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Mean Boundedness, Global Attractivity and Almost Periodic Sequence of Stochastic Neural Networks with Discrete-Time Analogue

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Abstract. A class of stochastic neural networks with discrete-time analogue is investigated in this paper. By employing contraction mapping principle and some stochastic analysis techniques, we establish some sufficient conditions for mean boundedness, global attractivity and almost periodic sequence of the model. An example and graphic illustrations are displayed to visually expound the main contributions. The research techniques in this literature are suitable for other stochastic models in science and engineering.

1. Introduction

Neural networks have been found useful in areas of signal processing, image processing, associative memories, pattern classification. So the dynamics and applications of neural networks arouse great interest by many authors, we can refer to [1–9]. However, in the applications of neural networks, it is very important to formulate a discrete-time system which is a discrete-time analogue of continuous-time neural network. The familiar schemes such as Euler scheme and Runge-Kutta scheme may show spurious equilibria or spurious stable behavior [10–12]. However, by using the discretization schemes introduced by [13–16], the convergent dynamics of the continuous-time neural networks are preserved in the discrete-time analogues for autonomous neural system. Especially, Huang et al. [17] considered the following neural network with piecewise constant argument

$$\mathrm{d}x_i(t) = -a_i([t])x_i(t)\mathrm{d}t + \sum_{j=1}^{m} b_{ij}([t])f_j(x_j([t]))\mathrm{d}t + I_i([t])\mathrm{d}t, \quad i = 1, 2, \ldots, m,$$

where $[t]$ denotes the integer part of t, $x_i(t)$ denotes the potential of the cell i at time t, $a_i(t)$ denotes the rate with which the cell i resets its potential to the resting state when isolated from other cells and inputs, $f_j(\cdot)$ denotes a non-linear output function, $b_{ij}(t)$ denotes the strengths of connectivity between the j-th cell and the i-th cell, $I_i(t)$ denotes the i-th component of an external input source introduced from outside the network to the cell i, $i, j = 1, 2, \ldots, m$. In [17], some sufficient conditions of existence and attractivity of an almost periodic sequence solution were given for the corresponding discrete-time analogue

$$x_i(n+1) = x_i(n)e^{-a_i(n)} + \frac{1 - e^{-a_i(n)}}{a_i(n)}\Big[\sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n)) + I_i(n)\Big], \quad i = 1, 2, \ldots, m. \tag{1}$$

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Stochastic differential equations are basically differential equations with an additional stochastic term. The deterministic term, which is common to ordinary differential equations, describes the "average" dynamical behaviour of the phenomenon under study and the stochastic term describes the "noise", i.e., the random perturbations that influence the phenomenon. Of course, in the particular case where such random perturbations are absent (deterministic case), the SDE becomes an ordinary differential equation. As the dynamical behaviour of many natural phenomena can be described by differential equations, SDEs have important applications in basically all fields of science and technology whenever we need to consider random perturbations in the environmental conditions (environment taken here in a very broad sense) that affect such phenomena in a relevant manner. The concept of almost periodic stochastic process is of great importance in probability for investigating stochastic process. Recently, the existence and stability of almost periodic solution to stochastic neural networks were considered [19–22]. In this paper, we consider the following stochastic neural networks

$$\mathrm{d}x_i(t)=-a_i(t)x_i(t)\mathrm{d}t+\sum_{j=1}^{m} b_{ij}(t)f_j(x_j(t))\mathrm{d}t+\sum_{j=1}^{m} c_{ij}(t)\sigma_j(x_j(t))\mathrm{d}B_{it}+I_i(t)\mathrm{d}t, \tag{2}$$

where $\sigma_j(\cdot)$ denotes a non-linear output function, B_{it} is the standard Brownian motion defined on a complete probability space, $i=1,2,\ldots,m$. The corresponding model of system (2) with piecewise constant argument is described as

$$\mathrm{d}x_i(t)=-a_i([t])x_i(t)\mathrm{d}t+\sum_{j=1}^{m} b_{ij}([t])f_j(x_j([t]))\mathrm{d}t+\sum_{j=1}^{m} c_{ij}([t])\sigma_j(x_j([t]))\mathrm{d}B_{it}+I_i([t])\mathrm{d}t, \tag{3}$$

where $i=1,2,\ldots,m$. For any $t\in\mathbb{R}$, there exists an integer $n\in\mathbb{Z}$ such that $n\le t<n+1$, where $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ denotes the set of integer numbers. Then (3) becomes

$$\mathrm{d}x_i(t)=-a_i(n)x_i(t)\mathrm{d}t+\sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n))\mathrm{d}t+\sum_{j=1}^{m} c_{ij}(n)\sigma_j(x_j(n))\mathrm{d}B_{it}+I_i(n)\mathrm{d}t, \quad i=1,2,\ldots,m. \tag{4}$$

An integration of (4) over $[n,t)$ and letting $t\to n+1$ lead to

$$x_i(n+1) \;=\; x_i(n)e^{-a_i(n)}+L_i(n)\sum_{j=1}^{m} c_{ij}(n)\sigma_j(x_j(n))+\frac{1-e^{-a_i(n)}}{a_i(n)}\Big[\sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n))+I_i(n)\Big], \tag{5}$$

where $L_i(n)=\dfrac{\int_n^{n+1} e^{a_i(n)s}\,\mathrm{d}B_{is}}{e^{a_i(n)(n+1)}}$, $i=1,2,\ldots,m, n\in\mathbb{Z}$. The non-autonomous difference equation (5) is a discrete-time analogue of (2). The existence of solution of (5) on $\mathbb{Z}$ is guaranteed by the existence of the solution of (2) on $\mathbb{R}$.

Remark 1.1. *If $c_{ij}\equiv 0$ in* (5)*, then stochastic model* (5) *is transformed into model* (1)*. So the model studied in this article extends the main researching model in literature* [17, 18]*.*

The main aim of this paper is to study mean boundedness, mean global attractivity and the existence of a unique mean almost periodic sequence to model (5). The main contributions of this literature are described as follows:

(1) A semi-discrete model is obtained for continuous-time stochastic cellular neural networks.

