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The Spectrum of a New Class of Sylvester-Kac Matrices

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Abstract. We define a new class of Sylvester-Kac matrices and calculate their spectra explicitly. We use the technique of the left eigenvectors to obtain the claim. We also provide some right eigenvectors which can be useful in applied computations. The main results are rather general and contain many known particular characterizations. Matrices belonging to this family represent a convenient test matrices for numerical eigenvalue computations with known spectrum.

1. Introduction

In 1854, J.J. Sylvester conjectured in [15] that the eigenvalues of the tridiagonal matrix

$$A_n = \begin{pmatrix} 0 & 1 & & & & \\ n & 0 & 2 & & & \\ & n-1 & \ddots & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & 0 & n \\ & & & & 1 & 0 \end{pmatrix},$$

were $n-2k$, for $k=0,1,\dots,n$. Since then, many extensions and proofs have been proposed. Perhaps the most pertinent results can be found in [2–4, 7–14] and references therein. The matrix A_n, which we call Sylvester-Kac matrix, became also known as Clement matrix due to the independent study of P.A. Clement in [6]. Throughout the text, all non-mentioned entries should be read as zero.

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Recently, R. Oste and J. Van der Jeugt [14] established a study of a new family of matrices. From this study we can deduce that the eigenvalues of

$$H_{2n}(a) = \begin{pmatrix} 0 & 1+a & & & & \\ 2n & 0 & 2 & & & \\ & 2n-1+a & 0 & 3+a & & \\ & & 2n-2 & \ddots & \ddots & \\ & & & \ddots & \ddots & 2n \\ & & & & 1+a & 0 \end{pmatrix}$$

are

$$\pm 2\sqrt{k(k+a)}, \quad \text{for } k = 0, 1, \ldots, n,$$

and of

$$H_{2n-1}(a) = \begin{pmatrix} 0 & 1+a & & & & \\ 2n-1+a & 0 & 2 & & & \\ & 2n-2 & 0 & 3+a & & \\ & & 2n-3+a & \ddots & \ddots & \\ & & & \ddots & \ddots & 2n-1 \\ & & & & 1 & 0 \end{pmatrix}$$

are

$$\pm(2k+a+1), \quad \text{for } k = 0, 1, \ldots, n-1.$$

It is interesting to notice that a possible extension of the first case can be

$$\tilde{H}_n(a) = \begin{pmatrix} 0 & 1\cdot r+a & & & & \\ n\cdot s & 0 & 2\cdot r & & & \\ & (n-1)\cdot s+a & 0 & 3\cdot r+a & & \\ & & (n-2)\cdot s & \ddots & \ddots & \\ & & & \ddots & \ddots & n\cdot r \\ & & & & 1\cdot s+a & 0 \end{pmatrix},$$

whose eigenvalues are

$$\pm\sqrt{2k(ar+as+2krs)}, \quad \text{for } k = 0, 1, \ldots, \ell.$$

However, for the second case, it seems not possible to advance a close formula.

When the eigenvalues, eigenvectors, determinant, and similar other notions involving spectral properties of a matrix are known, we refer to such a matrix as a test matrix. Test matrices are used to evaluate the accuracy of matrix inversion programs since the exact inverses are known (cf. e.g. [1, 14] and references therein). We believe that this family and the corresponding explicit eigenvalues will make a significant contribution to these types of special matrices.

Recently, Coelho, Dimitrov, and Rakai in [5] suggested a method for a fast estimation of the largest eigenvalue of an asymmetric tridiagonal matrix. The proposed procedure was based on the power method and the computation of the square of the original matrix. Then they provided numerical results with simulations in C/C++ implementation in order to demonstrate the effectiveness of the proposed method. They adopted the Sylvester-Kac test matrix [14] for comparing the power method and the proposed method performance.

In this spirit, a new family of Sylvester-Kac type matrices is defined in [8] and the corresponding spectrum is derived. Namely, the authors claim that the eigenvalues of

$$G_{2n}(a)=\begin{pmatrix} 0 & 1+a & & & & & & \\ 4n+2 & 0 & 2 & & & & & \\ & 4n+1+a & 0 & 3+a & & & & \\ & & 4n & 0 & 4 & & & \\ & & & 4n-1+a & \ddots & \ddots & & \\ & & & & \ddots & \ddots & 2n-1+a & \\ & & & & & 2n+4 & 0 & 2n \\ & & & & & & 2n+3+a & 0 \end{pmatrix},$$

are

$$\pm 2\sqrt{2k}\sqrt{a+2k}, \quad \text{for } k=0,1,\ldots,n,$$

and the eigenvalues of

$$G_{2n-1}(a)=\begin{pmatrix} 0 & 1+a & & & & & \\ 4n & 0 & 2 & & & & \\ & 4n-1+a & 0 & 3+a & & & \\ & & 4n-2 & \ddots & \ddots & & \\ & & & \ddots & \ddots & 2n-2 & \\ & & & & 2n+3+a & 0 & 2n-1+a \\ & & & & & 2n+2 & 0 \end{pmatrix},$$

are

$$\pm 2\sqrt{2k+1}\sqrt{a+2k+1}, \quad \text{for } k=0,1,\ldots,n-1.$$

It is interesting to notice that $\frac{1}{2}G_n(0)$ and the Sylvester-Kac matrix share exactly the same eigenvalues.

The authors used the left eigenvectors to prove the formulas for the eigenvalues. They also provide explicitly right eigenvectors of $G_{2n-1}(a)$ corresponding eigenvalues $\mp 2\sqrt{(2n-1)(a+2n-1)}$ as well as right eigenvectors of $G_{2n}(a)$ corresponding to eigenvalues $\mp 2\sqrt{2n(a+2n)}$.

In this manner, now going further, we generalize the matrix $G_n(a)$ by adding additional parameters and then derive all eigenvalues of this general matrix explicitly by using left-eigenvector trick. As a consequence, we evaluate its determinant. Finally we shall compute some right eigenvectors of the matrix.

