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On the Solvability of Certain (SSIE) and (SSE), with Operators of the Form $B(r,s,t)$

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Abstract. Given any sequence $z=(z_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_z for the set of all sequences $y=(y_n)_{n\geq 1}$ such that $y/z=(y_n/z_n)_{n\geq 1}\in E$; in particular, $\mathbf{s}_z^0$ denotes the set of all sequences y such that y/z tends to zero. Here, we deal with some extensions of *sequence spaces inclusion equations (SSIE) and sequence spaces equations (SSE) with operator.* They are determined by an inclusion or identity each term of which is a *sum* or a *sum of products of sets of the form* $(\chi_a)_\Lambda$ *and* $(\chi_x)_\Lambda$ where χ is any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$, a is a given sequence in U^+, x is the unknown, and Λ is an infinite matrix. Here, we explicitely calculate the inverse of the triangle $B(r,s,t)$ represented by the operator defined by $(B(r,s,t)\,y)_1=ry_1$, $(B(r,s,t)\,y)_2=ry_2+sy_1$ and $(B(r,s,t)\,y)_n=ry_n+sy_{n-1}+ty_{n-2}$ for all $n\geq 3$. Then we determine the set of all x that satisfy the *(SSIE)* $(\chi_x)_{\widetilde{B(r,s,t)}}\subset\chi_x$, and the (SSE) $(\chi_x)_{\widetilde{B(r,s,t)}}=\chi_x$, where $\chi\in\{\mathbf{s},\mathbf{s}^0\}$ and $\widetilde{B(r,s,t)}$ is the infinite tridiagonal matrix obtained from $B(r,s,t)$ by deleting its first row. For $\chi=\mathbf{s}^0$ the solvability of the (SSE) $(\chi_x)_{\widetilde{B(r,s,t)}}=\chi_x$ consists in determining the set of all $x\in U^+$ for which

$$\frac{ry_{n+1}+sy_n+ty_{n-1}}{x_n}\to 0 \iff \frac{y_n}{x_n}\to 0\ (n\to\infty)\ \text{for all } y.$$

1. Introduction.

As usual we denote by ω the set of all complex sequences $y=(y_n)_{n\geq 1}$ and by c_0, c and ℓ_∞ the subsets of all null, convergent and bounded sequences, respectively. Also let U^+ denote the set of all sequences $u=(u_n)_{n\geq 1}$ with $u_n>0$ for all n. Given a sequence $a\in\omega$ and a subset E of ω, Wilansky [23] introduced the notation $a^{-1}*E=\left\{y\in\omega : ay=(a_ny_n)_{n\geq 1}\in E\right\}$. In [7] we introduced the notations $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$ for the sets $\left((1/a_n)_{n\geq 1}\right)^{-1}*E$ for any sequence $a\in U^+$ and $E\in\{\ell_\infty,c_0,c\}$. In [8] we considered the sum $\chi_a+\chi'_b$ and the product $\chi_a*\chi'_b$, where χ and χ' are any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$. Then we gave characterizations of matrix transformations in the sets $\mathbf{s}_a+\left(\mathbf{s}_b^0\right)_{\Delta^q}$ and $\mathbf{s}_a+\left(\mathbf{s}_b^{(c)}\right)_{\Delta^q}$, where Δ is the *operator of the first difference*. In [15] we gave characterizations of the classes of matrix transformations from $(\mathbf{s}_a)_{\Delta^q}$ to χ_b, where χ is any

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of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$. In [18] we gave applications of the measure of noncompactness to operators on the spaces $\mathbf{s}_\alpha$, $\mathbf{s}_\alpha^0$, $\mathbf{s}_\alpha^{(c)}$ and ℓ_α^p to determine compact operators between some of these spaces. In [3, 12] we introduced the notion of *sequence spaces inclusion equations (SSIE) and sequence spaces equations (SSE), with operators* which are determined by an inclusion or identity each term of which is a *sum* or a *sum of products of sets of the form* $(\chi_a)_T$ *and* $\left(\chi_{f(x)}\right)_T$ where χ is any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$, a is a given sequence in U^+, x is the unknown, f maps U^+ to itself and T is a triangle. In [13] we dealt with the class of (SSIE) of the form $F \subset E_a + F'_x$ where $F \in \{c_0, \ell^p, w_0, w_\infty\}$ and $E, F' \in \{c_0, c, \ell_\infty, \ell^p, w_0, w_\infty\}$, $(p \geq 1)$. In [14] writing D_r for the diagonal matrix with $(D_r)_{nn} = r^n$, we dealt with the solvability of the (SSIE) using the operator of the first difference Δ, defined by $c \subset D_r * E_\Delta + c_x$ with $E = c_0$, or s_1. Then we dealt with the (SSIE) $c \subset D_r * E_{C_1} + s_x^{(c)}$ with $E = c_0$, c or s_1, and $s_1 \subset D_r * E_{C_1} + s_x$ where $E = c$, or s_1 and C_1 is the Cesàro operator defined by $(C_1)_n y = \left(\sum_{k=1}^n y_k\right)/n$. In [1] Altay and Başar defined the *generalized operator of the first difference* defined by $B(r,s)_n y = ry_n + sy_{n-1}$ for all $n \geq 2$ and $B(r,s)_1 y_1 = ry_1$. Then these authors dealt with the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c. Then, in [11, 17] we dealt with the (SSIE) $(\chi_x)_{B(r,s)} \subset (\chi_x)_{B(r',s')}$ and the (SSE) $(\chi_x)_{B(r,s)} = (\chi_x)_{B(r',s')}$, where $\chi = \mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$. Then we stated some results on the spectrum of $B(r,s)$ considered as an operator from χ_x to itself, where $\chi = \mathbf{s}$, or $\mathbf{s}^0$; and on the solvability of the (SSE) $\chi_a + \left(\mathbf{s}_x^{(c)}\right)_{B(r,s)} = \mathbf{s}_x^{(c)}$ where $\chi = \mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$ and x was the unknown. Note that for $\chi = \mathbf{s}^0$, the previous (SSE) consists in determining the set of all $x \in U^+$ such that $y_n/x_n \to l$ $(n \to \infty)$ if and only if there are u, v such that $y = u + v$ and $u_n/a_n \to 0$ and $(rv_n + sv_{n-1})/x_n \to l'$ $(n \to \infty)$ for all $y \in \omega$ and for some scalars l and l'. Then, in 2007 Furkan, Bilgic and Altay [4] dealt with the spectrum of the operator represented by the triangle

$$B(r,s,t) = \begin{pmatrix} r & & & \\ s & r & & 0 \\ t & s & r & \\ & \cdot & \cdot & \cdot \\ 0 & & \cdot & \cdot & \cdot \end{pmatrix}$$

over c_0 and c. Then, Bilgic and Furkan [2] dealt with the fine spectrum of $B(r,s,t)$ over the sequence spaces l_1 and bv. Finally, in 2010 Furkan, Bilgic and Başar [5] studied the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces l_p and bv_p.

In this paper, we extend some results stated in the papers [11, 17] and we consider an extension of the notion of (SSIE) and (SSE) where we use the operator $\Lambda = B\widetilde{(r,s,t)}$ obtained from $B(r,s,t)$ by deleting the first row of $B(r,s,t)$ which is not a triangle but an infinite tridiagonal matrix and we determine the sets of all positive sequences $x = (x_n)_{n\geq 1}$ for which $(\chi_x)_\Lambda \subset \chi_x$ and $(\chi_x)_\Lambda = \chi_x$, where χ is any of the symbols $\mathbf{s}$, or $\mathbf{s}^0$. In this way we are led to determine the set of all positive sequences x for which $\lim_{n\to\infty}(ry_{n+1} + sy_n + ty_{n-1})/x_n = 0$ if and only if $\lim_{n\to\infty} y_n/x_n = 0$ for all y. Notice that, if $r = 0$ then $\Lambda = B(0,s,t)$ is a triangle and we are refered to the papers [11, 17]. So the inclusion $(\chi_x)_\Lambda \subset \chi_x$ is associated with the statement $\lim_{n\to\infty}(sy_n + ty_{n-1})/x_n = 0$ implies $\lim_{n\to\infty} y_n/x_n = 0$ for all y.