(2) Mean boundedness and mean global attractivity of semi-discrete model (5) are researched.

(3) A decision theorem for the existence of a unique mean almost periodic sequence of semi-discrete model (5) is acquired. The research findings improve and extend the works in literature [17, 18].

The organization of the paper is as follows. In Section 2, some useful definitions and lemmas are listed. In Section 3, mean boundedness and mean global attractivity of semi-discrete model (5) are studied. In Section 4, the existence of a unique mean almost periodic sequence of semi-discrete model (5) is discussed. In Section 5, an illustrative example is provided to demonstrate the main results in this article. The conclusion and discussion are given in Section 6.

This article uses the following notations. Let $\mathbb{Z}$ denote the set of integers, $\mathbb{R}^n$ denote the n-dimensional real vector space, $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $L^1(\Omega, \mathbb{R}^n)$ denote the set of all integrable $\mathbb{R}^n$-valued random variables and $E(\cdot)$ be the expectation operator. We use $\mathbb{B}(\mathbb{Z}, L^1(\Omega, \mathbb{R}^n))$ to stand for the set of all bounded functions from $\mathbb{Z}$ to $L^1(\Omega, \mathbb{R}^n)$.

2. Preliminaries

Before we derive our main results, we shall introduce several basic definitions.

Definition 2.1. ([23]) *Suppose that $X \in L^1(\Omega, \mathbb{R}^n)$, then the number*

$$EX = \int_\Omega X \mathrm{d}P \tag{6}$$

is the expectation of X.

Definition 2.2. ([23]) *A bounded solution $X(n) = (x_1(n), x_2(n), ..., x_m(n))^T$ of* (5) *is said to be mean globally attractive if for any other solution $Y(n) = (y_1(n), y_2(n), ..., y_m(n))^T$ of* (1.4)*, we have*

$$\lim_{n\to\infty} E|x_i(n) - y_i(n)| = 0, \quad i = 1, 2, \ldots, m.$$

Definition 2.3. ([21]) *A real valued sequence $x(n)$ is called a mean almost periodic sequence if the ϵ-translation set*

$$E\{\epsilon, x\} = \{\tau \in \mathbb{Z} : E|x(n+\tau) - x(n)| < \epsilon \ for\ all\ n \in \mathbb{Z}\}$$

is a relaticely dense set in $\mathbb{Z}$ for all $\epsilon > 0$, that is, for any given $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains an integer $\tau \in E\{\epsilon, x\}$ such that

$$E|x(n+\tau) - x(n)| < \epsilon \ for\ all\ n \in \mathbb{Z}.$$

τ is called ϵ-translation number or ϵ-almost period. The collection of such sequences will be denoted $AP(\mathbb{Z})$.

Definition 2.4. ([23]) *Suppose that $\{\omega_t : t \in \mathbb{R}_+\}$ is an m-dimensional adaptive stochastic process and it satisfies the following conditions:*

(C_1) $\omega_0 = 0$, a.s.;

(C_2) Normality: if $0 \le s < t < \infty$, then $\omega_t - \omega_s \sim N\big(0, (t-s)\Sigma\big)$, $\Sigma = [\sigma_{ki}] \in \mathbb{R}^{m\times m}$ is a normal matrix;

(C_3) Incremental independence: if $0 \le s < t < \infty$, then $\omega_t - \omega_s$ is a process with stationary and mutually independent increments.

Then ω_t is said to be a Brownian movement or Wiener process; especially when $\Sigma = I$, then ω_t is said to be a standard Brownian movement.

Lemma 2.5. ([23]) *Suppose that $g \in L^2(J, \mathbb{R}^{m\times n}), p > 0$, then*

$$E\Big[\sup_{t\in J} |\int_{t_0}^t g(s) d\omega(s)|^p\Big] \le C_p E\Big[\int_{t_0}^T |g(t)|^2 dt\Big]^{\frac{p}{2}}, \tag{7}$$

where

$$C_p = \begin{cases} (32/p)^{p/2}, & 0 < p < 2, \\ 4, & p = 2, \\ \Big[\frac{p^{p+1}}{2(p-1)^{(p-1)}}\Big]^{\frac{p}{2}}, & p > 2. \end{cases}$$

Set

$$a_i^* = \sup_{n\in\mathbb{Z}} |a_i(n)|,\ \ b_{ij}^* = \sup_{n\in\mathbb{Z}} |b_{ij}(n)|,\ \ c_{ij}^* = \sup_{n\in\mathbb{Z}} |c_{ij}(n)|,\ \ I_i^* = \sup_{n\in\mathbb{Z}} |I_i(n)|,$$

$$P_i^* = \sum_{j=1}^{m} c_{ij}^*\sigma_j^*,\ \ \alpha_i = \sum_{j=1}^{m} b_{ij}^* L_j^f,\ \ \beta_i = \sum_{j=1}^{m} c_{ij}^* L_j^\sigma.$$

Throughout this letter, we suppose that the following conditions are satisfied:

(H_1) $|f_j(x)| \le f_j^*$, $|\sigma_j(x)| \le \sigma_j^*$ and $|f_j(x) - f_j(y)| \le L_j^f|x-y|$, $|\sigma_j(x)-\sigma_j(y)| \le L_j^\sigma|x-y|$ for all $x, y \in \mathbb{R}$, where $f_j^* > 0, \sigma_j^* > 0, L_j^f > 0, L_j^\sigma > 0$, $j = 1, 2, \ldots, m$.

(H_2) $a_{i*} = \inf_{n\in\mathbb{Z}} a_i(n) > 0$, $K_i^* = \sum_{j=1}^{m} b_{ij}^* f_j^* + I_i^* > 0$ and there exists a constant $r_i > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_i(l-k) = r_i$$

holds uniformly for all $l \in \mathbb{Z}$, $i = 1, 2, \ldots, m$.

(H_3) a_i, b_{ij}, c_{ij} and I_i are almost periodic sequences with real values, $i, j = 1, 2, \ldots, m$.

3. Mean boundedness and mean global attractivity

In applications, many important properties of semi-discrete model (5) depend on its boundedness and global attractivity. So it is worth investigating boundedness and global attractivity of semi-discrete model (5).