For any real numbers t and m, we define the following generalization of the Sylvester-Kac matrix $G_{2n}(a,t,m)$ of order $2n+1$ as following

$$G_{2n}(a,t,m)=\begin{pmatrix} m & 1+a & & & & & & \\ 4n+2 & t & 2 & & & & & \\ & 4n+1+a & m & 3+a & & & & \\ & & 4n & t & 4 & & & \\ & & & 4n-1+a & \ddots & \ddots & & \\ & & & & \ddots & \ddots & 2n-1+a & \\ & & & & & 2n+4 & t & 2n \\ & & & & & & 2n+3+a & m \end{pmatrix}$$

or in closed form

$$(G_{2n}(a,t,m))_{i,j} = \begin{cases} m & \text{if } i \text{ is odd,} \\ t & \text{if } i \text{ is even,} \\ i+a & \text{if } j=i+1 \text{ and } i \text{ is odd,} \\ i & \text{if } j=i+1 \text{ and } i \text{ is even,} \\ 4n+4-i & \text{if } i=j+1 \text{ and } i \text{ is even,} \\ 4n+4+a-i & \text{if } i=j+1 \text{ and } i \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

And the general Sylvester-Kac matrix $G_{2n-1}(a,t,m)$ of order $2n$ as following

$$G_{2n-1}(a,t,m) = \begin{pmatrix} m & 1+a & & & & & \\ 4n & t & 2 & & & & \\ & 4n-1+a & m & 3+a & & & \\ & & 4n-2 & \ddots & \ddots & & \\ & & & \ddots & \ddots & 2n-2 & \\ & & & & 2n+3+a & m & 2n-1+a \\ & & & & & 2n+2 & t \end{pmatrix}$$

or in closed form

$$(G_{2n-1}(a,t,m))_{i,j} = \begin{cases} m & \text{if } i \text{ is odd,} \\ t & \text{if } i \text{ is even,} \\ i+a & \text{if } j=i+1 \text{ and } i \text{ is odd,} \\ i & \text{if } j=i+1 \text{ and } i \text{ is even,} \\ 4n+2-i & \text{if } i=j+1 \text{ and } i \text{ is even,} \\ 4n+2+a-i & \text{if } i=j+1 \text{ and } i \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Since the main diagonal entries are consist of m and t, respectively, it is called a periodic generalization of the Sylvester-Kac matrix.

Our main purpose is to determine explicitly the spectrum of $G_n(a,t,m)$, denoted by $\lambda(G_n(a,t,m))$:

$$\lambda(G_{2\ell}(a,t,m)) = \left\{\frac{1}{2}\left(m+t \pm \sqrt{(m-t)^2 + 32ak + 64k^2}\right)\right\}_{k=1}^{\ell} \cup \{m\}$$

and

$$\lambda(G_{2\ell-1}(a,t,m)) = \left\{\frac{1}{2}\left(m+t \mp \sqrt{(m-t)^2 + 16(2k-1)(a+2k-1)}\right)\right\}_{k=1}^{\ell}.$$

When we take the periodicity parameters t and m as 0, then the matrix $G_n(a,t,m)$ is reduced to the Sylvester matrix $G_n(a)$.

We first determine the spectrum of $G_n(a,t,m)$ in the next section. In the third section, we provide the right eigenvectors which can be of independent interest both in pure and numerical applications. We use basically the left eigenvectors of the matrix and an inductive approach to reach our aims.

As we mentioned above, the Sylvester-Kac is part of a family of matrices known as test matrices, which are used to compare performance of methods. We hope that this new family of Sylvester-Kac type matrices defined here will contribute to the literature regarding special matrices with known eigenvalues as test matrices.

2. The spectrum of $G_n(a,t,m)$

In this section we find the spectrum of $G_n(a,t,m)$, denoted by $\lambda(G_n(a,t,m))$. For this purpose we shall focus on the matrix $G_n(a,t,0)$ and derive its spectrum $\lambda(G_n(a,t,0))$. Then using some transform trick, we will derive the spectrum of the matrix $G_n(a,t,m)$. In the end, we derive the determinant. For easy writing, we denote the matrix $G_n(a,t,0)$ by G_n unless we do not need special values of a and t.

Theorem 2.1. *Explicitly, the eigenvalues of G_n are*

$$\lambda\left(G_{2\ell}\left(a,t,0\right)\right)=\left\{\frac{1}{2}\left(t\pm\sqrt{t^2+32ak+64k^2}\right)\right\}_{k=1}^{\ell}\cup\{0\}$$

and

$$\lambda\left(G_{2\ell-1}\left(a,t,0\right)\right)=\left\{\frac{1}{2}\left(t\mp\sqrt{t^2+16\left(2k-1\right)\left(a+2k-1\right)}\right)\right\}_{k=1}^{\ell}.$$

We start by finding two eigenvalues of G_{2n} and then two left eigenvectors corresponding to each of them.

Lemma 2.2. *The matrix G_{2n} has the eigenvalues $\lambda^+=\frac{t+\sqrt{t^2+32an+64n^2}}{2}$ and $\lambda^-=\frac{t-\sqrt{t^2+32an+64n^2}}{2}$ with left $(2n+1)$-eigenvectors u^+ and u^- defined by*

$$u^+=\begin{pmatrix} 2n+1 & \frac{n\lambda^+}{2n} & 2n-1 & \frac{(n-1)\lambda^+}{2n} & \cdots & 5 & \frac{2\lambda^+}{2n} & 3 & \frac{\lambda^+}{2n} & 1\end{pmatrix}$$

and

$$u^-=\begin{pmatrix} 2n+1 & \frac{n\lambda^-}{2n} & 2n-1 & \frac{(n-1)\lambda^-}{2n} & \cdots & 5 & \frac{2\lambda^-}{2n} & 3 & \frac{\lambda^-}{2n} & 1\end{pmatrix},$$

respectively.