This paper is organized as follows. In Section 2, we recall some results on AK and BK spaces and on the set $S_{a,b}$. In Section 3, we consider the operator $C(\xi)$ and its inverse $\Delta(\xi)$, and recall the definitions and properties of the sets $\widehat{\Gamma}$, $\widehat{C}$, Γ and $\widehat{C_1}$. In Section 4, we recall some results on the *triangular Toeplitz matrices of* S_r and we consider the isomorphism φ from the *algebra of the power series* into the algebra $\overline{\mathcal{M}}$ of corresponding matrices. Then using φ we explicitely calculate the inverse of the infinite triple band matrix $B(r,s,t)$. In Section 5, we consider the infinite tridiagonal matrix $B\widetilde{(r,s,t)}$ obtained from $B(r,s,t)$ by deleting its first row, and determine the set of all x such that $(\chi_x)_{B\widetilde{(r,s,t)}} \subset \chi_x$ where χ is any of the symbols $\mathbf{s}$, or $\mathbf{s}^0$. Finally in Section 6 we deal with the (SSE) $(\chi_x)_{B\widetilde{(r,s,t)}} = \chi_x$ where χ is any of the symbols $\mathbf{s}$, or $\mathbf{s}^0$.

2. Notations and preliminary results

Let $A = (a_{nk})_{n,k\geq 1}$ be an infinite matrix and $y = (y_k)_{k\geq 1}$ be a sequence. Then we write

$$A_n y = \sum_{k=1}^{\infty} a_{nk} y_k \text{ for any integer } n \geq 1 \tag{1}$$

and $Ay = (A_n y)_{n\geq 1}$ provided all the series in (1) converge. Let E and F be any subsets of ω. Then we write (E, F), (see for instance [6]), for the class of all infinite matrices A for which the series in (1) converge for all $y \in E$ and all n, and $Ay \in F$ for all $y \in E$. So if $A \in (E, F)$ then we are led to the study of the operator $\Lambda = \Lambda_A : E \to F$ defined by $Ay = \Lambda y$ and we identify the operator Λ to the matrix A. A Banach space E of complex sequences is said to be a *BK space* if each projection $P_n : E \to \mathbb{C}$ defined by $P_n(y) = y_n$ for all $y = (y_n)_{n\geq 1} \in E$ is continuous. A BK space E is said to have *AK* if every sequence $y = (y_k)_{k\geq 1} \in E$ has a unique representation $y = \sum_{k=1}^{\infty} y_k e^{(k)}$ where $e^{(k)}$ is the sequence with 1 in the k-th position and 0 otherwise. To simplify the notations, we use the diagonal matrix D_a defined by $[D_a]_{nn} = a_n$ for all n, write $D_a * E = (1/a)^{-1} * E = \{(y_n)_{n\geq 1} \in \omega : (y_n/a_n)_{n\geq 1} \in E\}$ for any $a \in U^+$ and any $E \subset \omega$, and define $\mathbf{s}_a = D_a * \ell_\infty$, $\mathbf{s}_a^0 = D_a * c_0$ and $\mathbf{s}_a^{(c)} = D_a * c$, (see, for instance, [7, 9, 10, 18]). Each of the spaces $D_\alpha * \chi$, where $\chi \in \{\ell_\infty, c_0, c\}$, is a BK space normed by $\|\xi\|_{\mathbf{s}_a} = \sup_{n\geq 1}(|\xi_n|/a_n)$ and $\mathbf{s}_a^0$ has AK. Now, let $a = (a_n)_{n\geq 1}, b = (b_n)_{n\geq 1} \in U^+$. By $S_{a,b}$ we denote the set of all infinite matrices $\Lambda = (\lambda_{nk})_{n,k\geq 1}$ such that $\|\Lambda\|_{S_{a,b}} = \sup_{n\geq 1}\left(b_n^{-1}\sum_{k=1}^{\infty}|\lambda_{nk}|a_k\right) < \infty$. It is well known that $\Lambda \in (\mathbf{s}_a, \mathbf{s}_b)$ if and only if $\Lambda \in S_{a,b}$. So, we can write $(\mathbf{s}_a, \mathbf{s}_b) = S_{a,b}$. When $\mathbf{s}_a = \mathbf{s}_b$ we obtain the *Banach algebra with identity* $S_{a,b} = S_a$ (see [7]), normed by $\|\Lambda\|_{S_a} = \|\Lambda\|_{S_{a,a}}$. We also have $\Lambda \in (\mathbf{s}_a, \mathbf{s}_a)$ if and only if $\Lambda \in S_a$. If $a = (r^n)_{n\geq 1}$, the sets S_a, $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$ are denoted by S_r, $\mathbf{s}_r$, $\mathbf{s}_r^0$ and $\mathbf{s}_r^{(c)}$, respectively (see [8]). When $r = 1$, we obtain $\mathbf{s}_1 = \ell_\infty$, $\mathbf{s}_1^0 = c_0$ and $\mathbf{s}_1^{(c)} = c$, and witing $e = (1, 1, ...)$ we have $S_1 = S_e$. It is well known that $(\mathbf{s}_1, \mathbf{s}_1) = (c_0, \mathbf{s}_1) = (c, \mathbf{s}_1) = S_1$ (see, for instance, [23]). We also have $\Lambda \in (c_0, c_0)$ if and only if $\Lambda \in S_1$ and $\lim_{n\to\infty}\lambda_{nk} = 0$ for $k = 1, 2, ...$ In the sequel we use the next property. We have $\Lambda \in (\chi_a, \chi_b')$ if and only if $D_{1/b}\Lambda D_a \in (\chi_e, \chi_e')$ where χ, χ' are any of the symbols $\mathbf{s}^0$, $\mathbf{s}^{(c)}$, or $\mathbf{s}$. For any subset E of ω, we put $\Lambda E = \{\eta \in \omega : \eta = \Lambda y \text{ for some } y \in E\}$. If F is a subset of ω, we write $F(\Lambda) = F_\Lambda = \{y \in \omega : \Lambda y \in F\}$ *for the matrix domain of* Λ *in* F.

3. The operators $C(\xi)$, $\Delta(\xi)$ and the sets $\widehat{\Gamma}$, $\widehat{C}$, Γ and $\widehat{C_1}$

An infinite matrix $T = (t_{nk})_{n,k\geq 1}$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all n. Now let U be the set of all sequences $(u_n)_{n\geq 1} \in \omega$ with $u_n \neq 0$ for all n. If $\xi = (\xi_n)_{n\geq 1} \in U$, we define by $C(\xi)$ the triangle defined by $[C(\xi)]_{nk} = 1/\xi_n$ for $k \leq n$, (see, for instance, [9, 10], and [19, 21]). It is easy to see that the triangle $\Delta(\xi)$ whose the nonzero entries are defined by $[\Delta(\xi)]_{nn} = \xi_n$ and $[\Delta(\xi)]_{n,n-1} = \xi_{n-1}$ is the inverse of $C(\xi)$, that is, $C(\xi)(\Delta(\xi)y) = \Delta(\xi)(C(\xi)y) = y$ for all $y \in \omega$. If $\xi = e$ we obtain $\Delta(e) = \Delta$, where Δ is the well–known operator of the first difference defined by $\Delta_n y = y_n - y_{n-1}$ for all $y \in \omega$ and all $n \geq 1$, with the convention $y_0 = 0$. It is usual to write $\Sigma = C(e)$. We note that Δ and Σ are inverse to one another, and $\Delta, \Sigma \in S_R$ for any $R > 1$.

To simplify notation, for $\xi \in U^+$, we write $c_n(\xi) = \xi_n^{-1}\sum_{k=1}^{n}\xi_k$ for all n. We also consider the sets $\widehat{C}$ and $\widehat{C_1}$ of all positive sequences ξ such that $(c_n(\xi))_n \in c$, $\sup_n c_n(\xi) < \infty$, respectively. Then we write $\xi^\bullet = (\xi_n^\bullet)_{n\geq 1}$ where $\xi_n^\bullet = \xi_{n-1}/\xi_n$ with the convention $\xi_1^\bullet = 1/\xi_1$, and we define by $\widehat{\Gamma}$ and Γ the sets of all positive sequences such that $\lim_{n\to\infty}\xi_n^\bullet < 1$ and $\limsup_{n\to\infty}\xi_n^\bullet < 1$, respectively. Finally, by G_1 we define the set of all positive sequences such that $\xi_n \geq C\gamma^n$ for all n, and for some $C > 0$ and $\gamma > 1$. Note that if ξ and $\eta \in \widehat{C_1}$, then we have $\xi + \eta$ and $\xi\eta \in \widehat{C_1}$. It can easily be seen that $(R^n)_n \in \widehat{C_1}$ if and only if $R > 1$, and there is no real number α for which the sequence $(n^\alpha)_{n\geq 1}$ belongs to $\widehat{C_1}$.