Theorem 3.1. *Assume that* (H_1)-(H_2) *hold, suppose further that*

(H_4) $d = \max_{1\le i\le m}\left[e^{-a_{i*}} + \frac{1-e^{-a_{i*}}}{a_{i*}}\alpha_i + \frac{4(1-e^{-2a_i^*})^{\frac{1}{2}}}{\sqrt{a_{i*}}}\beta_i\right] < 1,$

then model (5) *has a unique mean bounded solution* $X(n)$, *which is mean globally attractive.*

Proof. By (H_2), it has

$$\exp\Big[-\sum_{k=1}^{n} a_i(l-k)\Big] = \exp(-nr_i + \sum_{k=1}^{n}\rho_i(l-k)), \tag{8}$$

where

$$\sum_{k=1}^{n}\rho_i(l-k) = -\sum_{k=1}^{n} a_i(l-k) + nr_i \to 0 \quad \text{as } n\to\infty,$$

$l \in \mathbb{Z}$. Since $\sum_{k=1}^{n}\rho_i(l-k)$ is bounded for $l \in \mathbb{Z}$, let $|\sum_{k=1}^{n}\rho_i(l-k)| \le \Delta_i$, where Δ_i is independent of any $l \in \mathbb{Z}$. (8) implies

$$\exp(-\sum_{k=1}^{n} a_i(l-k)) = \exp(-nr_i + \sum_{k=1}^{n}\rho_i(l-k)) \le e^{-nr_ie^{\Delta_i}} \to 0,$$

as $n \to \infty$. From model (5), it gets

$$x_i(n) = x_i(0)\exp(-\sum_{k=0}^{n-1} a_i(k)) + \sum_{k=1}^{n}\Big\{\frac{1-e^{-a_i(n-k)}}{a_i(n-k)}\exp\Big[-\sum_{p=1}^{k-1} a_i(n-p)\Big]$$

$$\Big[\sum_{j=1}^{m} b_{ij}(n-k)f_j(x_j(n-k)) + I_i(n-k)\Big]\Big\} + \sum_{k=1}^{n}\Big\{L_i(n-k)\exp\Big[-\sum_{p=1}^{k-1}a_i(n-p)\Big]$$
$$\sum_{j=1}^{m} c_{ij}(n-k)\sigma_j(x_j(n-k))\Big\}, \quad n\geq 1, i=1,2,\ldots,m.$$

Let $Y(n)=(y_1(n),\ldots,y_m(n))^T$ be another solution of (5) on $\mathbb{Z}$. The solution can be described as

$$\begin{aligned} y_i(n) &= \sum_{k=1}^{\infty}\Big\{\frac{1-e^{-a_i(n-k)}}{a_i(n-k)}\exp\Big[-\sum_{p=1}^{k-1}a_i(n-p)\Big]\Big[\sum_{j=1}^{m} b_{ij}(n-k)f_j(x_j(n-k)) + I_i(n-k)\Big]\Big\} \\ &\quad + \sum_{k=1}^{\infty}\Big\{L_i(n-k)\exp\Big[-\sum_{p=1}^{k-1}a_i(n-p)\Big]\sum_{j=1}^{m} c_{ij}(n-k)\sigma_j(x_j(n-k))\Big\}, \quad i=1,2,\ldots,m. \end{aligned}$$

By Lemma 2.5, it has

$$E|L_i(n)| = E\Big|\frac{\int_{n-k}^{n-k+1} e^{a_i(n)s}dB_{is}}{e^{a_i(n)(n+1)}}\Big| \leq \frac{\sqrt{32}}{e^{a_i(n)(n+1)}}E\Big[\int_n^{n+1} e^{2a_i(n)s}ds\Big]^{\frac{1}{2}} \leq \frac{4(1-e^{-2a_{i*}(n)})^{\frac{1}{2}}}{\sqrt{a_{i*}(n)}}, \quad i=1,2,\ldots,m.$$

Then

$$E|y_i(n)| \leq \sum_{k=1}^{\infty}\Big[\frac{K_i^*}{a_{i*}} + E|L_i(n-k)|P_i^*\Big]e^{-(k-1)r_i}e^{\Delta_i} = \Big[\frac{K_i^*}{a_{i*}} + \frac{4P_i^*}{\sqrt{a_{i*}}}\Big]\frac{e^{\Delta_i}}{1-e^{-r_i}} = M_i, \quad i=1,2,\ldots,m.$$

Set

$$\Omega = \{x=(x_1,x_2,...,x_m)\in\mathbb{R}^m : E|x_i|\leq M_i, i=1,2,\ldots,m\}.$$

Clearly, $Y(n)\subset\Omega$. Define

$$V(n) = \max_{1\leq i\leq m} E|x_i(n)-y_i(n)|, \quad i=1,2,\ldots,m.$$

Then

$$\begin{aligned} V(n+1) &= \max_{1\leq i\leq m} E|x_i(n+1)-y_i(n+1)| \\ &\leq \max_{1\leq i\leq m}\Big[e^{-a_i(n)}E|x_i(n)-y_i(n)| + \frac{1-e^{-a_i(n)}}{a_i(n)}\sum_{j=1}^{m}|b_{ij}(n)|L_j^f E|x_j(n)-y_j(n)| \\ &\quad + \sum_{j=1}^{m}|c_{ij}(n)|L_j^{\sigma}E\frac{|\int_n^{n+1}|x_i(n)-y_i(n)|e^{a_i(n)s}dB_{is}|}{e^{a_i(n)(n+1)}}\Big] \\ &\leq \max_{1\leq i\leq m}\Big[e^{-a_{i*}} + \frac{1-e^{-a_{i*}}}{a_{i*}}\alpha_i + \frac{4(1-e^{-2a_i^*})^{\frac{1}{2}}}{\sqrt{a_{i*}}}\beta_i\Big]V(n), \end{aligned}$$

So

$$V(n+1)\leq dV(n)$$

implies

$$0\leq V(n+1)\leq d^{n+1}V(0)\to 0 \text{ as } n\to\infty.$$

It leads to

$$\lim_{n\to\infty}\max_{1\leq i\leq m} E|x_i(n)-y_i(n)| = 0.$$

Then model (5) has a unique bounded solution $X(n)$, which is globally attractive. This completes the proof. □

4. Mean almost periodic sequence

In real-world applications, the system parameters of model (5) are considered to be periodic or almost periodic or asymptotically periodic [24, 25], since many important factors like habit, competition for limiting resources and available food are periodically forced. So it is worth investigating almost periodicity of semi-discrete model (5).