Proof. To prove our claim, it is sufficient to show that

$$u^+G_{2n}=\lambda^+u^+ \quad \text{and} \quad u^-G_{2n}=\lambda^-u^- .$$

From the definitions of G_{2n} and u^+, we should show that

$$\begin{aligned}(u^+G_{2n})_{1,1} &= (\lambda^+u^+)_{1,1},\\ (u^+G_{2n})_{1,2n+1} &= (\lambda^+u^+)_{1,2n+1}\end{aligned}$$

and

$$(u^+G_{2n})_{1,m}=(\lambda^+u^+)_{1,m}, \quad \text{for } 1<m<2n+1.$$

The first two claims are simple to check. For example, the first identity comes from

$$(u^+G_{2n})_{1,1}=(2n+1)\times 0+\frac{n\lambda^+}{2n}(4n+2)=(2n+1)\lambda^+$$

and

$$(\lambda^+u^+)_{1,1}=\lambda^+(2n+1).$$

We now focus on the case $2\le m\le 2n$. For even m, say $m=2k$, we consider $(u^+G_{2n})_{1,2k}=(\lambda^+u^+)_{1,2k}$. The product of u^+ by G_{2n} provides, for $1\le k\le n$,

$$\begin{aligned}(u^+G_{2n})_{1,2k} &= (2n+3-2k)(a+2k-1)+\frac{(n-k+1)\lambda^+t}{2n}+(2n+1-2k)(4n+3-2k+a)\\ &= 4(a+2n)(n-k+1)+\frac{(n-k+1)\lambda^+t}{2n}\\ &= (n-k+1)\left[4(a+2n)+\frac{\lambda^+t}{2n}\right],\end{aligned}$$

which, by the definition of λ^+, gives us

$$(u^+G_{2n})_{1,2k} = \frac{n+1-k}{2n}\,(\lambda^+)^2\,.$$

On the other hand, we see that

$$(\lambda^+u^+)_{1,2k} = (\lambda^+)^2\,\frac{n+1-k}{2n}\,,$$

as claimed. The other case, i.e., $u^-G_{2n} = \lambda^-u^-$, can be handled in a similar fashion. □

We shall now consider the matrix G_{2n}. Define the matrix Y of order $2n+1$ as

$$Y = \left(\begin{array}{cc|ccccccccc} 2n+1 & \frac{n\lambda^+}{2n} & 2n-1 & \frac{(n-1)\lambda^+}{2n} & \cdots & 5 & \frac{2\lambda^+}{2n} & 3 & \frac{\lambda^+}{2n} & 1 \\ 2n+1 & \frac{n\lambda^-}{2n} & 2n-1 & \frac{(n-1)\lambda^-}{2n} & \cdots & 5 & \frac{2\lambda^-}{2n} & 3 & \frac{\lambda^-}{2n} & 1 \\ \hline \multicolumn{2}{c|}{\mathbf{0}_{(2n-1)\times 2}} & \multicolumn{8}{c}{I_{2n-1}} \end{array}\right).$$

Similarly to the previous case, we obtain

$$Y^{-1} = \left(\begin{array}{cc|ccccccccccc} \frac{\lambda^-}{(2n+1)(2\lambda^- - t)} & \frac{\lambda^+}{(2n+1)(2\lambda^+ - t)} & -\frac{2n-1}{2n+1} & 0 & -\frac{2n-3}{2n+1} & 0 & \cdots & 0 & \frac{-3}{2n+1} & 0 & \frac{-1}{2n+1} \\ \frac{2}{2\lambda^+ - t} & \frac{2}{2\lambda^- - t} & 0 & -\frac{n-1}{n} & 0 & -\frac{n-2}{n} & \cdots & \frac{-2}{n} & 0 & \frac{-1}{n} & 0 \\ \hline \multicolumn{2}{c|}{\mathbf{0}_{(2n-1)\times 2}} & \multicolumn{9}{c}{I_{2n-1}} \end{array}\right).$$

Therefore, G_{2n} is similar to $D = YG_{2n}Y^{-1}$ where

$$D = \left(\begin{array}{cc|c} \lambda^+ & 0 & \\ 0 & \lambda^- & 0_{2\times(2n-1)} \\ \hline \frac{2(a+4n+1)}{2\lambda^+ - t} & \frac{2(a+4n+1)}{2\lambda^- - t} & \\ & 0_{(2n-2)\times 2} & Q \end{array}\right),$$

where $Q = \left(Q_{i,j}\right)$ is the matrix, of order $2n-1$, given by

$$Q_{i,j} = \begin{cases} t & \text{if } i = j \text{ is even,} \\ i+a+2 & \text{if } j = i+1 \text{ and } i > 1 \text{ is odd,} \\ i+2 & \text{if } j = i+1 \text{ and } i \text{ is even,} \\ 4n+2-i & \text{if } i = j+1 \text{ and } i \text{ is even,} \\ 4n+2+a-i & \text{if } i = j+1 \text{ and } i \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

with the exceptional entries $Q_{1,2j} = -\frac{(n-j)(a+4n+1)}{n}$ for $j > 1$ and $Q_{1,2} = a+3-\frac{(n-1)(a+4n+1)}{n}$.

Clearly it has the form

$$Q = \begin{pmatrix} 0 & a+3-\frac{(n-1)(a+4n+1)}{n} & 0 & \frac{-(n-2)(a+4n+1)}{n} & \cdots & & 0 & -\frac{a+4n+1}{n} & 0 \\ 4n & t & 4 & & & & & & \\ & a+4n-1 & 0 & a+5 & & & & & \\ & & 4n-2 & t & 6 & & & & \\ & & & a+4n-3 & 0 & \ddots & & & \\ & & & & \ddots & \ddots & 2n-2 & & \\ & & & & & \ddots & 0 & a+2n-1 & \\ & & & & & & 2n+4 & t & 2n \\ & & & & & & & a+2n+3 & 0 \end{pmatrix}.$$

Similarly to the previous case, we shall give two eigenvalues of G_{2n-1} and then two corresponding left eigenvectors associated to each of them.

Lemma 2.3. *The matrix G_{2n-1} has the eigenvalues*

$$\mu^+ = \frac{1}{2}\left(t + \sqrt{t^2 + 16\,(2n-1)\,(a+2n-1)}\right) \quad \textit{and} \quad \mu^- = \frac{1}{2}\left(t - \sqrt{t^2 + 16\,(2n-1)\,(a+2n-1)}\right)$$

with left $2n$-eigenvectors v^+ and v^- defined by

$$v^+ = \begin{pmatrix} 2n & \frac{(2n-1)\mu^+}{2(2n-1)} & 2n-2 & \frac{(2n-3)\mu^+}{2(2n-1)} & \cdots & 4 & \frac{3\mu^+}{2(2n-1)} & 2 & \frac{\mu^+}{2(2n-1)} \end{pmatrix}$$

and

$$v^- = \begin{pmatrix} 2n & \frac{(2n-1)\mu^-}{2(2n-1)} & 2n-2 & \frac{(2n-3)\mu^-}{2(2n-1)} & \cdots & 4 & \frac{3\mu^-}{2(2n-1)} & 2 & \frac{\mu^-}{2(2n-1)} \end{pmatrix},$$

respectively.

