_{\mu,q}(K, u) \cdot \rho_L(u)^{-q} \cdot \rho_L(u)^q]^{n/(n+q)} d\sigma(u) \\ &\leq \left(\frac{1}{n} \int_{S^{n-1}} \rho_L(u)^n d\sigma(u)\right)^{q/(n+q)} \cdot \left(\frac{1}{n+2} \int_{S^{n-1}} \frac{g_{\mu,q}(K, u)}{\rho_L(u)^q} d\sigma(u)\right)^{n/(n+q)} \\ &= |L|^{q/(n+q)} \cdot \tau_{1,q}(K, L^\circ)^{n/(n+q)}. \end{aligned} \tag{30}$$

And equality holds if and only if $\rho_L(u)^{n+q} = g_{\mu,q}(K, u)$ for $u \in S^{n-1}$. Taking minimize both sides of (30), one has

$$\left(\frac{1}{n+2}\right)^{n/(n+q)} \left(\frac{1}{n}\right)^{q/(n+q)} \int_{S^{n-1}} g_{\mu,q}(K, u)^{n/(n+q)} d\sigma(u) \leq F_{1,q}(K). \tag{31}$$

Since $K \in \zeta_{\mu,q}$, then there exists a star body $Q \in S_0$ such that

$$\rho_Q(u) = g_{\mu,q}(K, u)^{1/(n+q)},$$

for $u \in S^{n-1}$. And by (26), then

$$\begin{aligned} & \left(\frac{1}{n+2}\right)^{n/(n+q)} \left(\frac{1}{n}\right)^{q/(n+q)} \int_{S^{n-1}} g_{\mu,q}(K, u)^{n/(n+q)} d\sigma(u) \\ &= \tau_{1,q}(K, Q^\circ)^{n/(n+q)} \cdot |Q|^{q/(n+q)} \geq F_{1,q}(K). \end{aligned} \tag{32}$$

Combining with (31) and (32), one gets

$$F_{1,q}(K) = \left(\frac{1}{n+2}\right)^{n/(n+q)} \left(\frac{1}{n}\right)^{q/(n+q)} \int_{S^{n-1}} g_{\mu,q}(K, u)^{n/(n+q)} d\sigma(u).$$

In the case of $-n \neq q < 0$ can be obtained by using the same lines. \square

For $K \in \zeta_{\mu,q}$, and $-n \neq q \in \mathbb{R}$, one can define $\Theta_{\mu,q}K \in S_0$, the torsional rigidity q -curvature image of K , by

$$g_{\mu,q}(K, u) = \frac{n+2}{n|\Theta_{\mu,q}K|} \rho_{\Theta_{\mu,q}K}(u)^{n+q}, u \in S^{n-1}. \tag{33}$$

By Theorem 4.3, one can also get

$$F_{1,q}(K) = \left(\tau_{1,q}(K, (\Theta_{\mu,q}K)^\circ) \right)^{n/(n+q)} |\Theta_{\mu,q}K|^{q/(n+q)}. \tag{34}$$

For $-n \neq q \in \mathbb{R}$, $u \in S^{n-1}$,

$$\eta_{\mu,q} = \left\{ K \in \mathcal{G}_0^+ : \exists Q \in \mathcal{K}_0, \text{ s.t. } g_{\mu,q}(K, u) = \rho_Q(u)^{n+q} \right\}.$$

Obviously, $\eta_{\mu,q} \subseteq \zeta_{\mu,q}$. In particular, we have $B_2^n \in \eta_{\mu,q}$ that implies $\eta_{\mu,q} \neq \emptyset$. The following proposition provide a simple method for calculating $H_{1,q}(K)$ for $K \in \eta_{\mu,q}$.