Let

$$C_i = \frac{1-e^{-a_i^*}}{a_{i*}(1-e^{-a_{i*}})} K_i^* + \frac{4P_i^*}{\sqrt{a_{i*}}(1-e^{-a_{i*}})}, \quad i = 1,2,\ldots,m.$$

Theorem 4.1. *If* (H_1), (H_3) *and* (H_4) *hold, then model* (5) *admits a unique mean almost periodic sequence.*

Proof. Let

$$\Omega = \Big\{X \in AP(\mathbb{Z}) : E|x_i| \le C_i, i = 1,2,\ldots,m\Big\}.$$

Defining T on Ω by $TX = U = (u_1,\ldots,u_m)^T$, where

$$\begin{aligned} u_i(n+1) &= x_i(n)e^{-a_i(n)} + \frac{\int_n^{n+1} e^{a_i(n)s}\,\mathrm{d}B_{is}}{e^{a_i(n)(n+1)}} \sum_{j=1}^m c_{ij}(n)\sigma_j(x_j(n)) \\ &\quad + \frac{1-e^{-a_i(n)}}{a_i(n)}\Big[\sum_{j=1}^m b_{ij}(n)f_j(x_j(n)) + I_i(n)\Big], \quad n \in \mathbb{Z}, i = 1,2,\ldots,m. \end{aligned}$$

Next, $T : \Omega \to \Omega$ will be proved. It follows that

$$\begin{aligned} E|Tx_i(n)| &= E|u_i(n+1)| \\ &= E\Big|x_i(n)e^{-a_i(n)} + \frac{\int_n^{n+1} e^{a_i(n)s}\,\mathrm{d}B_{is}}{e^{a_i(n)(n+1)}} \sum_{j=1}^m c_{ij}(n)\sigma_j(x_j(n)) + \frac{1-e^{-a_i(n)}}{a_i(n)}\Big[\sum_{j=1}^m b_{ij}(n)f_j(x_j(n)) + I_i(n)\Big]\Big| \\ &\le E\Big|x_i(n)e^{-a_i(n)} + \frac{1-e^{-a_i(n)}}{a_i(n)}\Big[\sum_{j=1}^m b_{ij}^* f_j^* + I_i^*\Big] + \frac{\int_n^{n+1} e^{a_i(n)s}\,\mathrm{d}B_{is}}{e^{a_i(n)(n+1)}} \sum_{j=1}^m c_{ij}^*\sigma_j^*\Big| \\ &= E\Big|x_i(n)e^{-a_i(n)}\Big| + \frac{1-e^{-a_i(n)}}{a_i(n)} K_i^* + \frac{P_i^*}{e^{a_i(n)(n+1)}} E\Big|\int_n^{n+1} e^{a_i(n)s}\,\mathrm{d}B_{is}\Big| \\ &\le e^{-a_i(n)}E|x_i(n)| + \frac{1-e^{-a_i(n)}}{a_i(n)} K_i^* + \frac{P_i^*}{e^{a_i(n)(n+1)}} \cdot \sqrt{32} \cdot E\Big[\int_n^{n+1} e^{2a_i(n)t}\,\mathrm{d}t\Big]^{\frac{1}{2}} \\ &\le e^{-a_i(n)}M_i + \frac{1-e^{-a_i(n)}}{a_i(n)} K_i^* + \frac{P_i^*}{e^{a_i(n)(n+1)}} \cdot \sqrt{32} \cdot \Big[\frac{1}{2a_i(n)} e^{2a_i(n)(n+1) - e^{2a_i(n)n}}\Big]^{\frac{1}{2}} \\ &\le e^{-a_{i*}}M_i + \frac{1-e^{-a_i^*}}{a_{i*}} K_i^* + \frac{4P_i^*}{\sqrt{a_{i*}}} \\ &= C_i, \quad i = 1,2,\ldots,m. \end{aligned} \tag{9}$$

For any $\epsilon > 0$, from (H_1), (H_3) and literature [22], there must exist positive constants λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, λ_7, λ_8 such that

$$E\Big|x_i(n+\tau)e^{-a_i(n+\tau)} - x_i(n)e^{-a_i(n)}\Big| \le E\Big|x_i(n+\tau)(e^{-a_i(n+\tau)} - e^{-a_i(n)})\Big| + E\Big|(x_i(n+\tau) - x_i(n))e^{-a_i(n)}\Big|$$

$$\leq \lambda_1\varepsilon + \lambda_2\varepsilon, \tag{10}$$

$$\begin{aligned}
&E\ \left|L_i(n+\tau)\sum_{j=1}^{m} c_{ij}(n+\tau)\sigma_j(x_j(n+\tau)) - L_i(n)\sum_{j=1}^{m} c_{ij}(n)\sigma_j(x_j(n))\right| \\
&\leq\ E\left|(L_i(n+\tau)-L_i(n))\sum_{j=1}^{m} c_{ij}(n+\tau)\sigma_j(x_j(n+\tau))\right| + E\left|L_i(n)\sum_{j=1}^{m}(c_{ij}(n+\tau)-c_{ij}(n))\sigma_j(x_j(n+\tau))\right| \\
&\quad +E\left|L_i(n)\sum_{j=1}^{m} c_{ij}(n)(\sigma_j(x_j(n+\tau))-\sigma_j(x_j(n)))\right| \\
&\leq\ \lambda_3\varepsilon + \lambda_4\varepsilon + \lambda_5\varepsilon,
\end{aligned} \tag{11}$$

$$E\left|\frac{1-e^{-a_i(n+\tau)}}{a_i(n+\tau)}I_i(n+\tau) - \frac{1-e^{-a_i(n)}}{a_i(n)}I_i(n)\right| \leq \lambda_6\varepsilon, \tag{12}$$