Now our purpose is to find similar matrices to G_{2n} and G_{2n-1}, respectively. We start with the matrix G_{2n-1}.

Define a matrix T of order $2n$ as shown

$$T = \left(\begin{array}{cc|ccccccc} 2n & \frac{(2n-1)\mu^+}{2(2n-1)} & 2n-2 & \frac{(2n-3)\mu^+}{2(2n-1)} & \cdots & 4 & \frac{3\mu^+}{2(2n-1)} & 2 & \frac{\mu^+}{2(2n-1)} \\ 2n & \frac{(2n-1)\mu^-}{2(2n-1)} & 2n-2 & \frac{(2n-3)\mu^-}{2(2n-1)} & \cdots & 4 & \frac{3\mu^-}{2(2n-1)} & 2 & \frac{\mu^-}{2(2n-1)} \\ \hline \multicolumn{2}{c|}{\mathbf{0}_{2(n-1)\times 2}} & \multicolumn{7}{c}{I_{2n-2}} \end{array}\right),$$

where $\mathbf{0}_{2(n-1)\times 2}$ is the $2(n-1)\times 2$ zero matrix and I_{2n-2} is the identity matrix of order $2n-2$. Its inverse is

$$T^{-1} = \left(\begin{array}{cc|ccccccccc} \frac{\mu^-}{2n(2\mu^- - t)} & \frac{\mu^+}{2n(2\mu^+ - t)} & -\frac{n-1}{n} & 0 & -\frac{n-2}{n} & 0 & \cdots & \frac{-2}{n} & 0 & \frac{-1}{n} & 0 \\ \frac{2}{2\mu^+ - t} & \frac{2}{2\mu^- - t} & 0 & -\frac{2n-3}{2n-1} & 0 & -\frac{2n-5}{2n-1} & \cdots & 0 & \frac{-3}{2n-1} & 0 & \frac{-1}{2n-1} \\ \hline \multicolumn{2}{c|}{\mathbf{0}_{2(n-1)\times 2}} & \multicolumn{9}{c}{I_{2n-2}} \end{array}\right).$$

We can check that G_{2n-1} is similar to the matrix

$$E = \left(\begin{array}{cc|c} \mu^+ & 0 & \\ 0 & \mu^- & \mathbf{0}_{2\times(2n-2)} \\ \hline \frac{2(a+4n-1)}{2\mu^+ - t} & \frac{2(a+4n-1)}{2\mu^- - t} & \\ \multicolumn{2}{c|}{\mathbf{0}_{(2n-3)\times 2}} & W \end{array}\right),$$

where $W = \left(W_{i,j}\right)$ is the block of order $2n-2$ defined by

$$W_{i,j} = \begin{cases} t & \text{if } i = j \text{ is even}, \\ i+a+2 & \text{if } j = i+1 \text{ and } i > 1 \text{ is odd}, \\ i+2 & \text{if } j = i+1 \text{ and } i \text{ is even}, \\ 4n-i & \text{if } i = j+1 \text{ and } i \text{ is even}, \\ 4n+a-i & \text{if } i = j+1 \text{ and } i \text{ is odd}, \\ 0 & \text{otherwise}, \end{cases}$$

with the exceptional entries $W_{1,2j} = -\frac{(2n-2j-1)(a+4n-1)}{2n-1}$ for $j > 1$ and $W_{1,2} = a + 3 - \frac{(2n-3)(a+4n-1)}{2n-1}$. Clearly it has the form

$$W = \begin{pmatrix} 0 & a+3-\frac{(2n-3)(a+4n-1)}{2n-1} & 0 & \frac{-(2n-5)(a+4n-1)}{2n-1} & 0 & \cdots & 0 & \frac{-(a+4n-1)}{2n-1} \\ 4n-2 & t & 4 & & & & & \\ & 4n-3+a & 0 & 5+a & & & & \\ & & 4n-4 & t & 6 & & & \\ & & & 4n-3+a & \ddots & \ddots & & \\ & & & & \ddots & t & 2n-2 & \\ & & & & & 2n+3+a & 0 & 2n-1+a \\ & & & & & & 2n+2 & t \end{pmatrix},$$

since $E = TG_{2n-1}T^{-1}$. Consequently, μ^+ and μ^- are eigenvalues of E.

To compute the remaining eigenvalues of G_{2n-1} and G_{2n}, we proceed providing some auxiliary results.

Define an upper triangular matrix U_{2n} of order $2n$ with

$$U_{i,i} = \frac{(2n+1-\lfloor i/2\rfloor)(a+4n+1-2\lfloor (i-1)/2\rfloor)}{(n+1)(a+2n+3)}, \quad \text{for } 1 \le i \le 2n$$

and

$$U_{i,i+2r} = \frac{(2n+1-2r-i)(a+4n+3)}{(n+1)(a+2n+3)}, \quad \text{for } 1 \le i \le 2n-2 \text{ and } 1 \le r \le n-1,$$

and 0 otherwise, where $\lfloor\cdot\rfloor$ stands for the floor function.