By ([7], Proposition 2.1, p. 1786) and ([16], Proposition 2.2 p. 88) we obtain the following lemma.

Lemma 3.1. *We have* $\widehat{C} = \widehat{\Gamma} \subset \Gamma \subset \widehat{C_1} \subset G_1$.

Concerning the identity $(\chi_a)_\Delta = \chi_a$ for $\chi = \mathbf{s}$ or $\mathbf{s}^0$ it was shown in [8], Proposition 9, pp. 300-301] the following results.

Lemma 3.2. *Let $a \in U^+$ and let χ be any of the symbols* $\mathbf{s}$ *or* $\mathbf{s}^0$. *Then the following statements are equivalent (i) $a \in \widehat{C_1}$. (ii) $(\chi_a)_\Delta = \chi_a$. (iii) $(\chi_a)_\Delta \subset \chi_a$. (iv) The operator $\Delta \in (\chi_a, \chi_a)$ is surjective.*

Lemma 3.3. *([8], Proposition 9, p. 300) For each $a \in \omega$ we have $a \in \widehat{\Gamma}$ if and only if $\left(\mathbf{s}_a^{(c)}\right)_\Delta = \mathbf{s}_a^{(c)}$.*

In the following we consider the sets $\widehat{C_2}$ and $\widetilde{C_2}$ of all positive sequences x that satisfy $(1/x_n)\sum_{k=1}^{n}(n-k+1)x_k = O(1)$ and $(1/x_n)\sum_{k=1}^{n}(n-k+1)x_{k-1} = O(1)$ $(n \to \infty)$, with the convention $x_0 = 1$, respectively. We obtain the following result.

Lemma 3.4. *We have $\widehat{C_2} = \widetilde{C_2} = \widehat{C_1}$.*

Proof. First we show $\widehat{C_1} = \widehat{C_2}$. Let $x \in \widehat{C_1}$. By Lemma 3.2 with $\chi = \mathbf{s}$ we have $x \in \widehat{C_1}$ implies $(\mathbf{s}_x)_\Delta = \mathbf{s}_x$ and trivially we obtain $(\mathbf{s}_x)_{\Delta^2} = ((\mathbf{s}_x)_\Delta)_\Delta = (\mathbf{s}_x)_\Delta = \mathbf{s}_x$. Then we have $\Delta^{-2} = \Sigma^2 \in (\mathbf{s}_x, \mathbf{s}_x)$ and since $D_{1/x}\Sigma^2 D_x$ is the triangle defined by $\left[D_{1/x}\Sigma^2 D_x\right]_{nk} = (n-k+1)x_k/x_n$, for $k \leq n$, we deduce $x \in \widehat{C_2}$. So we have shown $\widehat{C_1} \subset \widehat{C_2}$. Now since $n-k+1 \geq 1$ for $k = 1, 2, \ldots, n$ and for all n we easily see that $x_n^{-1}\sum_{k=1}^{n}(n-k+1)x_k \geq x_n^{-1}\sum_{k=1}^{n}x_k$ for all n, and trivially we obtain $\widehat{C_2} \subset \widehat{C_1}$ and since $\widehat{C_1} \subset \widehat{C_2}$ we conclude $\widehat{C_1} = \widehat{C_2}$. Now, we show $\widehat{C_2} = \widetilde{C_2}$. For this, notice that for every n we have

$$\sum_{k=1}^{n-1}(n-k+1)x_k = \sum_{k=1}^{n-1}(n-k)x_k + \sum_{k=1}^{n-1}x_k = \sum_{k=2}^{n}(n-k+1)x_{k-1} + \sum_{k=1}^{n-1}x_k. \tag{2}$$

Now, let $x \in \widetilde{C_2}$. Then we have $x \in \widehat{C_1}$ since

$$\frac{1}{x_n}\sum_{k=1}^{n}x_k - 1 = \frac{1}{x_n}\sum_{k=2}^{n}x_{k-1} \leq \frac{1}{x_n}\sum_{k=2}^{n}(n-k+1)x_{k-1} \leq K$$

for all n for some $K > 0$. Then we have $\widetilde{C_2} \subset \widehat{C_1} = \widehat{C_2}$. Now we show $\widehat{C_2} \subset \widetilde{C_2}$. For this, we let $x \in \widehat{C_2}$. Then, by (2) we have

$$\sigma_n = \frac{1}{x_n}\sum_{k=2}^{n}(n-k+1)x_{k-1} = \frac{1}{x_n}\sum_{k=1}^{n-1}(n-k+1)x_k - \frac{1}{x_n}\sum_{k=1}^{n-1}x_k$$

and $\sigma \in \ell_\infty$. We conclude $x \in \widetilde{C_2}$ and $\widehat{C_2} \subset \widetilde{C_2}$ and we have shown $\widehat{C_2} = \widetilde{C_2}$. □

4. Calculation of the inverse of the triple band matrix $B(r, s, t)$ using the isomorphism φ

4.1. Triangular Toeplitz matrices of S_r and power series.

A *Toeplitz matrix* is an infinite matrix whose the entries are of the form $(\mathcal{M})_{nk} = \mathfrak{a}_{k-n}$ with $n, k \geq 1$. Here we focus on *triangular Toeplitz matrices* and consider $\mathcal{M}$ as an operator mapping s_r into itself, with $r > 0$. Let

$$f(u) = \sum_{k=0}^{\infty}\mathfrak{a}_k u^k \tag{3}$$

be a power series defined in the open disk $|u| < R$. We can associate with f the *upper infinite triangular Toeplitz matrix* $\mathcal{M} = \varphi(f) \in \cap_{0<r<R}S_r$ defined by $\varphi(f) = \begin{pmatrix} \mathfrak{a}_0 & \mathfrak{a}_1 & \mathfrak{a}_2 & . \\ & \mathfrak{a}_0 & \mathfrak{a}_1 & . \\ 0 & & \mathfrak{a}_0 & . \\ & & & . \end{pmatrix}$. For practical reasons, we write

$\varphi[f(u)]$ instead of $\varphi(f)$. So we can associate with 1 the matrix I and we can associate with u^k for k integer, the matrix whose the only nonzero entries are equal to 1 and are on the diagonal of equation $m = n + k$. From [22] we obtain the next result.

Lemma 4.1. *The map $\varphi : f \mapsto \mathcal{M}$ is an isomorphism from the algebra of the power series defined in $|u| < R$ into the algebra $\overline{\mathcal{M}}$ of the corresponding matrices.*

4.2. Application to the calculation of the inverse of the infinite tridiagonal matrix $B(r,s,t)$.

In this section we explicitly calculate the inverse of the infinite triangular Toeplitz matrix $B(r,s,t)$ using the function φ. The triangle $B(r,s,t)$ is represented by the operator defined by $(B(r,s,t)\,y)_1 = ry_1$, $(B(r,s,t)\,y)_2 = ry_2 + sy_1$ and $(B(r,s,t)\,y)_n = ry_n + sy_{n-1} + ty_{n-2}$ for all $n \geq 3$, where r, s, t are complex numbers. Throughout this paper we assume, except in special cases, that r, s and t are nonzero real numbers. Since $[B(r,s,t)]^T = \varphi\left(r + su + tu^2\right)$, we associate with the matrix $B(r,s,t)$ the equation

$$b(u) = r + su + tu^2 = 0. \tag{4}$$

We denote by u_1 and u_2 the roots of (4). Since r, $t \neq 0$ all the roots of (4) are distinct from zero. We can state the next result where we let $\Delta = s^2 - 4tr$.

