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Differences of Composition Operators From Analytic Besov Spaces Into Little Bloch Type Spaces

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Abstract. We present some characterizations for the compactness of the difference of two composition operators acting between analytic Besov spaces and the weighted little Bloch type space over the unit disk.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and dA the normalized area measure on $\mathbb{D}$ (i.e., $A(\mathbb{D}) = 1$). Let $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$. When $1 < p < \infty$, a function $f \in H(\mathbb{D})$ is said to be in the analytic Besov space $B^p(\mathbb{D}) = B^p$ if and only if

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty. \tag{1}$$

The following functional

$$\|f\|_p = |f(0)| + \left[\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) \right]^{1/p},$$

is a norm on B^p.

In the case $p = 1$, condition (1) is satisfied by only constant functions. Thus the definition of the space B^1 is complicated and there are several ways to define B^1. If $1 < p < \infty$, it is well known that $f \in B^p$ is equivalent to

$$\int_0^1 M_p^p(r, f'')(1 - r)^{p-1} dr < \infty,$$

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where

$$M_p^p(r,F) = \frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^p d\theta, \quad r \in (0,1).$$

In the case $p = 1$, the above condition becomes

$$\int_0^1 M_1(r, f'')dr < \infty.$$

Hence we can define the space B^1 by the condition

$$\int_{\mathbb{D}} |f''(z)|dA(z) < \infty.$$

For $w \in \mathbb{D}$, let $\alpha_w(z)$ be the conformal automorphism of $\mathbb{D}$ defined by

$$\alpha_w(z) = \frac{w - z}{1 - \overline{w}z}, \quad z \in \mathbb{D}.$$

Each function $f \in B^1$ has an atomic decomposition, that is, there exist sequences $(c_j)_{j\in\mathbb{N}} \in l^1$ and $(w_j)_{j\in\mathbb{N}} \subset \mathbb{D}$ such that

$$f(z) = c_0 + \sum_{j=1}^{\infty} c_j \alpha_{w_j}(z), \quad z \in \mathbb{D}.$$

By using this representation, a norm $\|\cdot\|_1$ on B^1 is defined by

$$\|f\|_1 = \inf \sum_{j=0}^{\infty} |c_j|,$$

where the infimum is taken over all $(c_j)_{j\in\mathbb{N}} \in l^1$ satisfying the above atomic decomposition for a given $f \in B^1$. It is known that $\|f\|_1$ is comparable to

$$|f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(z)|dA(z).$$

For more details about analytic Besov spaces, we can refer to the monograph [17].

Next we will introduce the weighted Bloch type space. Throughout this paper, let ν be a positive continuous radial function on $\mathbb{D}$. Here "radial" means that $\nu(z) = \nu(|z|)$ for $z \in \mathbb{D}$. The weighted Bloch type space $\mathcal{B}_\nu$ is the space of all $f \in H(\mathbb{D})$ which satisfy $\sup_{z\in\mathbb{D}} \nu(z)|f'(z)| < \infty$, and the little Bloch type space $\mathcal{B}_{\nu,0}$ consists of all $f \in \mathcal{B}_\nu$ satisfying $\nu(z)|f'(z)| \to 0$ as $|z| \to 1^-$. It is easy to see that the space $\mathcal{B}_{\nu,0}$ is a closed subspace in $\mathcal{B}_\nu$. The Bloch type spaces have appeared in studies of composition, differentiation and integral operators. For instance, S. Stević and his collaborators have many studies about these operators, as well as product type operators containing them, acting from or to Bloch type spaces, as well as other spaces with weight functions; see [3, 4, 8, 11–14] and the related references therein.