For example, when $n = 4$, we have

$$U_8 = \begin{pmatrix} \frac{9(a+17)}{5(a+11)} & 0 & \frac{6(a+19)}{5(a+11)} & 0 & \frac{4(a+19)}{5(a+11)} & 0 & \frac{2(a+19)}{5(a+11)} & 0 \\ 0 & \frac{8(a+17)}{5(a+11)} & 0 & \frac{5(a+19)}{5(a+11)} & 0 & \frac{3(a+19)}{5(a+11)} & 0 & \frac{1(a+19)}{5(a+11)} \\ 0 & 0 & \frac{8(a+15)}{5(a+11)} & 0 & \frac{4(a+19)}{5(a+11)} & 0 & \frac{2(a+19)}{5(a+11)} & 0 \\ 0 & 0 & 0 & \frac{7(a+15)}{5(a+11)} & 0 & \frac{3(a+19)}{5(a+11)} & 0 & \frac{1(a+19)}{5(a+11)} \\ 0 & 0 & 0 & 0 & \frac{7(a+13)}{5(a+11)} & 0 & \frac{2(a+19)}{5(a+11)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{6(a+13)}{5(a+11)} & 0 & \frac{1(a+19)}{5(a+11)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6(a+11)}{5(a+11)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5(a+11)}{5(a+11)} \end{pmatrix}.$$

For odd orders, we define an upper triangular matrix U_{2n+1} of order $2n+1$ with

$$U_{i,i} = \frac{(2n+2-\lfloor i/2\rfloor)(a+4n+3-2\lfloor (i-1)/2\rfloor)}{(n+2)(a+2n+3)}, \quad \text{for } 1 \le i \le 2n+1$$

and

$$U_{i,i+2r} = \frac{(2n+2-2r-i)(a+4n+5)}{(n+2)(a+2n+3)}, \quad \text{for } 1 \le i \le 2n \text{ and } 1 \le r \le n,$$

and 0 otherwise.

For example, when $n = 3$, we get

$$U_7 = \begin{pmatrix} \frac{8(a+15)}{5(a+9)} & 0 & \frac{5(a+17)}{5(a+9)} & 0 & \frac{3(a+17)}{5(a+9)} & 0 & \frac{1(a+17)}{5(a+9)} \\ 0 & \frac{7(a+15)}{5(a+9)} & 0 & \frac{4(a+17)}{5(a+9)} & 0 & \frac{2(a+17)}{5(a+9)} & 0 \\ 0 & 0 & \frac{7(a+13)}{5(a+9)} & 0 & \frac{3(a+17)}{5(a+9)} & 0 & \frac{1(a+17)}{5(a+9)} \\ 0 & 0 & 0 & \frac{6(a+13)}{5(a+9)} & 0 & \frac{2(a+17)}{5(a+9)} & 0 \\ 0 & 0 & 0 & 0 & \frac{6(a+11)}{5(a+9)} & 0 & \frac{1(a+17)}{5(a+9)} \\ 0 & 0 & 0 & 0 & 0 & \frac{5(a+11)}{5(a+9)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5(a+9)}{5(a+9)} \end{pmatrix}.$$

A routine calculation leads us to the inverse matrix $U_{2n}^{-1} = \left(C_{ij}\right)$, with

$$C_{ii} = \frac{a+2n+3}{a+4n+1-2\lfloor (i-1)/2 \rfloor} \times \frac{n+1}{2n+1-\lfloor i/2 \rfloor}$$

for $1 \le i \le 2n$,

$$C_{i,i+2} = -\frac{(a+2n+3)(a+4n+3)}{(a+4n+1-2\lfloor (i-1)/2 \rfloor)(a+4n-1-2\lfloor (i-1)/2 \rfloor)} \times \frac{(n+1)(2n-1-i)}{(2n+1-\lfloor i/2 \rfloor)(2n-\lfloor i/2 \rfloor)},$$

for $1 \le i \le 2n-2$, while, for $1 < r \le n-1$,

$$\begin{aligned} C_{i,i+2r} &= -(a+2n+3)(a+4n+3)\prod_{t=1}^{r-1}(a+2t+1) \times \prod_{t=0}^{r} \frac{1}{(a+4n+1-2t-2\lfloor (i-1)/2 \rfloor)} \\ &\quad \times (n+1)(2n+1-2r-i) \times (r-1)! \binom{\lfloor (i+1)/2 \rfloor + r - 1}{r-1} \times \prod_{t=1}^{r+1} \frac{1}{(2n+2-t-\lfloor i/2 \rfloor)} \end{aligned}$$

and 0 otherwise.

On the other hand, $U_{2n+1}^{-1} = \left(S_{ij}\right)$ is

$$S_{ii} = \frac{a+2n+3}{a+4n+3-2\lfloor (i-1)/2 \rfloor} \times \frac{n+2}{2n+2-\lfloor i/2 \rfloor}$$

for $1 \le i \le 2n+1$,

$$S_{i,i+2} = -\frac{(a+2n+3)(a+4n+5)}{(a+4n+3-2\lfloor (i-1)/2 \rfloor)(a+4n+1-2\lfloor (i-1)/2 \rfloor)} \times \frac{(n+2)(2n-i)}{(2n+1-\lfloor i/2 \rfloor)(2n-\lfloor i/2 \rfloor)}$$

for $1 \le i \le 2n-1$, and

$$\begin{aligned} S_{i,i+2r} &= -(a+2n+3)(a+4n+5)\prod_{t=1}^{r-1}(a+2t+1) \times \prod_{t=0}^{r} \frac{1}{(a+4n+3-2t-2\lfloor (i-1)/2 \rfloor)} \\ &\quad \times (n+2)(2n+2-2r-i) \times (r-1)! \binom{\lfloor (i+1)/2 \rfloor + r - 1}{r-1} \times \prod_{t=1}^{r+1} \frac{1}{(2n+3-t-\lfloor i/2 \rfloor)}, \end{aligned}$$

for $1 < r \le n$, and 0 otherwise.

Taking into account the definition of U_n, we clearly have

$$G_{2n-2} = U_{2n-1}\,QU_{2n-1}^{-1} \quad \text{and} \quad G_{2n-1} = U_{2n}\,W\,U_{2n}^{-1}.$$

For the readers convenience, we give a sketch proof for the equality $G_{2n-1} = U_{2n}\,W\,U_{2n}^{-1}$ as a showcase to show how such similar equalities could be proven. But later we leave some similar equalities without giving proofs.

We want to prove the equality

$$G_{2n-1} = U_{2n} W U_{2n}^{-1}$$

or equivalently

$$G_{2n-1} U_{2n} = U_{2n} W.$$

Denote $G_{2n-1}U_{2n}$ and $U_{2n}W$ by $A_n = \left(A_{i,j}\right)$ and $B_n = \left(B_{i,j}\right)$, respectively. Thus, we have to prove that $A_n = B_n$.