$$\begin{aligned}
&E\left|\frac{1-e^{-a_i(n+\tau)}}{a_i(n+\tau)}\sum_{j=1}^{m} b_{ij}(n+\tau)f_j(x_j(n+\tau)) - \frac{1-e^{-a_i(n)}}{a_i(n)}\sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n))\right| \\
&\leq\ E\left|\sum_{j=1}^{m}\frac{1-e^{-a_i(n+\tau)}}{a_i(n+\tau)}b_{ij}(n+\tau)(f_j(x_j(n+\tau))-f_j(x_j(n)))\right| \\
&\quad +E\left|\sum_{j=1}^{m}[\frac{1-e^{-a_i(n+\tau)}}{a_i(n+\tau)}b_{ij}(n+\tau) - \frac{1-e^{-a_i(n)}}{a_i(n)}b_{ij}(n)]f_j(x_j(n))\right| \\
&\leq\ \lambda_7\varepsilon + \lambda_8\varepsilon,
\end{aligned} \tag{13}$$

where $n \in \mathbb{Z}$, $i = 1, 2, \ldots, m$.

From (10)-(13), we obtain

$$\begin{aligned}
E|Tx_i(n+\tau) - Tx_i(n)| =\ & E\left|x_i(n+\tau)e^{-a_i(n+\tau)} + L_i(n+\tau)\sum_{j=1}^{m} c_{ij}(n+\tau)\sigma_j(x_j(n+\tau))\right. \\
&+\frac{1-e^{-a_i(n+\tau)}}{a_i(n+\tau)}\Big[\sum_{j=1}^{m} b_{ij}(n+\tau)f_j(x_j(n+\tau)) + I_i(n+\tau)\Big] \\
&-x_i(n)e^{-a_i(n)} - L_i(n)\sum_{j=1}^{m} c_{ij}(n)\sigma_j(x_j(n)) \\
&\left.-\frac{1-e^{-a_i(n)}}{a_i(n)}\Big[\sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n)) + I_i(n)\Big]\right| \\
\leq\ & E\left|x_i(n+\tau)e^{-a_i(n+\tau)} - x_i(n)e^{-a_i(n)}\right| + E\left|L_i(n+\tau)\sum_{j=1}^{m} c_{ij}(n+\tau)\sigma_j(x_j(n+\tau))\right. \\
&\left.-L_i(n)\sum_{j=1}^{m} c_{ij}(n)\sigma_j(x_j(n))\right| + E\left|\frac{1-e^{-a_i(n+\tau)}}{a_i(n+\tau)}\right. \\
&\Big[\sum_{j=1}^{m} b_{ij}(n+\tau)f_j(x_j(n+\tau)) + I_i(n+\tau)\Big]
\end{aligned}$$

$$-\frac{1-e^{-a_i(n)}}{a_i(n)}\Big[\sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n)) + I_i(n)\Big]\Big|$$
$$\le (\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8)\varepsilon,$$

which implies that Tx_i is mean almost periodic. So $T:\Omega\to\Omega$.

For any $X = (x_1(n),\dots,x_m(n))^T, \tilde{X} = (\tilde{x}_1(n),\dots,\tilde{x}_m(n))^T \in \Omega$, letting $TX = U, T\tilde{X} = \tilde{U}$, where $U(n) = (u_1(n),\dots,u_m(n))^T$ and $\tilde{U} = (\tilde{u}_1(n),\dots,\tilde{u}_m(n))^T$. Define

$$\|TX - T\tilde{X}\| = \sup_{n\in\mathbb{Z}}\max_{1\le i\le m} E|u_i(n)-\tilde{u}_i(n)|, \quad \|X-\tilde{X}\| = \sup_{n\in\mathbb{Z}}\max_{1\le i\le m} E|x_i(n)-\tilde{x}_i(n)|.$$

From the proof of Theorem 3.1, it gets

$$\max_{1\le i\le m} E|u_i(n+1)-\tilde{u}_i(n+1)| \le \max_{1\le i\le m}\Big[e^{-a_i(n)} + \frac{1-e^{-a_i(n)}}{a_i(n)}\alpha_i + \frac{4(1-e^{-2a_i(n)})^{\frac{1}{2}}}{\sqrt{a_i(n)}}\beta_i\Big]\|X-\tilde{X}\|, \quad n\in\mathbb{Z}.$$

Then

$$\|TX - T\tilde{X}\| \le d\|X-\tilde{X}\|.$$

By (H_4), $d\in(0,1)$ and T is a contraction. Therefore, there exists a unique mean almost periodic sequence of model (5) by employing Banach contraction mapping principle. The proof of Theorem 4.1 is completed. □

Remark 4.2. *If the stochastic terms are vanished in model* (5)*, i.e.,* $\beta_i = 0 (i = 1,2,\dots,n)$ *in assumption* (H_4)*, then the obtained Theorems* 3.1 *and* 4.1 *are changed into the corresponding results in Refs.* [17, 18]*. So the current research findings complement and extend the corresponding works in literatures* [17, 18].

Remark 4.3. *Observing the conditions of Theorem* 3.1 *and* 4.1*,* (H_4) *is more complicated then other conditions. To ensure that model* (5) *has a unique bounded globally attractive mean almost periodic sequence, in application, the following rules should be followed:*

Rule 1. *The coefficients* a_i *of model* (5) *are best to be selected to meet* (H_4) *with large enough numbers,* $i = 1,2,\dots,n$.

Rule 2. *Choosing the activation functions* f_j *and* σ_j *in model* (5) *with some small enough constants* L_j^f *and* L_j^σ*,* (H_4) *is easier to be satisfied,* $j = 1,2,\dots,n$.