Lemma 4.2. *If $\Delta \neq 0$, then $u_1 = \left(-s - \sqrt{\Delta}\right)/2t$ and $u_2 = \left(-s + \sqrt{\Delta}\right)/2t$ are the real or complex roots of (4). Then the inverse of $B(r,s,t)$ is a triangle whose the nonzero entries are defined for $k \leq n$, in the following way.*

(i) If $\Delta \neq 0$, then we have

$$\left([B(r,s,t)]^{-1}\right)_{nk} = \begin{cases} -\dfrac{u_2^{k-n-1} - u_1^{k-n-1}}{\sqrt{\Delta}} & \text{if } \Delta > 0, \\ \dfrac{i\left(u_2^{k-n-1} - u_1^{k-n-1}\right)}{\sqrt{-\Delta}} & \text{if } \Delta < 0. \end{cases}$$

(ii) If $\Delta = 0$, then $u_1 = -s/2t$ is the double root of (4) and the non-zero entries of the inverse of $B(r,s,t)$ are defined by

$$\left([B(r,s,t)]^{-1}\right)_{nk} = \frac{1}{r}(n-k+1)\,u_1^{k-n}.$$

Proof. (i) We have $\Delta = s^2 - 4tr \neq 0$ and $B(r,s,t)^T = \varphi\left(tu^2 + su + r\right) = \varphi[t(u-u_1)(u-u_2)]$, where $u_1 = -\alpha_1 - s/2t$, $u_2 = \alpha_1 - s/2$ are the roots of $b(u) = 0$. Then we have $\alpha_1 = \sqrt{\Delta}/2t$ if $\Delta > 0$, and $\alpha_1 = i\sqrt{-\Delta}/2t$ if $\Delta < 0$. By Lemma 4.1 we have $\left[B(r,s,t)^T\right]^{-1} = \varphi\left(\left(tu^2 + su + r\right)^{-1}\right) = \varphi\left[(t(u-u_1)(u-u_2))^{-1}\right]$, but

$$R(u) = \frac{1}{t(u-u_1)(u-u_2)} = \frac{1}{t\frac{r}{t}}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_1^{-j}u_2^{j-k}\right)u^k \text{ for } |u| < \min(|u_1|, |u_2|).$$

Since trivially we have $[B(r,s,t)]^{-1} = \left(\left[B(r,s,t)^T\right]^{-1}\right)^T$, we obtain

$$\begin{aligned} \left([B(r,s,t)]^{-1}\right)_{nk} &= \frac{1}{r}u_2^{-(n-k)}\sum_{j=0}^{n-k}\left(\frac{u_2}{u_1}\right)^j \\ &= \frac{1}{r}u_2^{k-n}\left[1 - \left(\frac{u_2}{u_1}\right)^{n-k+1}\right]\frac{1}{1 - \dfrac{u_2}{u_1}} \\ &= \frac{1}{r}\frac{u_1u_2}{u_1 - u_2}\left(u_2^{k-n-1} - u_1^{k-n-1}\right), \end{aligned}$$

and

$$\left([B(r,s,t)]^{-1}\right)_{nk} = -\frac{1}{2t\alpha_1}\left(u_2^{k-n-1} - u_1^{k-n-1}\right) \text{ for } k \leq n. \tag{5}$$

Since we have $-2t\alpha_1 = -\sqrt{\Delta}$ if $\Delta > 0$ and $-2t\alpha_1 = -i\sqrt{-\Delta}$ if $\Delta < 0$ we conclude (i) holds.

(ii) Here $\Delta = 0$ and $u_1 = u_2 = -s/2t$. We have $\left[B(r,s,t)^T\right]^{-1} = \varphi\left[\left(tu^2 + su + r\right)^{-1}\right] = \varphi\left(1/t(u-u_1)^2\right)$, and

$$R(u) = \frac{1}{t(u-u_1)^2} = \frac{4t}{s^2}\sum_{k=0}^{\infty}\frac{k+1}{u_1^k}u^k \text{ for } |u| < |u_1|.$$

Since $\Delta = 0$ we have $4t/s^2 = 1/r$, and $\left([B(r,s,t)]^{-1}\right)_{nk} = r^{-1}(n-k+1)u_1^{k-n}$ for $k \leq n$. This completes the proof. □

In all that follows, when $\Delta < 0$, we write $u_1 = \rho e^{i\theta}$ and $u_2 = \overline{u_1} = \rho e^{-i\theta}$ with $\rho > 0$ and $\theta \notin \pi\mathbb{Z}$ for the roots of the equation in (4). We then obtain another expression of the inverse of $B(r,s,t)$, which is given in the next result.

Corollary 4.3. *Assume $\Delta < 0$, and let $u_1 = \rho e^{i\theta}$ be a root of (4). Then the inverse of $B(r,s,t)$ is a triangle whose the non-zero entries are given by*

$$\left([B(r,s,t)]^{-1}\right)_{nk} = \frac{1}{r}\frac{\sin(n-k+1)\theta}{\rho^{n-k}\sin\theta}.$$

Proof. For $\Delta < 0$ we have $u_1 = \rho e^{i\theta}$ with $\rho > 0$ and $\theta \neq m\pi$ for all integer m. By (5) we successively obtain $u_2 = \overline{u_1}$, $u_1u_2 = \rho^2$, $u_1 - u_2 = -2\alpha_1 = 2i\rho\sin\theta$, $u_2^{k-n-1} - u_1^{k-n-1} = -2i\rho^{k-n-1}\sin[(k-n-1)\theta]$ and

$$\begin{aligned}\left([B(r,s,t)]^{-1}\right)_{nk} &= -\frac{1}{r}\frac{\rho^2}{2i\rho\sin\theta}2i\rho^{k-n-1}\sin[(k-n-1)\theta] \\ &= \frac{1}{r}\rho^{k-n}\frac{\sin[(n-k+1)\theta]}{\sin\theta} \text{ for } k \leq n. \quad \square\end{aligned}$$

Remark 4.4. *In the case when $\Delta \neq 0$, by elementary calculations the inverse of $B(r,s,t)$ is the triangle whose the nonzero entries are given by,*

$$\left([B(r,s,t)]^{-1}\right)_{nk} = \begin{cases} -\dfrac{(2t)^{n-k+1}}{\sqrt{\Delta}}\left[\left(\dfrac{-1}{s+\sqrt{\Delta}}\right)^{n-k+1} - \left(\dfrac{1}{-s+\sqrt{\Delta}}\right)^{n-k+1}\right] & \text{if } \Delta > 0, \\ \dfrac{i(2t)^{n-k+1}}{\sqrt{-\Delta}}\left[\left(\dfrac{-1}{s+i\sqrt{-\Delta}}\right)^{n-k+1} - \left(\dfrac{1}{-s+i\sqrt{-\Delta}}\right)^{n-k+1}\right] & \text{if } \Delta < 0. \end{cases}$$

5. Application to the (SSIE) $(\chi_x)_{\widetilde{B(r,s,t)}} \subset \chi_x$ where $\chi = \mathbf{s}$, or $\mathbf{s^0}$

In this section, we consider the tridiagonal matrix $\widetilde{B(r,s,t)}$ obtained from $B(r,s,t)$ by deleting the first row and we determine the sets of all $x \in U^+$ such that $(\mathbf{s}_x)_{\widetilde{B(r,s,t)}} \subset \mathbf{s}_x$ and $\left(\mathbf{s}_x^0\right)_{\widetilde{B(r,s,t)}} \subset \mathbf{s}_x^0$, respectively. The previous problems consists in determining the set of all $x \in U^+$ for which $(ry_{n+1} + sy_n + ty_{n-1})/x_n = \kappa(1)$ implies $y_n/x_n = \kappa(1)$ $(n \to \infty)$ for all y where κ is either of the symbols o, or O.