The matrix $U_n = \left(U_{i,j}\right)$ is an upper triangular matrix and its almost half of entries at the upper band are zero. The matrix $G_{2n-1} = \left(G_{i,j}\right)$ is a tridiagonal matrix and the matrix $W = \left(W_{i,j}\right)$ is almost a tridiagonal matrix as well as it has first row entries. Considering these facts and from a matrix multiplication, we write the entries of the matrix A_n as

$$\begin{array}{lll} A_{1,j} & = G_{1,2}U_{2,j} & \text{for even } j, \\ A_{i,i-1} & = G_{i,i-1}U_{i-1,i-1} & \text{for } 2 \le i \le n, \\ A_{i,j} & = G_{i,i-1}U_{i-1,j} + G_{i,i+1}U_{i+1,j} & \text{for odd } i \text{ and even } j, \text{ or, vice versa}, \\ A_{i,j} & = t \cdot U_{i,j} & \text{for even } i \text{ and } j, \\ A_{i,j} & = 0 & \text{for odd } i \text{ and } j, \\ A_{i,j} & = 0 & \text{for } i > j+1. \end{array}$$

And similarly we write the entries of the matrix B_n as

$$\begin{array}{lll} B_{1,j} & = U_{1,j-1}W_{j-1,j} + U_{1,j+1}W_{j+1,j} - U_{1,1}z_{j/2} & \text{for even } j, \\ B_{i,i-1} & = U_{i,i}W_{i,i-1} & \text{for } 2 \le i \le n, \\ B_{i,j} & = U_{i,j-1}W_{j-1,j} + U_{i,j+1}W_{j+1,j} & \text{for odd } i \text{ and even } j, \text{ or, vice versa}, \\ B_{i,j} & = t \cdot U_{i,j} & \text{for even } i \text{ and } j, \\ B_{i,j} & = 0 & \text{for odd } i \text{ and } j, \\ B_{i,j} & = 0 & \text{for } i > j+1, \end{array}$$

where $z_j = -(2n-2j+1)(a+4n+3)/(2n+1)$.

In order to prove $A_n = B_n$, we shall chose the first two of entries and leave the others to the reader to do not bother. Now we show that

$$A_{1,j} = B_{1,j}.$$

By using the definitions of the matrices G_{2n-1}, U_{2n} and W, first consider

$$A_{1,2j} = g_{1,2}u_{2,2j} = (a+1)\,u_{2,2+2(j-1)},$$

which by taking $i = 2, r = j-1$, gives

$$A_{1,j} = (a+1) \times \frac{(2n+1-2j-2)(a+4n+3)}{(n+1)(a+2n+3)}.$$

Next, we consider

$$\begin{aligned} B_{i,2j} &= U_{1,2j-1}W_{2j-1,2j} + U_{1,2j+1}W_{2j+1,2j} - U_{1,1}k_j \\ &= \frac{(2n-2j+2)(a+4n+3)(a+2j+1)}{(n+1)(a+2n+3)} + \frac{(2n-2j)(a+4n+3)(4n-2j+3+a)}{(n+1)(a+2n+3)} \\ &\quad - \frac{(2n+1)(a+4n+1)}{(n+1)(a+2n+3)} \times \frac{(2n-2j+1)(a+4n+3)}{(2n+1)}, \end{aligned}$$

which, by summing the first two rational statements, gives

$$\begin{aligned} B_{i,2j} &= \frac{(a+4n+3)\cdot 2\,(2n-2j+1)\,(a+2n+1)}{(n+1)\,(a+2n+3)} - \frac{(a+4n+1)\,(2n-2j+1)\,(a+4n+3)}{(n+1)\,(a+2n+3)} \\ &= \frac{(a+1)\,(2n-2j+1)\,(a+4n+3)}{(n+1)\,(a+2n+3)}, \end{aligned}$$

which equals $A_{1,2j}$ as claimed.

As a second showcase, we shall prove that $A_{i,i-1} = B_{i,i-1}$ for only odd integers i. In that case, we have to prove that $A_{2i+1,2i} = B_{2i+1,2i}$. Now consider

$$\begin{aligned} A_{2i+1,2i} &= G_{2i+1,2i}U_{2i,2i} \\ &= (4n+a-2i+1)\times\frac{(2n+1-i)\,(a+4n+1-2\,(i-1))}{(n+1)\,(a+2n+3)} \\ &= (4n+a-2i+1)\times\frac{(2n+1-i)\,(a+4n-2i+3)}{(n+1)\,(a+2n+3)}. \end{aligned}$$

On the other hand, consider

$$\begin{aligned} B_{2i+1,2i} &= U_{i,i}W_{i,i-1} \\ &= \frac{(2n+1-i)\,(a+4n-2i+1)}{(n+1)\,(a+2n+3)}\times(a+4n-2i+3)\,, \end{aligned}$$

which gives the claimed result, $A_{2i+1,2i} = B_{2i+1,2i}$.

The remaining cases

$$\begin{array}{lll} A_{i,j} &= G_{i,i-1}U_{i-1,j} + G_{i,i+1}U_{i+1,j} & \text{for odd } i \text{ and even } j, \text{ or, vice versa,} \\ A_{i,j} &= t\cdot U_{i,j} & \text{for even } i \text{ and } j, \end{array}$$

could be proven similarly.

If we define the matrix of order n

$$M_n = \left(\begin{array}{c|c} I_2 & \mathbf{0}_{2\times(n-2)} \\ \hline \mathbf{0}_{(n-2)\times 2} & U_{n-2} \end{array}\right),$$

then we get

$$M_{2n}EM_{2n}^{-1} = \left(\begin{array}{cc|c} \mu^+ & 0 & \mathbf{0}_{2\times(2n-2)} \\ 0 & \mu^- & \\ \hline \frac{(a+4n-3)(a+4n-1)}{(a+2n+1)\left(2\mu^+-t\right)}\frac{2(2n-1)}{n} & \frac{(a+4n-3)(a+4n-1)}{(a+2n+1)\left(2\mu^--t\right)}\frac{2(2n-1)}{n} & \\ \mathbf{0}_{(2n-3)\times 2} & & U_{2n-2}WU_{2n-2}^{-1} \end{array}\right)$$

and

$$M_{2n+1}DM_{2n+1}^{-1} = \left(\begin{array}{cc|c} \lambda^+ & 0 & \mathbf{0}_{2\times(2n-1)} \\ 0 & \lambda^- & \\ \hline \frac{(a+4n-1)(a+4n+1)}{(a+2n+1)(2\lambda^+-t)}\frac{4n}{n+1} & \frac{(a+4n-3)(a+4n-1)}{(a+2n+1)(2\lambda^--t)}\frac{4n}{n+1} & \\ \mathbf{0}_{(2n-2)\times 2} & & U_{2n-1}QU_{2n-1}^{-1} \end{array}\right).$$