Rule 3. *Selecting all coefficients in model* (5) *excluding* $a_i (i = 1,2,\dots,n)$ *with some small enough constants,* (H_4) *is easier to be satisfied.*

5. A numerical illustrative example

Example 5.1. *In this section, an example is provided to illustrate the results in previous sections. Considering the following semi-discrete stochastic neural networks*

$$\begin{cases} x_1(n+1) = x_1(n)e^{-(2+\sin\sqrt{2}n)} + \frac{1-e^{-(2+\sin\sqrt{2}n)}}{2+\sin\sqrt{2}n}\{0.01\sin n f(x_1(n)) + 0.01\cos n f(x_2(n)) + \sin\sqrt{5}n\} \\ \qquad +L_1(n)\{(0.006+0.005\sin\sqrt{3}n)\sigma(x_1(n)) + (0.006+0.005\cos n)\sigma(x_2(n))\} \\ x_2(n+1) = x_2(n)e^{-(2+\cos n)} + \frac{1-e^{-(2+\cos n)}}{2+\cos n}\{0.01\cos n f(x_1(n)) + 0.01\sin n f(x_2(n)) + \cos\sqrt{2}n\} \\ \qquad +L_2(n)\{(0.005+0.004\cos\sqrt{2}n)\sigma(x_1(n)) + (0.005+0.004\sin\sqrt{3}n)\sigma(x_2(n))\}, \end{cases} \tag{14}$$

where

$$a_1(n) = 2+\sin\sqrt{2}n,\ b_{11}(n) = 0.01\sin n,\ b_{12}(n) = 0.01\cos n,\ I_1(n) = \sin\sqrt{5}n,$$
$$c_{11}(n) = 0.006+0.005\sin\sqrt{3}n,\ c_{12}(n) = 0.006+0.005\cos n,\ a_2(n) = 2+\cos n,\ b_{21}(n) = 0.01\cos n,$$
$$b_{22}(n) = 0.01\sin n,\quad I_2(n) = \cos\sqrt{2}n,\ c_{21}(n) = 0.005+0.004cos\sqrt{2}n,\ c_{22}(n) = 0.005+0.004sin\sqrt{3}n,$$
$$L_1(n) = \frac{\int_n^{n+1} e^{(2+\sin\sqrt{2}n)s}\,\mathrm{d}B_{1s}}{e^{(2+\sin\sqrt{2}n)(n+1)}},\ L_2(n) = \frac{\int_n^{n+1} e^{(2+\cos n)s}\,\mathrm{d}B_{2s}}{e^{(2+\cos n)(n+1)}},\ f_i(x) = \sigma_i(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}},\ i = 1,2.$$

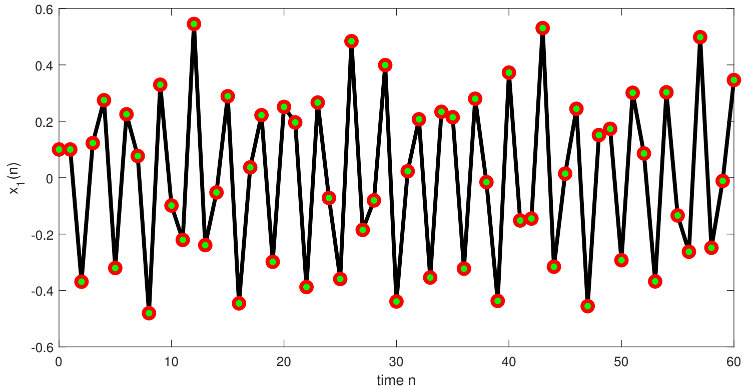

Figure 1 Mean almost periodicity of state variable x_1 for model (14)

By simple calculation, we can see that the conditions $(H_1), (H_2), (H_3)$ are satisfied and we can obtain that $L_1^f = L_2^f = L_1^\sigma = L_2^\sigma = 4$.

Now we will show that the condition (H_4) is satisfied too.

$$a_{1*} = \inf_{n\in\mathbb{Z}} a_1(n) = 1,\ \ b_{11}^* = \sup_{n\in\mathbb{Z}} |b_{11}(n)| = 0.01,\ \ b_{12}^* = \sup_{n\in\mathbb{Z}} |b_{12}(n)| = 0.01,$$

$$c_{11}^* = \sup_{n\in\mathbb{Z}} |c_{11}(n)| = 0.011,\ \ c_{12}^* = \sup_{n\in\mathbb{Z}} |c_{12}(n)| = 0.011,$$

$$a_{2*} = \inf_{n\in\mathbb{Z}} a_2(n) = 1,\ \ b_{21}^* = \sup_{n\in\mathbb{Z}} |b_{21}(n)| = 0.01,\ \ b_{22}^* = \sup_{n\in\mathbb{Z}} |b_{22}(n)| = 0.01,$$

$$c_{21}^* = \sup_{n\in\mathbb{Z}} |c_{21}(n)| = 0.009,\ \ c_{22}^* = \sup_{n\in\mathbb{Z}} |c_{22}(n)| = 0.009.$$

Because

$$\alpha_1^* = b_{11}^* L_1^f + b_{12}^* L_2^f = 0.08,\ \ \beta_1^* = c_{11}^* L_1^\sigma + c_{12}^* L_2^\sigma = 0.088,$$

$$\alpha_2^* = b_{21}^* L_1^f + b_{22}^* L_2^f = 0.08,\ \ \beta_2^* = c_{21}^* L_1^\sigma + c_{22}^* L_2^\sigma = 0.072,$$

we have

$$\begin{aligned} e^{-a_{1*}} + \frac{1-e^{-a_{1*}}}{a_{1*}}\alpha_1^* + \frac{4(1-e^{-2a_{1*}})^{\frac{1}{2}}}{\sqrt{a_{1*}}}\beta_1^* &= e^{-1} + \frac{1-e^{-1}}{1} \times 0.08 + \frac{4\times(1-e^{-2})^{\frac{1}{2}}}{1} \times 0.088 \\ &= e^{-1} + 0.08 - 0.08e^{-1} + 0.352(1-e^{-2})^{\frac{1}{2}} \\ &\le 0.08 + 0.92e^{-1} + 0.352 < 1, \end{aligned}$$

$$\begin{aligned} e^{-a_{2*}} + \frac{1-e^{-a_{2*}}}{a_{2*}}\alpha_2^* + \frac{4(1-e^{-2a_{2*}})^{\frac{1}{2}}}{\sqrt{a_{2*}}}\beta_2^* &= e^{-1} + \frac{1-e^{-1}}{1} \times 0.08 + \frac{4\times(1-e^{-2})^{\frac{1}{2}}}{1} \times 0.072 \\ &= e^{-1} + 0.08 - 0.08e^{-1} + 0.288(1-e^{-2})^{\frac{1}{2}} \\ &\le 0.08 + 0.92e^{-1} + 0.288 < 1. \end{aligned}$$

This shows that the condition (H_4) is satisfied. So by Theorems 1-2, there exists a unique mean globally attractive almost periodic sequence of model (14), which can be seen in Figures 1-6.