Proposition 4.4. *If $-n \neq q \in \mathbb{R}$, $K \in \eta_{\mu,q}$, then $H_{1,q}(K) = F_{1,q}(K)$.*

Proof. First, we will show that $\Theta_{\mu,q}K \in \mathcal{K}_0$ if $K \in \eta_{\mu,q}$. Assume $K \in \eta_{\mu,q}$, then there exists a convex body $Q \in \mathcal{K}_0$ such that $g_{\mu,q}(K, u) = \rho_Q(u)^{n+q}$ for $u \in S^{n-1}$, combining with (33), one has

$$\frac{n+2}{n|\Theta_{\mu,q}K|} \rho_{\Theta_{\mu,q}K}(u)^{n+q} = \rho_Q(u)^{n+q} \text{ for } u \in S^{n-1},$$

thus

$$\Theta_{\mu,q}K = \left(\frac{n|\Theta_{\mu,q}K|}{n+2} \right)^{1/(n+q)} Q \in \mathcal{K}_0.$$

Next, we prove $H_{1,q}(K) = F_{1,q}(K)$.

If $q = 0$, it is easy to prove that $H_{1,0}(K) = F_{1,0}(K)$.

If $q > 0$, by (24) and (26), one gets $H_{1,q}(K) \geq F_{1,q}(K)$. By $\Theta_{\mu,q}K \in \mathcal{K}_0$, (24) and (34), one has

$$F_{1,q}(K) \geq H_{1,q}(K).$$

Thus $H_{1,q}(K) = F_{1,q}(K)$.

If $-n \neq q < 0$, by (25), (27), (34) and $\Theta_{\mu,q}K \in \mathcal{K}_0$, one gets

$$H_{1,q}(K) \leq F_{1,q}(K) \leq H_{1,q}(K).$$

Therefore, $H_{1,q}(K) = F_{1,q}(K)$. \square

In addition, one can easily get the following isoperimetric type inequality.

Theorem 4.5. *Let $K \in \mathcal{K}_0$, the center or Santaló point of K is the origin o .*

(i) For $q \geq 0$,

$$\frac{F_{1,q}(K)}{F_{1,q}(B_K)} \leq \frac{H_{1,q}(K)}{H_{1,q}(B_K)} \leq \left(\frac{\tau(K)}{\tau(B_K)} \right)^{n/(n+q)}.$$

(ii) For $-n < q < 0$,

$$\frac{F_{1,q}(K)}{F_{1,q}(B_K)} \geq \frac{H_{1,q}(K)}{H_{1,q}(B_K)} \geq \left(\frac{\tau(K)}{\tau(B_K)} \right)^{n/(n+q)}.$$

(iii) For $q < -n$, there exists a constant $a > 0$ such that

$$\frac{F_{1,q}(K)}{F_{1,q}(B_K)} \geq \frac{H_{1,q}(K)}{H_{1,q}(B_K)} \geq a^{nq/(n+q)} \left(\frac{\tau(K)}{\tau(B_K)} \right)^{n/(n+q)}.$$

Proof. (i) If $q = 0$, since $F_{1,0}(K) = H_{1,0}(K) = \tau(K)$ for $K \in \mathcal{K}_0$, then obviously

$$\frac{F_{1,q}(K)}{F_{1,q}(B_K)} = \frac{H_{1,q}(K)}{H_{1,q}(B_K)} = \left(\frac{\tau(K)}{\tau(B_K)} \right)^{n/(n+q)}$$

holds. If $q > 0$, by (i) in Theorem 3.8, (28) and (29), one has

$$\frac{\frac{\tau(K)^{1-(1/q)}}{\omega_n^{1/n}} F_{1,q}(K)^{(n+q)/nq}}{\frac{\tau(B_K)^{1-(1/q)}}{\omega_n^{1/n}} F_{1,q}(B_K)^{(n+q)/nq}} \leq \frac{\frac{\tau(K)^{1-(1/q)}}{\omega_n^{1/n}} H_{1,q}(K)^{(n+q)/nq}}{\frac{\tau(B_K)^{1-(1/q)}}{\omega_n^{1/n}} H_{1,q}(B_K)^{(n+q)/nq}} \leq \frac{\tau(K)}{\tau(B_K)}.$$