5.1. General case.

Here we consider the infinite tridiagonal matrix $B\widetilde{(r,s,}t)$ obtained from $B(r,s,t)$ by deleting the first row, that is, $B\widetilde{(r,s,}t) = \begin{pmatrix} s & r & & & 0 \\ t & s & r & & \\ & t & s & r & \\ & & . & . & . \\ 0 & & & . & . \end{pmatrix}$. The operator associated with the matrix $B\widetilde{(r,s,}t)$ is defined by $\left[B\widetilde{(r,s,}t)y\right]_1 = sy_1 + ry_2$, and $\left[B\widetilde{(r,s,}t)y\right]_n = ty_{n-1} + sy_n + ry_{n+1}$ for all $n \geq 2$. Consider the sets $\widehat{\mathcal{S}} = \left\{x \in U^+ : (\mathbf{s}_x)_{B\widetilde{(r,s,t)}} \subset \mathbf{s}_x\right\}$ and $\widehat{\mathcal{S}^0} = \left\{x \in U^+ : \left(\mathbf{s}_x^0\right)_{B\widetilde{(r,s,t)}} \subset \mathbf{s}_x^0\right\}$. We then have $x \in \widehat{\mathcal{S}}$ if and only if the condition $\left|ry_{n+1} + sy_n + ty_{n-1}\right|/x_n \leq K_1$ implies $\left|y_n\right|/x_n \leq K_2$ for all y, for all n and for some K_1 and $K_2 > 0$. Similarly we have $x \in \widehat{\mathcal{S}^0}$ if and only if the condition $(ry_{n+1} + sy_n + ty_{n-1})/x_n \to 0$ implies $y_n/x_n \to 0$ $(n \to \infty)$ for all y. Now, we state the next result, where we associate with the sequence $x \in U^+$ the sequence x^- defined by $[x^-]_n = x_{n-1}$ for all $n \geq 1$ with the convention $[x^-]_1 = 1$. So we write $x_0 = 1$ in all that follows.

Lemma 5.1. *(i) We have $x \in \widehat{\mathcal{S}}$ if and only if $(\mathbf{s}_{x^-})_{B(r,s,t)} \subset \mathbf{s}_x$. (ii) We have $x \in \widehat{\mathcal{S}^0}$ if and only if $\left(\mathbf{s}_{x^-}^0\right)_{B(r,s,t)} \subset \mathbf{s}_x^0$.*

Proof. We have $\left(B\widetilde{(r,s,}t)y\right)_{n-1} = (B(r,s,t)y)_n$ for all $n \geq 2$ and for all y. Then we have $x_n^{-1}\left(B\widetilde{(r,s,}t)y\right)_n = O(1)$ if and only if $x_{n-1}^{-1}(B(r,s,t)y)_n = O(1)$ $(n \to \infty)$, and $x \in \widehat{\mathcal{S}}$ if and only if $x_{n-1}^{-1}(B(r,s,t)y)_n = O(1)$ implies $y_n/x_n = O(1)$ $(n \to \infty)$ for all $y \in \omega$, and we conclude for (i). (ii) can be shown in a similar way. □

From Lemma 4.2 we obtain the following results where we use the convention $x_0 = 1$.

Proposition 5.2. *(i) Assume $\Delta \neq 0$ and let u_1 and u_2 denote the roots of (4).*

a) $x \in \widehat{\mathcal{S}}$ if and only if

$$\sup_n \left(\frac{1}{x_n}\sum_{k=1}^{n}\left|u_2^{k-n-1} - u_1^{k-n-1}\right| x_{k-1}\right) < \infty. \tag{6}$$

b) $x \in \widehat{\mathcal{S}^0}$ if and only if (6) holds and

$$\lim_{n\to\infty}\frac{1}{x_n}\left(u_2^{k-n-1} - u_1^{k-n-1}\right)x_{k-1} = 0 \ \ (n \to \infty) \text{ for } k = 1, 2, ... \tag{7}$$

(ii) Assume $\Delta = 0$, and let u_1 be the double root of (4). Then $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$ and $x \in \widehat{\mathcal{S}}$ if and only if $(|u_1|^n x_n)_{n\geq 1} \in \widehat{C_1}$, that is,

$$\frac{1}{|u_1|^n x_n}\sum_{k=1}^{n}|u_1|^k x_k = O(1) \ \ (n \to \infty).$$

Proof. (i) a) From Lemma 5.1 we have $x \in \widehat{\mathcal{S}}$ if and only if $(\mathbf{s}_{x^-})_{B(r,s,t)} \subset \mathbf{s}_x$. This means $[B(r,s,t)]^{-1} \in (\mathbf{s}_{x^-}, \mathbf{s}_x)$, and $D_{1/x}[B(r,s,t)]^{-1}D_{x^-} \in S_1$. By Part (i) of Lemma 4.2 we obtain (6). b) we have $x \in \widehat{\mathcal{S}^0}$ if and only if $\left(\mathbf{s}_{x^-}^0\right)_{B(r,s,t)} \subset \mathbf{s}_x^0$ and

$$D_{1/x}[B(r,s,t)]^{-1}D_{x^-} \in (c_0, c_0). \tag{8}$$

From the characterization of (c_0, c_0) and Part (i) of Lemma 4.2, we conclude (8) holds if and only if (6) and (7) hold.

(ii) First, by Lemma 4.2 (ii), we easily see that $x \in \widehat{\mathcal{S}}$ if and only if

$$\sup_n \left(\frac{1}{|u_1|^n x_n} \sum_{k=1}^{n} (n-k+1) |u_1|^k x_{k-1} \right) < \infty. \tag{9}$$

This means $(|u_1|^n x_n)_{n \geq 1} \in \widetilde{C_2}$ and since by Lemma 3.4, we have $\widetilde{C_2} = \widehat{C_1}$, then $(|u_1|^n x_n)_{n \geq 1} \in \widehat{C_1}$. This shows $x \in \widehat{\mathcal{S}}$ if and only if $(|u_1|^n x_n)_{n \geq 1} \in \widehat{C_1}$. It remains to show $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$. Trivially we have $x \in \widehat{\mathcal{S}^0}$ if and only if (8) holds, which is equivalent to (9) and

$$\lim_{n \to \infty} \frac{1}{r u_1^n x_n} (n-k+1) u_1^k x_{k-1} = 0 \text{ for } k = 1, 2, ... \tag{10}$$

and since (9) is equivalent to $(|u_1|^n x_n)_{n \geq 1} \in \widehat{C_1}$ we have shown $\widehat{\mathcal{S}^0} \subset \widehat{\mathcal{S}}$. Now we show $\widehat{\mathcal{S}} \subset \widehat{\mathcal{S}^0}$. Take $x \in \widehat{\mathcal{S}}$. As we have just seen we have $(|u_1|^n x_n)_{n \geq 1} \in \widehat{C_1}$. Now, since by Lemma 3.1 we have $\widehat{C_1} \subset G_1$, there are $\gamma > 1$ and $K > 0$ such that $|u_1|^n x_n \geq K\gamma^n$ for all n, and since

$$\frac{n-k+1}{\left|u_1^n\right| x_n} \leq \frac{n}{K\gamma^n} \text{ for } k = 1, 2, ..., n, \text{ and for all } n,$$

we deduce (10) holds and $x \in \widehat{\mathcal{S}^0}$. We conclude $\widehat{\mathcal{S}} \subset \widehat{\mathcal{S}^0}$. So we have shown $\widehat{\mathcal{S}^0} = \widehat{\mathcal{S}}$. □

We immediately deduce the following,

Corollary 5.3. *If (6) holds and $x_n u_j^n \to \infty$ $(n \to \infty)$ for $j = 1, 2$, then $x \in \widehat{\mathcal{S}^0}$.*

Proof. This result is a direct consequence of the fact that the condition $x_n u_j^n \to \infty$ $(n \to \infty)$ for $j = 1, 2$, implies (7). □

Remark 5.4. *From the characterization of (c_0, c) and the proof of (ii) in Proposition 5.2 it can easily be seen that the set $\widehat{\mathcal{S}^{0,c}} = \widehat{\mathcal{S}^0}$ where*

$$\widehat{\mathcal{S}^{0,c}} = \left\{ x \in U^+ : \left(\mathbf{s}_x^0\right)_{\widetilde{B(r,s,t)}} \subset \mathbf{s}_x^{(c)} \right\}.$$

5.2. Relations between the sets $\widehat{\mathcal{S}}$, $\widehat{\mathcal{S}^0}$ and $\widehat{C_\alpha}$ for $\alpha \neq 0$.