So far, we derived the identities

$$\begin{aligned} D &= YG_{2n}Y^{-1}, \\ E &= TG_{2n-1}T^{-1}, \\ G_{2n-2} &= U_{2n-1}\,QU_{2n-1}^{-1}, \\ G_{2n-1} &= U_{2n-2}\,W\,U_{2n-2}^{-1}. \end{aligned}$$

From the definition of G_n, both $M_{2n}EM_{2n}^{-1}$ and $M_{2n+1}DM_{2n+1}^{-1}$ can be rewritten in the following lower triangular block form

$$\left(\begin{array}{cc|c} \mu^+ & 0 & \multirow{2}{*}{0} \\ 0 & \mu^- & \\ \hline \multicolumn{2}{c|}{*} & G_{2n-1} \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} \lambda^+ & 0 & \multirow{2}{*}{0} \\ 0 & \lambda^- & \\ \hline \multicolumn{2}{c|}{*} & G_{2n-2} \end{array}\right), \tag{1}$$

respectively.

From (1), we get the recurrences on $n > 0$,

$$\det G_{2n} = \lambda^+\lambda^- \det G_{2n-2}, \text{ with } \det G_0 = 1$$

and

$$\det G_{2n+1} = \mu^+\mu^- \det G_{2n-1}, \text{ with } \det G_1 = 1.$$

Finally, we obtain Theorem 2.1.

Let I_n be the identity matrix of order n. For any real number m, if we consider the summation $G_n\left(a, \hat{t}, 0\right) + mI_n$, then we get that $G_n\left(a, \hat{t}, 0\right) + mI_n = G_n\left(a, \hat{t} + m, m\right)$ and its spectrum as

$$\begin{aligned} \lambda\left(G_{2\ell}\left(a, \hat{t}, 0\right) + mI_{2\ell}\right) &= \lambda\left(G_{2\ell}\left(a, \hat{t} + m, m\right)\right) \\ &= \left\{m + \frac{1}{2}\hat{t} \pm \frac{1}{2}\sqrt{\left(\hat{t}\right)^2 + 32ak + 64k^2}\right\}_{k=1}^{\ell} \cup \{m\} \end{aligned}$$

and

$$\begin{aligned} \lambda\left(G_{2\ell-1}\left(a, \hat{t}, 0\right) + mI_{2\ell-1}\right) &= \lambda\left(G_{2\ell-1}\left(a, \hat{t} + m, m\right)\right) \\ &= \left\{m + \frac{1}{2}\hat{t} \mp \frac{1}{2}\sqrt{\left(\hat{t}\right)^2 + 16\left(2k-1\right)\left(a + 2k - 1\right)}\right\}_{k=1}^{\ell}. \end{aligned}$$

After this, we denote $\hat{t} + m$ by a new parameter t and then we get the matrix $G_n\left(a, t, m\right)$, which is our main matrix. We see the fact that the terms in square roots in the above expressions for the eigenvalues of the matrix $G_n\left(a, \hat{t} + m, m\right)$ only depend on the parameter $\hat{t}$, not the other parameter m. Considering this fact and the transformation $\hat{t} + m \to t$, we get the claimed result that the spectrum of $G_n(a, t, m)$ is given by

$$\lambda\left(G_{2\ell}\left(a, t, m\right)\right) = \left\{\frac{1}{2}\left(m + t \pm \sqrt{\left(t - m\right)^2 + 32ak + 64k^2}\right)\right\}_{k=1}^{\ell} \cup \{m\}$$

and

$$\lambda\left(G_{2\ell-1}\left(at, m\right)\right) = \left\{\frac{1}{2}\left(m + t \mp \sqrt{\left(t - m\right)^2 + 16\left(2k-1\right)\left(a + 2k - 1\right)}\right)\right\}_{k=1}^{\ell}.$$

So, we can compute the determinant of the matrix $G_n(a,t,m)$ as following

$$\begin{aligned}\det G_{2n}(a,t,m) &= m\prod_{k=1}^{n}\left[\frac{1}{2}\left(m+t\pm\sqrt{(t-m)^2+32ak+64k^2}\right)\right]\\ &= m\prod_{k=1}^{n}\left(mt-8ak-16k^2\right)\end{aligned}$$

and

$$\begin{aligned}\det G_{2n-1}(a,t,m) &= \prod_{k=1}^{n}\left[\frac{1}{2}\left(m+t\mp\sqrt{(t-m)^2+16(2k-1)(a+2k-1)}\right)\right]\\ &= \prod_{k=1}^{n}\left[mt-4(2k-1)(a+2k-1)\right].\end{aligned}$$

3. The right eigenvectors

We used the left eigenvectors in the previous section to prove the formulas for the eigenvalues of the matrix $G_n(a,t,m)$. In this final section, we first consider the matrix $G_n(a,t,0)$, or shortly G_n, and provide explicitly right eigenvectors corresponding to λ^+ and λ^-. We realize how these eigenvectors can be complicated, and this fact can be important to the interested readers. Thus, we only note the right eigenvectors of the matrix $G_n(a,t,m)$ by again using the same transform.

So, regarding the right eigenvectors, the formulas seem rather intricate and providing a compact formulation of them seems difficult to achieve. This is mainly due to the fact that they include combinatorial expressions with certain rational coefficients.