After simple calculation, one can get the following result:

$$\frac{F_{1,q}(K)}{F_{1,q}(B_K)} \leq \frac{H_{1,q}(K)}{H_{1,q}(B_K)} \leq \left(\frac{\tau(K)}{\tau(B_K)} \right)^{n/(n+q)}.$$

On the basis of the same way, again, using Theorem 3.8, (28) and (29), according to Theorem 3.9, one can prove (ii) and (iii). \square

The following theorem gives the cyclic inequality for $H_{1,r}(K)$.

Theorem 4.6. Let $K \in \mathcal{K}_0$.

(i) If $-n < t < 0 < r < s$, or $-n < s < 0 < r < t$, then

$$H_{1,r}(K) \leq (H_{1,t}(K))^{(r-s)(n+t)/(t-s)(n+r)} (H_{1,s}(K))^{(r-t)(n+s)/(s-t)(n+r)}.$$

(ii) If $-n < t < r < s < 0$, or $-n < s < r < t < 0$, then

$$H_{1,r}(K) \leq (H_{1,t}(K))^{(r-s)(n+t)/(t-s)(n+r)} (H_{1,s}(K))^{(r-t)(n+s)/(s-t)(n+r)}.$$

(iii) If $t < r < -n < s < 0$, or $s < r < -n < t < 0$, then

$$H_{1,r}(K) \geq (H_{1,t}(K))^{(r-s)(n+t)/(t-s)(n+r)} (H_{1,s}(K))^{(r-t)(n+s)/(s-t)(n+r)}.$$

Proof. First of all, let $K, L \in \mathcal{K}_0, s, r, t \in \mathbb{R}$ such that $0 < \frac{t-r}{t-s} < 1$, by the integral form of the Hölder inequality, one has

$$\begin{aligned} \tau_{1,r}(K, L) &= \frac{1}{n+2} \int_{S^{n-1}} h_L(u)^r h_K(u)^{1-r} d\mu_\tau(K, u) \\ &\leq \frac{1}{n+2} \left(\int_{S^{n-1}} h_L(u)^s h_K(u)^{1-s} d\mu_\tau(K, u) \right)^{(r-t)/(s-t)} \\ &\quad \cdot \left(\int_{S^{n-1}} h_L(u)^t h_K(u)^{1-t} d\mu_\tau(K, u) \right)^{(r-s)/(t-s)} \\ &= (\tau_{1,s}(K, L))^{(r-t)/(s-t)} (\tau_{1,t}(K, L))^{(r-s)/(t-s)}. \end{aligned} \tag{35}$$

(i) If $-n < t < 0 < r < s$, then

$$\frac{(r-s)(n+t)}{(t-s)(n+t)} > 0 \text{ and } \frac{(r-t)(n+s)}{(s-t)(n+r)} > 0.$$

Combining with Definition 4.1 and (35), one has

$$\begin{aligned} H_{1,r}(K) &= \inf_{L \in \mathcal{K}_0} \left\{ \tau_{1,r}(K, L)^{n/(n+r)} |L^\circ|^r / (n+r) \right\} \\ &\leq \inf_{L \in \mathcal{K}_0} \left\{ [\tau_{1,t}(K, L)^{n/(n+t)} |L^\circ|^t / (n+t)]^{(r-s)(n+t)/(t-s)(n+r)} \right. \\ &\quad \cdot \left. [\tau_{1,s}(K, L)^{n/(n+s)} |L^\circ|^s / (n+s)]^{(r-t)(n+s)/(s-t)(n+r)} \right\} \\ &\leq \sup_{L \in \mathcal{K}_0} \left\{ \tau_{1,t}(K, L)^{n/(n+t)} |L^\circ|^t / (n+t) \right\}^{(r-s)(n+t)/(t-s)(n+r)} \\ &\quad \cdot \inf_{L \in \mathcal{K}_0} \left\{ \tau_{1,s}(K, L)^{n/(n+s)} |L^\circ|^s / (n+s) \right\}^{(r-t)(n+s)/(s-t)(n+r)} \\ &= H_{1,t}(K)^{(r-s)(n+t)/(t-s)(n+r)} \cdot H_{1,s}(K)^{(r-t)(n+s)/(s-t)(n+r)}. \end{aligned}$$