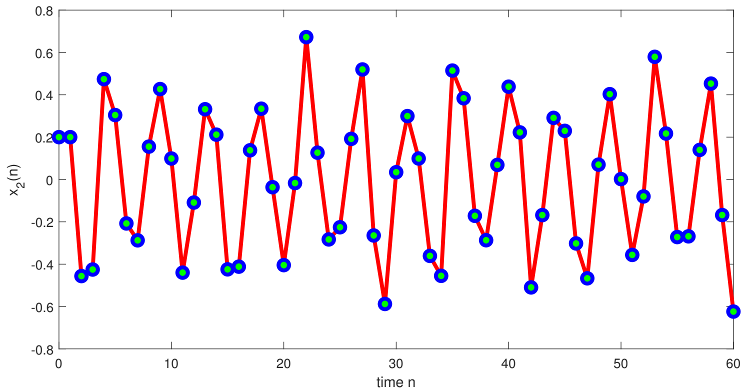

Figure 2 Mean almost periodicity of state variable x_2 for model (14)

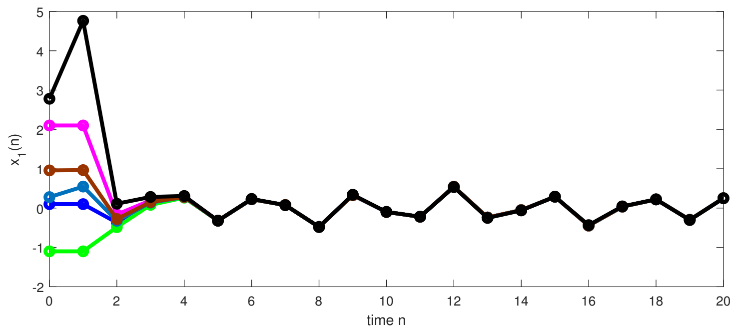

Figure 3 Mean global attractivity of state variable x_1 for model (14)

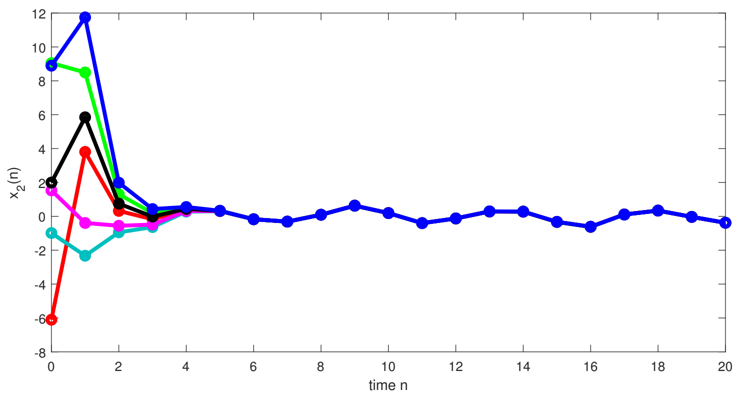

Figure 4 Mean global attractivity of state variable x_2 for model (14)

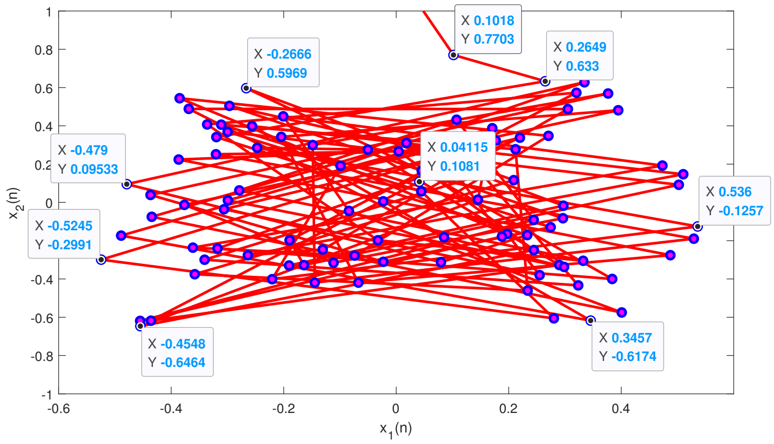

Figure 5 Phase response of state variables $(x_1, x_2)^T$ of model (14)

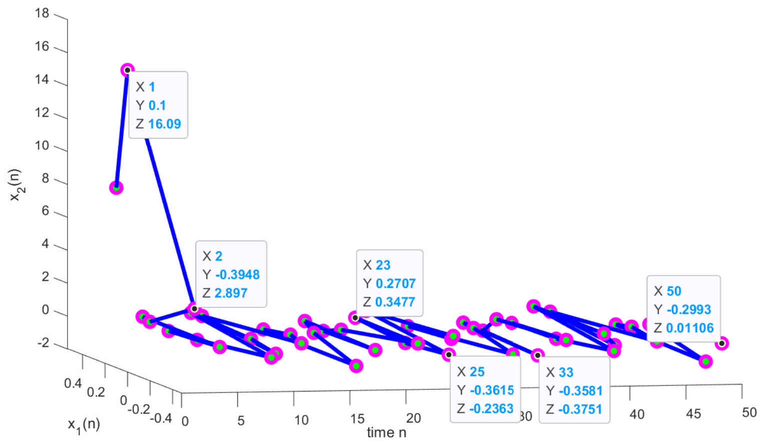

Figure 6 State trajectory in 3D space of state variables $(x_1, x_2)^T$ of model (14)

6. Conclusion and discussion

In this paper, the stochastic perturbation

$$\sum_{j=1}^{m} c_{ij}(t)\sigma_j(x_j(t))\mathrm{d}B_{it}$$

is added to the following differential equation

$$\mathrm{d}x_i(t) = -a_i(t)x_i(t)\mathrm{d}t + \sum_{j=1}^{m} b_{ij}(t)f_j(x_j(t))\mathrm{d}t + I_i(t)\mathrm{d}t, \quad i = 1, 2, \ldots, n.$$

We can summarize as follows:

(D_1) A non-autonomous difference equation corresponding to system (2) is given by model (5).