One of the major subjects in the fields of analytic function spaces and operator theory are composition operators. For an analytic self-map φ of $\mathbb{D}$, the composition operator C_φ is defined by $C_\varphi f = f \circ \varphi$ ($f \in H(\mathbb{D})$). This composition operator has been studied extensively on various analytic function spaces. The aim of these studies is to explore the relation between operator-theoretic behaviors of C_φ and function-theoretic properties of the map φ. Over the past few decades, a considerable number of studies have been conducted on the difference of composition operators on analytic function spaces. Shapiro and Sundberg [9] and MacCluer et al. [6] studied a compact difference of composition operators on the Hardy spaces and topological structures of the space of composition operators. Hosokawa and Ohno [1] have considered the same operator acting on the Bloch spaces. They used the pseudo-hyperbolic metric to give equivalent conditions for the compactness of the difference of composition operators. After that, several authors [2, 5, 15, 16, 18] have studied the difference of composition type operators acting between two different analytic function spaces.

Motivated by Zhu and Yang's results [18] on the difference of composition operators from the weighted Bergman space into Bloch space, we have investigated recently this type operator from the analytic Besov space B^p into the Bloch type space $\mathcal{B}_\nu$ in [10]. In that paper, we dealt with the case $C_\varphi - C_\psi : B^p \to \mathcal{B}_\nu$ only. The results in [2, 5, 15, 16, 18], as well as our recent ones in [10], do not deal with the case when the range space of $C_\varphi - C_\psi$ is a little-type space. This case remains as a matter to be discussed further. Hence the purpose of this paper is to describe equivalent conditions for the compactness of $C_\varphi - C_\psi : B^p \to \mathcal{B}_{\nu,0}$. When we consider the case that the range space of $C_\varphi - C_\psi$ is different from its domain space, we have to take notice of the boundedness of it because a pair $\{\varphi, \psi\}$ does not always induce the bounded difference of composition operators. In Section 3, we will give characterizations for the boundedness of $C_\varphi - C_\psi$ which the range space is $\mathcal{B}_{\nu,0}$. By applying this result for the boundedness, we will describe characterizations for the compactness of $C_\varphi - C_\psi$. Section 4 is devoted to explain the details of them.

Throughout this paper, the notation $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$. Of course, the constant C is independent of a function f, a point $z \in \mathbb{D}$ and related parameters $\{t, r\}$. Moreover, if both $A \lesssim B$ and $B \lesssim A$ hold, then one says that $A \approx B$.

2. Preliminaries

We will need the following results in Section 3 and 4. The following lemma is folklore, but we include a proof of it for completeness.

Lemma 2.1. *Let $1 \leq p < \infty$ and $f \in B^p$. Then*

$$|f'(z)| \lesssim \frac{\|f\|_p}{1 - |z|^2}$$

for all $z \in \mathbb{D}$.

Proof. We have to consider the following two cases: $p \neq 1$, and $p = 1$. For the case $p \neq 1$, by the definition of the space B^p, $f \in B^p$ if and only if f' belongs to the classical weighted Bergman space $L^p_a(dA_{p-2})$. Hence f' has the following point evaluation estimate:

$$|f'(z)| \leq \frac{\|f'\|_{L^p_a(dA_{p-2})}}{1 - |z|^2}$$

for all $z \in \mathbb{D}$. Since $\|f'\|_{L^p_a(dA_{p-2})} \leq C\|f\|_p$, we obtain the desired estimate. To prove the case $p = 1$, we use the atomic decomposition of $f \in B^1$. If $f \in B^1$, we can choose sequences $(c_j)_{j\in\mathbb{N}} \in l^1$ and $(w_j)_{j\in\mathbb{N}} \subset \mathbb{D}$ such that $f = c_0 + \sum c_j \alpha_{w_j}$. Thus we have $|f(z)| \lesssim \sum |c_j|$ for all $z \in \mathbb{D}$. By taking the infimum with respect to all such representation of f, we obtain $|f(z)| \lesssim \|f\|_1$ for all $z \in \mathbb{D}$. An application of Cauchy's estimate to f' on the circle with center at z and radius $(1 - |z|)/2$ shows $|f'(z)| \lesssim \|f\|_1/(1 - |z|^2)$ for all $z \in \mathbb{D}$. □

Lemma 2.2. *Let $1 \leq p < \infty$ and $f \in B^p$. Then*

$$|(1 - |z|^2)f'(z) - (1 - |w|^2)f'(w)| \lesssim \|f\|_p \rho(z, w)$$

for all $\{z, w\} \subset \mathbb{D}$. Here $\rho(z, w)$ denotes the pseudohyperbolic distance for $z, w \in \mathbb{D}$, that is, $\rho(z, w) = \left|\frac{z-w}{1-z\overline{w}}\right|$.