The case $-n < s < 0 < r < t$ can be proved follow along the same lines.

(ii) If $-n < t < r < s < 0$, then

$$\frac{(r-s)(n+t)}{(t-s)(n+r)} > 0 \text{ and } \frac{(r-t)(n+s)}{(s-t)(n+r)} > 0.$$

Combining with Definition 4.1 and (35), one has

$$\begin{aligned} H_{1,r}(K) &= \sup_{L \in \mathcal{K}_0} \left\{ \tau_{1,r}(K, L)^{n/(n+r)} |L^\circ|^r / (n+r) \right\} \\ &\leq \sup_{L \in \mathcal{K}_0} \left\{ [\tau_{1,t}(K, L)^{(r-s)/(t-s)} \tau_{1,s}(K, L)^{(r-t)/(s-t)}]^{n/(n+r)} |L^\circ|^r / (n+r) \right\} \\ &\leq \sup_{L \in \mathcal{K}_0} \left\{ \tau_{1,t}(K, L)^{n/(n+t)} |L^\circ|^t / (n+t) \right\}^{(r-s)(n+t)/(t-s)(n+r)} \\ &\quad \cdot \sup_{L \in \mathcal{K}_0} \left\{ \tau_{1,s}(K, L)^{n/(n+s)} |L^\circ|^s / (n+s) \right\}^{(r-t)(n+s)/(s-t)(n+r)} \\ &= H_{1,t}(K)^{(r-s)(n+t)/(t-s)(n+r)} \cdot H_{1,s}(K)^{(r-t)(n+s)/(s-t)(n+r)}. \end{aligned}$$

By transposing s and t , the case $-n < s < r < t < 0$ can be proved.

(iii) If $t < r < -n < s < 0$, then

$$\frac{(r-s)(n+t)}{(t-s)(n+r)} > 0 \text{ and } \frac{(r-t)(n+s)}{(s-t)(n+r)} < 0.$$

Combining with Definition 4.1 and (35), one has

$$\begin{aligned} H_{1,r}(K) &= \sup_{L \in \mathcal{K}_0} \left\{ \tau_{1,r}(K, L)^{n/(n+r)} |L^\circ|^r / (n+r) \right\} \\ &\geq \sup_{L \in \mathcal{K}_0} \left\{ [\tau_{1,t}(K, L)^{(r-s)/(t-s)} \tau_{1,s}(K, L)^{(r-t)/(s-t)}]^{n/(n+r)} |L^\circ|^r / (n+r) \right\} \\ &\geq \sup_{L \in \mathcal{K}_0} \left\{ \tau_{1,t}(K, L)^{n/(n+t)} |L^\circ|^t / (n+t) \right\}^{(r-s)(n+t)/(t-s)(n+r)} \\ &\quad \cdot \sup_{L \in \mathcal{K}_0} \left\{ \tau_{1,s}(K, L)^{n/(n+s)} |L^\circ|^s / (n+s) \right\}^{(r-t)(n+s)/(s-t)(n+r)} \\ &= H_{1,t}(K)^{(r-s)(n+t)/(t-s)(n+r)} \cdot H_{1,s}(K)^{(r-t)(n+s)/(s-t)(n+r)}. \end{aligned}$$

By transposing s and t , the case $s < r < -n < t < 0$ can be proved. \square

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