In this subsection we establish a relation between the sets $\widehat{\mathcal{S}}$, or $\widehat{\mathcal{S}^0}$ and the set $\widehat{C_\alpha} = D_{(|\alpha|^n)_{n \geq 1}} * \widehat{C_1}$.

5.2.1. Case $\Delta \geq 0$.

For any nonzero real number α, we write

$$\widehat{C_\alpha} = D_{(|\alpha|^n)_{n \geq 1}} * \widehat{C_1} = \left\{ x \in U^+ : (x_n / |\alpha|^n)_{n \geq 1} \in \widehat{C_1} \right\},$$

that is,

$$\widehat{C_\alpha} = \left\{ x \in U^+ : \sup_n \left(\frac{|\alpha|^n}{x_n} \sum_{k=1}^{n} \frac{x_k}{|\alpha|^k} \right) < \infty \right\}.$$

Note that $\widehat{C_\alpha} = \widehat{C_{|\alpha|}}$. It is trivial that if x and $x' \in \widehat{C_\alpha}$ then we have $x + x' \in \widehat{C_\alpha}$. We may state the following result where we confine our study to the case when $\Delta \geq 0$.

Theorem 5.5. *Let $u_1 \neq u_2$ be the roots of (4) for $\Delta > 0$ and let $u_1 = u_2 = -s/2t$ be the double root of (4) for $\Delta = 0$. We have:*

(i)

$$\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \begin{cases} \widehat{C}_{\max(|1/u_1|,|1/u_2|)} & \text{if } \Delta > 0, \\ \widehat{C}_{1/u_1} & \text{if } \Delta = 0. \end{cases} \tag{11}$$

(ii)

$$\overline{\lim_{n\to\infty}} x_n^{\bullet} < \min(|u_1|, |u_2|) \text{ implies } x \in \widehat{\mathcal{S}}, \text{ for } \Delta > 0, \tag{12}$$

and

$$\overline{\lim_{n\to\infty}} x_n^{\bullet} < |u_1| \text{ implies } x \in \widehat{\mathcal{S}}, \text{ for } \Delta = 0.$$

Proof. (i) First we show $\widehat{\mathcal{S}} = \widehat{C}_{\max(|1/u_1|,|1/u_2|)}$ for $\Delta > 0$. By Proposition 5.2 we have $x \in \widehat{\mathcal{S}}$ if and only if (6) holds. As we have seen, since $r \neq 0$, we have u_1 and $u_2 \neq 0$, and since s and t are different from zero, then we have $-s/t = u_1 + u_2 \neq 0$, so $|u_i| > 0$ for $i = 1, 2$, and $|u_1| \neq |u_2|$. Now we consider the case $0 < |u_1| < |u_2|$. For any given n and for $k = 1, 2, ..., n$, we successively obtain $0 < |u_1/u_2| < 1$, $|u_1/u_2|^{n-k+1} \leq |u_1/u_2|$, $1 - |u_1/u_2| \leq \left|1 - (u_1/u_2)^{n-k+1}\right| \leq 2$, and

$$\left|u_1^{k-n-1}\right|\left(1 - \left|\frac{u_1}{u_2}\right|\right) \leq \left|u_2^{k-n-1} - u_1^{k-n-1}\right| = \left|u_1^{k-n-1}\right|\left|1 - \left(\frac{u_1}{u_2}\right)^{n-k+1}\right| \leq 2\left|u_1^{k-n-1}\right|.$$

Then we have

$$\begin{aligned} \left(1 - \left|\frac{u_1}{u_2}\right|\right)\frac{1}{x_n |u_1|^{n+1}}\sum_{k=1}^{n} |u_1|^k x_{k-1} &\leq \frac{1}{x_n}\sum_{k=1}^{n}\left|u_2^{k-n-1} - u_1^{k-n-1}\right| x_{k-1} \\ &\leq 2\frac{1}{|u_1|}\frac{1}{x_n |u_1|^n}\sum_{k=1}^{n} |u_1|^k x_{k-1} \text{ for all } n. \end{aligned}$$

So, the statement in (6) holds if and only if $|u_1|^{-n} x_n^{-1} \sum_{k=1}^{n} |u_1|^k x_{k-1} = O(1)$ $(n \to \infty)$, that is, $x \in \widehat{C}_{1/u_1}$. We conclude $\widehat{\mathcal{S}} = \widehat{C}_{1/u_1}$. By similar arguments as those used above we can show that $0 < |u_2| < |u_1|$ implies $\widehat{\mathcal{S}} = \widehat{C}_{1/u_2}$. So we have shown $\widehat{\mathcal{S}} = \widehat{C}_{\max(|1/u_1|,|1/u_2|)}$.

Now show $\widehat{\mathcal{S}^0} = \widehat{C}_{\max(|1/u_1|,|1/u_2|)}$. From Proposition 5.2 we have $x \in \widehat{\mathcal{S}_0}$ if and only if $x \in \widehat{\mathcal{S}}$ and (7) holds. But as we have just seen we have $\widehat{\mathcal{S}} = \widehat{C}_{\max(|1/u_1|,|1/u_2|)}$, so $x \in \widehat{C}_{\max(|1/u_1|,|1/u_2|)}$ implies $\left(x_n \left|u_j\right|^n\right)_{n\geq 1} \in \widehat{C_1}$ for $j = 1$, 2. Now since by Lemma 3.1, we have $\widehat{C_1} \subset G_1$, there are $C > 0$ and $\gamma > 1$ such that $x_n \left|u_j\right|^n > C\gamma^n$ for all n and for $j = 1, 2$. Then we have

$$\lim_{n\to\infty}\frac{1}{x_n}\frac{1}{u_j^{n-k+1}}x_{k-1} = \lim_{n\to\infty}\frac{1}{x_n}\frac{1}{u_j^n}x_{k-1}u_j^{k-1} = 0 \text{ for all } k \text{ and for } j = 1, 2.$$

So we have shown (7) holds and $\widehat{\mathcal{S}_0} = \widehat{\mathcal{S}} = \widehat{C}_{\max(|1/u_1|,|1/u_2|)}$.

The case $\Delta = 0$ follows from Proposition 5.2. This concludes the proof of (i).

(ii) Case $\Delta > 0$. Since $\Gamma \subset \widehat{C_1}$, the condition $\overline{\lim}_{n\to\infty} x_n^{\bullet} < \min(|u_1|, |u_2|)$ successively implies $\overline{\lim}_{n\to\infty} x_n^{\bullet} < |u_i|$, $\left(x_n u_j^n\right)_{n\geq 1} \in \Gamma$ for $j = 1, 2, ...$, and $x \in \widehat{\mathcal{S}}$. This completes the proof. The case $\Delta = 0$ can be shown similarly. □

When $r = 0$ the previous results was extended in [11] in the following way.

Remark 5.6. *Let $r = 0$ and $s \neq 0$. Then $B\widetilde{(0,s,t)}$ is a triangle and by ([11], Proposition 5.8, p. 47) we have*

$$(\chi_x)_{B\widetilde{(0,s,t)}} \subset \chi_x \Longleftrightarrow x \in \widehat{C}_{1/w},$$

where χ is any of the symbols $\mathbf{s}^0$, or $\mathbf{s}$ and w is the root of the equation $s + tu = 0$.

When $\chi = \mathbf{s}^{(c)}$ we obtain the next result.