(D_2) In (H_4), the stochastic perturbation leads to a change (i.e., β_i). In view of Theorem 3.1 and Theorem 4.1, the stochastic perturbation has a negative effect on the existence of mean almost periodic solutions and mean global attractivity of model (5).

(D_3) If the stochastic perturbation is removed from model (5), we can know that the work of paper [17] is a special case of this article, and this paper is the extension of [17, 18].

(D_4) The approaches used in this article could be applied to research other stochastic models in science and engineering, such as biomathematical model [24], fractional dynamic model [25], impulsive model [26], etc.

References

[1] M. Kalpana, P. Balasubramaniam, K. Ratnavelu, Direct delay decomposition approach to synchronization of chaotic fuzzy cellular neural networks with discrete, unbounded distributed delays and Markovian jumping parameters, Appl. Math. Comput. 254 (2015) 291-304.

[2] M.J. Bohner, I.M. Stamova, An impulsive delay discrete stochastic neural network fractional-order model and applications in finance, Filomat 32 (2018) 6339-6352.

[3] B. Du, Design of exponential state estimators for neutral-type neural networks with mixed time delays, Filomat 30 (2016) 3435-3449.

[4] Sitian Qin, Jun Wang, Xiaoping Xue, Convergence and attractivity of memristor-based cellular neural networks with time delays, Neural Networks 63 (2015) 223-233 63 (2015) 223-233.

[5] H.Z. Qu, T.W. Zhang, J.W. Zhou, Global stability analysis of S-asymptotically ω-periodic oscillation in fractional-order cellular neural networks with time variable delays, Neurocomputing 399 (2020) 390-398.

[6] L.O. Chua, L. Yang, Cellular neural networks: theory, IEEE Trans. Circuits Syst. 35 (1988) 1257-1272.

[7] L.O. Chua, L. Yang, Cellular neural networks: applications, IEEE Trans. Circuits Syst. 35 (1988) 1273-1290.

[8] T.W. Zhang, S.F. Han, J.W. Zhou, Dynamic behaviours for semi-discrete stochastic Cohen-Grossberg neural networks with time delays, Journal of the Franklin Institute 357 (2020) 13006-13040.

[9] T.W. Zhang, J.W. Zhou, Exponentially stable periodic oscillation and Mittag-Leffler stabilization for fractional-order impulsive control neural networks with piecewise caputo derivatives, IEEE Transactions on Cybernetics, 2021, in press, doi: 10.1109/TCYB.2021.3054946.

[10] A. Iserles, Stability and dynamics of numerical methods for nonlinear ordinary differential equations, IMA J. Numer. Anal. 10 (1990) 1-30.

[11] R.E. Michens, Nonstandard Finite Difference Models of Differential Equations, World Scientific, Singapore, 1994.

[12] M. Prufer, Turbulence in multistep methods for initial value problems, SIAM J. Appl. Math. 45 (1985) 32-69.

[13] S. Mohamad, Global exponential stability in continuous-time and discrete-time delayed bidirectional neural networks, Physica D 159 (2001) 233-251.

[14] S. Mohamad, K. Gopalsamy, Dynamics of a class of discrete-time neural networks and their continuous-time counterparts, Math. Comput. Simul. 53 (2000) 1-39.

[15] S. Mohamad, A.G. Naim, Discrete-time analogues of integro-differential equations modelling bidirectional neural networks, J. Comput. Appl. Math. 138 (2002) 1-20.

[16] S.F. Han, T.W. Zhang, G.X. Liu, Stochastic dynamics of discrete-time fuzzy random BAM neural networks with time delays, Mathematical Problems in Engineering 2019 (2019) 9416234.

[17] Z.K. Huang, X.H. Wang, F. Gao, The existence and global attractivity of almost periodic sequence solution of discrete-time neural networks, Phys. Lett. A 350 (2006) 182-191.

[18] Z.K. Huang, Y.H. Xia, X.H. Wang, The existence and exponential attractivity of κ-almost periodic sequence solution of discrete time neural networks, Nonlinear Dynamics 50 (2007) 13-26.

[19] P. H. Bezandry, T. Diagana, Existence of almost periodic solutions to some stochastic diferential equations, Appl. Anal. 86 (2007) 819-827.

[20] J.F. Cao, Q. Yang, Z. Huang, On almost periodic mild solutions for stochastic functional diferential equations, Nonlinear Analysis: Real World Applications 13 (2012) 275-286.

[21] P. H. Bezandry, T. Diagana, Almost periodic stochastic processes, Springer, New York, 2011.

[22] T. Diagana, M.M. Mbaye, Square-mean almost periodic solutions to some singular stochastic differential equations, Applied Mathematics Letters 54 (2016) 48-53.

[23] S. G. Hu, C. M. Huang, F. K. Wu, Stochastic differential equations, Science Press, Beijing, 2008.

[24] T.W. Zhang, L. Yang, L.J. Xu, Stage-structured control on a class of predator-prey system in almost periodic environment, International Journal of Control 93 (2020) 1442-1460.

[25] T.W. Zhang, L.L. Xiong, Periodic motion for impulsive fractional functional differential equations with piecewise Caputo derivative, Applied Mathematics Letters 101 (2020) 106072.

[26] M.S. Ali, G. Narayanan, V. Shekher, A. Alsaedi, B. Ahmad, Global Mittag-Leffler stability analysis of impulsive fractional-order complex-valued BAM neural networks with time varying delays, Communications in Nonlinear Science and Numerical Simulation 83 (2020) 105088.