Notice that the matrix G_{2n-1} has the eigenvalues

$$\frac{1}{2}\left(t+\sqrt{t^2+16(2n-1)(a+2n-1)}\right)\quad\text{and}\quad\frac{1}{2}\left(t-\sqrt{t^2+16(2n-1)(a+2n-1)}\right)$$

associated with the following eigenvectors

$$\frac{1}{b(n-1,n-1)}\begin{pmatrix} c(n-1,0)\,\frac{s(a;0,n-2)}{s(a;n+1,2n-1)}\lambda^+\\ b(n-1,0)\,\frac{s(a;1,n-1)}{s(a;n+1,2n-1)}\\ c(n-1,1)\,\frac{s(a;1,n-2)}{s(a;n+1,2n-2)}\lambda^+\\ b(n-1,1)\,\frac{s(a;2,n-1)}{s(a;n+1,2n-2)}\\ \vdots\\ c(n-1,k-1)\,\frac{s(a;k-1,n-2)}{s(a;n+1,2n-k)}\lambda^+\\ b(n-1,k-1)\,\frac{s(a;k,n-1)}{s(a;n+1,2n-k-1)}\\ \vdots\\ c(n-1,n-1)\,\frac{s(a;n-1,n-2)}{s(a;n+1,n)}\lambda^+\\ b(n-1,n-1)\,\frac{s(a;n,n-1)}{s(a;n+1,n)} \end{pmatrix}\begin{matrix}\to \text{1st row}\\ \to \text{2nd row}\\ \to \text{3rd row}\\ \to \text{4th row}\\ \\ \to (2k-1)\,\text{st row}\\ \to (2k)\,\text{th row}\\ \\ \to (2n-1)\,\text{st row}\\ \to (2n)\,\text{th row}\end{matrix}$$

and

$$\frac{1}{b(n-1,n-1)}\begin{pmatrix} -c(n-1,0)\,\frac{s(a;0,n-2)}{s(a;n+1,2n-1)}\lambda^- \\ b(n-1,0)\,\frac{s(a;1,n-1)}{s(a;n+1,2n-1)} \\ -c(n-1,1)\,\frac{s(a;1,n-2)}{s(a;n+1,2n-2)}\lambda^- \\ b(n-1,1)\,\frac{s(a;2,n-1)}{s(a;n+1,2n-2)} \\ \vdots \\ -c(n-1,k-1)\,\frac{s(a;k-1,n-2)}{s(a;n+1,2n-k)}\lambda^- \\ b(n-1,k-1)\,\frac{s(a;k,n-1)}{s(a;n+1,2n-k-1)} \\ \vdots \\ -c(n-1,n-1)\,\frac{s(a;n-1,n-2)}{s(a;n+1,n)}\lambda^- \\ b(n-1,n-1)\,\frac{s(a;n,n-1)}{s(a;n+1,n)} \end{pmatrix},$$

respectively, for $1 \le k \le n$, where

$$\begin{aligned} b(n,k) &= 4(2n-1)\binom{2n}{k}\frac{2n-2k+1}{2n-k+1}, \\ c(n,k) &= \binom{2n+1}{k}\frac{2(n-k+1)}{2n-k+2} \end{aligned}$$

and

$$\begin{aligned} s(a;m,n) &= \prod_{k=m}^{n}(a+2k+1) \\ &= (a+1)(a+3)(a+5)\cdots(a+2n+1). \end{aligned}$$

Similarly, the matrix G_{2n} of order $2n+1$ has the eigenvalues

$$2\sqrt{(2n-1)(a+2n-1)} \quad \text{and} \quad -2\sqrt{(2n-1)(a+2n-1)}$$

and the following corresponding eigenvectors

$$\frac{1}{c(n-1,n-1)}\begin{pmatrix} d(n,0)\,\frac{s(a;0,n-1)}{s(a;n+1,2n)} \\ c(n-1,0)\,\frac{s(a;1,n-1)}{s(a;n+1,2n)}\lambda^+ \\ d(n,1)\,\frac{s(a;1,n-1)}{s(a;n+1,2n-1)} \\ c(n-1,1)\,\frac{s(a;2,n-1)}{s(a;n+1,2n-1)}\lambda^+ \\ \vdots \\ c(n-1,k-1)\,\frac{s(a;k,n-1)}{s(a;n+1,2n-k+1)}\lambda^+ \\ d(n,k)\,\frac{s(a;k,n-1)}{s(a;n+1,2n-k)} \\ \vdots \\ c(n-1,n-1)\,\frac{s(a;n,n-1)}{s(a;n+1,n+1)}\lambda^+ \\ d(n,n)\,\frac{s(a;n,n-1)}{s(a;n+1,n)} \end{pmatrix} \begin{matrix} \to \text{1st row} \\ \to \text{2nd row} \\ \to \text{3rd row} \\ \to \text{4th row} \\ \\ \to (2k)\text{ th row} \\ \to (2k+1)\text{ th row} \\ \\ \to (2n)\text{ th row} \\ \to (2n+1)\text{ th row} \end{matrix}$$

and

$$\frac{1}{c(n-1,n-1)}\begin{pmatrix} d(n,0)\,\frac{s(a;0,n-1)}{s(a;n+1,2n)} \\ -c(n-1,0)\,\frac{s(a;1,n-1)}{s(a;n+1,2n)}\lambda^{-} \\ d(n,1)\,\frac{s(a;1,n-1)}{s(a;n+1,2n-1)} \\ -c(n-1,1)\,\frac{s(a;2,n-1)}{s(a;n+1,2n-1)}\lambda^{-} \\ \vdots \\ -c(n-1,k-1)\,\frac{s(a;k,n-1)}{s(a;n+1,2n-k+1)}\lambda^{-} \\ d(n,k)\,\frac{s(a;k,n-1)}{s(a;n+1,2n-k)} \\ \vdots \\ -c(n-1,n-1)\,\frac{s(a;n,n-1)}{s(a;n+1,n+1)}\lambda^{-} \\ d(n,n)\,\frac{s(a;n,n-1)}{s(a;n+1,n)} \end{pmatrix}$$

for $0 \leq k \leq n$ respectively, where $c(n,k)$ and $s(a;k,n)$ are defined as before and

$$d(n,k) = \binom{2n}{k}\frac{2n-2k+1}{2n-k+1}\,.$$

We would like to finish the paper mentioning that for the right eigenvectors of the matrix $G_n(a,t,m)$, it is enough to take $t-m$ instead of the parameter t in each eigenvector of the matrix $G_n(a,t,0)$ given in this section.

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