Remark 5.7. *We may deal with the inclusion $\left(\mathbf{s}_x^{(c)}\right)_{B\widetilde{(r,s,t)}} \subset \mathbf{s}_x^{(c)}$ where $\Delta = s^2 - 4rt > 0$. We have $\left(\mathbf{s}_x^{(c)}\right)_{B\widetilde{(r,s,t)}} \subset \mathbf{s}_x^{(c)}$ if and only if $\left(\mathbf{s}_{x^-}^{(c)}\right)_{B(r,s,t)} \subset \mathbf{s}_x^{(c)}$ and*

$$D_{1/x}\left[B(r,s,t)\right]^{-1} D_{x^-} \in (c,c). \tag{13}$$

Then from the characterization of (c,c) it can easily be seen that the condition in (13) is equivalent to

$$\lim_{n\to\infty}\left(\frac{1}{x_n}\sum_{k=1}^{n}\left(u_2^{k-n-1} - u_1^{k-n-1}\right)x_{k-1}\right) = l, \tag{14}$$

and

$$\lim_{n\to\infty}\frac{1}{x_n}\left(u_2^{k-n-1} - u_1^{k-n-1}\right)x_{k-1} = l_k \tag{15}$$

for some scalars l and l_k with $k = 2, \ldots$. By similar arguments as those used in the proof of Theorem 5.5 (ii) we conclude that by Lemma 3.1 the condition $\lim_{n\to\infty} x_n^{\bullet} < \min(|u_1|, |u_2|)$ implies $x \in D_{|u_i|} * \widehat{C}$ with $i = 1, 2$, and the conditions in (14) and (15) hold and $\left(\mathbf{s}_x^{(c)}\right)_{B\widetilde{(r,s,t)}} \subset \mathbf{s}_x^{(c)}$. So, we have shown that if the condition $\lim_{n\to\infty} x_n^{\bullet} < \min(|u_1|, |u_2|)$ holds, then we have $\left(\mathbf{s}_x^{(c)}\right)_{B\widetilde{(r,s,t)}} \subset \mathbf{s}_x^{(c)}$ which means that the condition $(ry_{n+1} + sy_n + ty_{n-1})/x_n \to L_1$ implies $y_n/x_n \to L_2$ $(n \to \infty)$ for all y and for some scalars L_1 and L_2.

When $r, s, t \in \mathbb{C}$ we obtain the next remark.

Remark 5.8. *Assume $\Delta \neq 0$ and let r, s and t be nonzero complex numbers. Then, the roots u_1 and u_2 of (4), can be written in the form $u_j = \rho_j e^{i\theta_j}$ for $j = 1, 2$. In the case when $|u_1| \neq |u_2|$, (that is, $\rho_1 \neq \rho_2$), by similar arguments as those used in Theorem 5.5 we have $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C}_{\max(1/\rho_1, 1/\rho_2)}$, and the condition $\overline{\lim}_{n\to\infty} x_n^{\bullet} < \min(\rho_1, \rho_2)$ implies $x \in \widehat{\mathcal{S}}$. We obtain a similar result in the case $\Delta = 0$, that is, $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C}_{1/\rho_1}$ where $1/\rho_1 = |2t/s|$.*

From Theorem 5.5 we also obtain the next result.

Corollary 5.9. *Assume $r/t > 1$. If $s = -(r + t)$, then we have $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C_1}$, moreover if $\overline{\lim}_{n\to\infty} x_n^{\bullet} < 1$ then $x \in \widehat{\mathcal{S}}$.*

Proof. From the hypotheses, the solutions of the equation $tu^2 - (r + t)u + r = 0$ are $u_1 = 1$ and $u_2 = r/t > 1$ and since $\max(|1/u_1|, |1/u_2|) = 1$, we have $\widehat{C}_{\max(|1/u_1|, |1/u_2|)} = \widehat{C_1}$. □

Since trivially we have $x = (n^{\alpha} R^n)_{n\geq 1} \in \Gamma \subset \widehat{C_1}$ for any given real number α and $R > 1$, we immediately deduce the following.

Example 5.10. *For any given reals R and α with $R > 1$, we have $\left|2y_{n+1} - 3y_n + y_{n-1}\right| \leq K_1 n^{\alpha} R^n$ implies $\left|y_n\right| \leq K_2 n^{\alpha} R^n$ for all y, for all n and for some K_1 and $K_2 > 0$.*

By Theorem 5.5 we obtain the next corollary.

Corollary 5.11. *Assume* $\Delta > 0$. *Then* $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$ *and we have*

$$|u_1| = \left|\left(-s - \sqrt{\Delta}\right)/2t\right| > 1 \text{ and } |u_2| = \left|\left(-s + \sqrt{\Delta}\right)/2t\right| > 1 \quad (16)$$

if and only if the next statement holds

$$ry_{n+1} + sy_n + ty_{n-1} \to 0 \text{ if and only if } y_n \to 0 \ (n \to \infty) \text{ for all } y. \quad (17)$$

Proof. The identity $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$ follows from Theorem 5.5. The sufficiency in statement (17) is trivially true. So it is enough to show that (16) holds if and only if $e \in \widehat{C_\nu}$, where $\nu = \max(|1/u_1|, |1/u_2|)$. We have $e \in \widehat{C_\nu}$ if and only if $(\nu^{-n})_{n\geq 1} \in \widehat{C_1}$ and as we have seen $(\nu^{-n})_{n\geq 1} \in \widehat{C_1}$ if and only if $\nu < 1$. We conclude from the equivalence of $\nu < 1$ and the condition in (16). This completes the proof. □

Example 5.12. *Since* $u_1 = 2$ *and* $u_2 = -3$ *are the roots of the equation* $u^2 + u - 6 = 0$, *we have* $6y_{n+1} - y_n - y_{n-1} \to 0$ *if and only if* $y_n \to 0$ $(n \to \infty)$ *for all* y.

5.2.2. Case $\Delta < 0$.

Here we assume $\Delta < 0$, then $u_1 = \rho e^{i\theta}$ and $u_2 = \overline{u}_1$ are the roots of equation (4). Consider the next conditions,

$$\sup_n \left(\frac{1}{\rho^n x_n} \sum_{k=1}^{n} |\sin(n-k+1)\theta| \rho^k x_{k-1} \right) < \infty \quad (18)$$

and

$$\overline{\lim_{n\to\infty}} x_n^\bullet < \rho. \quad (19)$$

Proposition 5.13. *Assume* $\Delta < 0$ *and let* $u_1 = \rho e^{i\theta}$ *be a root of equation (4). We have:*

(i) a) $x \in \widehat{\mathcal{S}}$ *if and only if condition (18) holds.*

b) $x \in \widehat{\mathcal{S}^0}$ *if and only if conditions (18) and (7) hold.*

(ii)

$$\widehat{C_{1/\rho}} \subset \widehat{\mathcal{S}^0} \subset \widehat{\mathcal{S}}. \quad (20)$$

(iii) The condition in (19) implies $x \in \widehat{\mathcal{S}^0}$.

Proof. (i) follows from Lemma 4.3 and from the characterization of (c_0, c_0). (ii) The inclusion $\widehat{\mathcal{S}^0} \subset \widehat{\mathcal{S}}$ is an immediate consequence of Proposition 5.2. Now, we let $x \in \widehat{C_{1/\rho}}$. Then (18) holds since $|\sin(n-k+1)\theta| \leq 1$ for all n, k. Then we successively obtain $(x_n\rho^n)_{n\geq 1} \in \widehat{C_1}$, $x_n\rho^n \to \infty$ $(n \to \infty)$ and $(x_n\rho^n)^{-1} \to 0$ $(n \to \infty)$, for $j = 1, 2$, and (7) holds. We conclude $x \in \widehat{C_{1/\rho}}$ implies (7), that is, $x \in \widehat{\mathcal{S}_0}$. (iii) By Lemma 3.1 we have $\Gamma \subset \widehat{C_1}$ and $D_{(1/\rho^n)_{n\geq 1}} * \Gamma \subset \widehat{C_{1/\rho}}$. So, the result follows from (ii) and from the equivalence of $x \in D_{(1/\rho^n)_{n\geq 1}} * \Gamma$ and (19). This completes the proof. □

As an immediate consequence of Proposition 5.13 we obtain the next corollary.

Corollary 5.14. *Assume* $\Delta < 0$ *and let* $u_1 = \rho e^{i\theta}$ *with* $\rho > 0$ *and* $\theta \neq m\pi$ *for* $m \in \mathbb{Z}$, *be a root of equation (4).*

(i) Let $(x_n\rho^n)_{n\geq 1} \in \widehat{C_1}$. *Then we have* $\left(\rho^2 y_{n+1} - 2\rho\cos\theta y_n + y_{n-1}\right)/x_n \to 0$ *implies* $y_n/x_n \to 0$ $(n \to \infty)$ *for all* y.

(ii) For any $\rho > 1$, *we have* $\rho^2 y_{n+1} - 2\rho\cos\theta y_n + y_{n-1} \to 0$ *implies* $y_n \to 0$ $(n \to \infty)$ *for all* y.

Proof. (i) is a direct consequence of Proposition 5.13. (ii) The conditions $x = e$ and $\rho > 1$ together imply $x \in \widehat{C_{1/\rho}}$ and $e \in \widehat{\mathcal{S}^0}$. □

Now we state the next elementary example.

Example 5.15. *If $\overline{\lim}_{n\to\infty} x_n^\bullet < 1$, then we have $(y_{n+1} + y_n + y_{n-1})/x_n \to 0$ implies $y_n/x_n \to 0$ $(n \to \infty)$ for all y. This result follows from the fact that (19) implies $x \in \widehat{C_{1/\rho}}$ and from Corollary 5.14, where $u_1 = e^{2i\pi/3}$ is a root of the equation $u^2 + u + 1 = 0$. It can easily be seen that for any given $R > 1$ and α real, we have $y_{n+1} + y_n + y_{n-1} = o(R^n/n^\alpha)$ implies $y_n = o(R^n/n^\alpha)$ $(n \to \infty)$ for all y.*

6. Application to the (SSE) $(\chi_x)_{\widetilde{B(r,s,t)}} = \chi_x$ for $\chi \in \{\mathbf{s}, \mathbf{s}^0\}$

Now, we consider the (SSE) $(\chi_x)_{\widetilde{B(r,s,t)}} = \chi_x$, where $\chi = \mathbf{s}$, or $\mathbf{s}^0$. For $\chi = \mathbf{s}^0$ this means that the condition $\lim_{n\to\infty} y_n/x_n = 0$ holds if and only if

$$\lim_{n\to\infty} (ry_{n+1} + sy_n + ty_{n-1})/x_n = 0\,(n \to \infty)$$

for all y. We define by $\mathbf{s}^-$ the set of all $x \in U^+$ that satisfy the condition $x_n \le Cx_{n-1}$ for some $C > 0$ and for all n, that is,

$$1/x^\bullet \in \ell_\infty, \tag{21}$$

and we let $\widehat{\mathbb{S}} = \left\{x \in U^+ : (\mathbf{s}_x)_{\widetilde{B(r,s,t)}} = \mathbf{s}_x\right\}$ and $\widehat{\mathbb{S}^0} = \left\{x \in U^+ : \left(\mathbf{s}_x^0\right)_{\widetilde{B(r,s,t)}} = \mathbf{s}_x^0\right\}$. We immediately obtain the following theorem.

Theorem 6.1. *(i) Assume $\Delta \neq 0$. Then we have:*
a) $x \in \widehat{\mathbb{S}}$ if and only if conditions (6) and (21) hold.
b) $x \in \widehat{\mathbb{S}^0}$ if and only if conditions (6), (21) and (7) hold.
(ii) Assume $\Delta = 0$, and let u_1 be the double root of (4). Then we have $\widehat{\mathbb{S}} = \widehat{\mathbb{S}^0} = \widehat{C_{|1/u_1|}} \cap \mathbf{s}^-$.

Proof. (i) a) We have $x \in \widehat{\mathbb{S}}$ if and only if $\mathbf{s}_x \subset (\mathbf{s}_x)_{\widetilde{B(r,s,t)}}$ and $(\mathbf{s}_x)_{\widetilde{B(r,s,t)}} \subset \mathbf{s}_x$. We have $\mathbf{s}_x \subset (\mathbf{s}_x)_{\widetilde{B(r,s,t)}}$ if and only if $\mathbf{s}_x \subset (\mathbf{s}_{x^-})_{B(r,s,t)}$ and $B(r,s,t) \in (\mathbf{s}_x, \mathbf{s}_{x^-})$. Then, the last condition is equivalent to

$$(|r|\,x_n + |s|\,x_{n-1} + |t|\,x_{n-2})/x_{n-1} = O(1) \quad (n \to \infty),$$

and to $K_1 \le x_n^\bullet \le K_2$ for all n and for some K_1 and $K_2 > 0$. Then, by Proposition 5.2 we have $(\mathbf{s}_x)_{\widetilde{B(r,s,t)}} \subset \mathbf{s}_x$ if and only if (6) holds and the condition in (6) implies $\left|u_2^{-1} - u_1^{-1}\right| x_n^\bullet = O(1)$ $(n \to \infty)$ and $x_n^\bullet = O(1)$ $(n \to \infty)$. We conclude that the equation $(\mathbf{s}_x)_{\widetilde{B(r,s,t)}} = \mathbf{s}_x$ is equivalent to the conditions in (6) and (21). So we have shown (i) a). The statement in (i) b) can be shown in a similar way. Statement (ii) is a consequence of Theorem 5.5 and of the equivalence of the inclusion $\mathbf{s}_x \subset (\mathbf{s}_x)_{\widetilde{B(r,s,t)}}$ and condition (21). □

More precisely from Theorem 5.5, Proposition 5.13 and Theorem 6.1, we obtain the following results.

Corollary 6.2. *(i) Let u_1 and u_2 be the roots of (4) whenever $\Delta > 0$, and let $u_1 = u_2 = -s/2t$ be the double root of (4) for $\Delta = 0$. Then we have*

$$\widehat{\mathbb{S}} = \widehat{\mathbb{S}^0} = \begin{cases} \widehat{C_{\max(|1/u_1|,|1/u_2|)}} \cap \mathbf{s}^- & \text{if } \Delta > 0, \\ \widehat{C_{1/u_1}} \cap \mathbf{s}^- & \text{if } \Delta = 0. \end{cases}$$

(ii) Assume $\Delta < 0$ and denote by $u = \rho e^{i\theta}$ a root of equation (4). Then we have

$$\widehat{C_{1/\rho}} \cap \mathbf{s}^- \subset \widehat{\mathbb{S}} \subset \widehat{\mathbb{S}}.$$

Using Corollary 5.9 we obtain the following corollary.

Corollary 6.3. *Assume $s = -(r + t)$ and $r/t > 1$. Then we have:*
(i) $\widehat{\mathbb{S}^0} = \widehat{C_1} \cap \mathbf{s}^-$.
(ii) For any $x \in U^+$ the condition

$$0 < \lim_{n\to\infty} x_n^\bullet < 1 \tag{22}$$

implies $x \in \widehat{\mathbb{S}^0}$.

Proof. (i) is a direct consequence of Corollary 5.9 and Part (i) of Corollary 6.2. Statement (ii). Let $x \in U^+$ such that condition (22) holds. Then $\lim_{n\to\infty} x_n^\bullet < 1$ implies $x \in \widehat{C_1}$ since $\widehat{\Gamma} \subset \widehat{C_1}$. On the other hand since $x_n^\bullet > 0$ for all n, the condition $\lim_{n\to\infty} x_n^\bullet > 0$ implies there is $K > 0$ such that $x_n^\bullet \geq K$ for all n. We conclude $x \in \widehat{C_1} \cap \mathbf{s}^- = \widehat{\mathbb{S}^0}$. This concludes the proof of (ii). □

Now we state another application that can be considered as a corollary.

Corollary 6.4. *For any given real number $\theta \neq k\pi$, $k \in \mathbb{Z}$, and for any $x \in U^+$, the condition in (22) implies the equivalence*

$$\left(y_{n+1} - 2\cos\theta y_n + y_{n-1}\right)/x_n \to 0 \text{ if and only if } y_n/x_n \to 0 \ (n \to \infty) \text{ for all } y. \tag{23}$$

Proof. Here, we have $\mathbf{\Delta} < 0$ and $u_1 = \rho e^{i\theta}$ with $\rho = 1$ is a root of equation (4) with $r = t = 1$ and $s = -2\cos\theta$. Now, assume x satisfies the condition in (22). As we have just seen, $\lim_{n\to\infty} x_n^\bullet < 1$ implies $x \in \widehat{C_1}$ and $\lim_{n\to\infty} x_n^\bullet > 0$ implies $x \in \mathbf{s}^-$. We conclude $x \in \widehat{C_1} \cap \mathbf{s}^-$ and by Part (ii) of Corollary 6.2 the statement in (23) holds. This concludes the proof. □

Example 6.5. *From Corollary 6.4 with $\theta = 2\pi/3$, we easily see that under (22) the condition $\left(y_{n+1} + y_n + y_{n-1}\right)/x_n \to 0$ holds if and only if $y_n/x_n \to 0$ $(n \to \infty)$ for all y.*

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