Proof. In [1, Proposition 2.2], Hosokawa and Ohno proved that

$$|(1 - |z|^2)f'(z) - (1 - |w|^2)f'(w)| \lesssim \rho(z, w) \sup_{\zeta\in\mathbb{D}}(1 - |\zeta|^2)|f'(\zeta)|$$

for f belonging to the Bloch space $\mathcal{B}$ and $\{z, w\} \subset \mathbb{D}$. Since Lemma 2.1 implies that $B^p \subset \mathcal{B}$ $(1 \leq p < \infty)$ and $\sup_{\zeta\in\mathbb{D}}(1 - |\zeta|^2)|f'(\zeta)| \lesssim \|f\|_p$, the desired estimate can be verified by the above estimate. □

A compact subset of $\mathcal{B}_{\nu,0}$ can be characterized as following. The same result for the usual little Bloch space $\mathcal{B}_0$ was proved by Madigan and Matheson [7]. By a slightly modification of their proof, we can prove the following lemma.

Lemma 2.3. *A closed subset L in $\mathcal{B}_{\nu,0}$ is compact if and only if it is a bounded subset in $\mathcal{B}_\nu$ and satisfies*

$$\lim_{|z|\to 1^-} \sup_{f\in L} \nu(z)|f'(z)| = 0.$$

The following result is appeared in our previous work [10]. We will need it in the compactness argument in Section 4.

Theorem 2.4. *Let $1 \le p < \infty$ and $\{\varphi, \psi\}$ a pair of analytic self-maps of $\mathbb{D}$. Then the following statements are equivalent:*

(i) $C_\varphi - C_\psi : B^p \to \mathcal{B}_\nu$ *is bounded,*

(ii) φ *and* ψ *satisfy the following two conditions:*

$$\sup_{z\in\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\rho(\varphi(z),\psi(z)) < \infty$$

and

$$\sup_{z\in\mathbb{D}} \left|\frac{\nu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\nu(z)\psi'(z)}{1-|\psi(z)|^2}\right| < \infty,$$

(iii) φ *and* ψ *satisfy the following two conditions:*

$$\sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi'(z)|}{1-|\psi(z)|^2}\rho(\varphi(z),\psi(z)) < \infty$$

and

$$\sup_{z\in\mathbb{D}} \left|\frac{\nu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\nu(z)\psi'(z)}{1-|\psi(z)|^2}\right| < \infty.$$

3. Boundedness of $C_\varphi - C_\psi$

Before considering the compactness of $C_\varphi - C_\psi$, we have to mention the boundedness of it. The following Theorem 3.1 can be found in [1, Theorem 3.4]. They proved the result for the case that $C_\varphi - C_\psi$ is acting on the little Bloch space $\mathcal{B}_0$. Under the assumption on the boundedness of $C_\varphi - C_\psi$ and the density of the polynomial set in the domain space, we can generalize their result as following.

Theorem 3.1. *Let X be a Banach space of analytic functions over $\mathbb{D}$ such that the polynomial set is dense in X. For each pair $\{\varphi, \psi\}$ of analytic self-maps of $\mathbb{D}$ with $C_\varphi - C_\psi : X \to \mathcal{B}_\nu$ is bounded, the following conditions are equivalent:*

(a) $C_\varphi - C_\psi : X \to \mathcal{B}_{\nu,0}$ *is bounded,*

(b) $\varphi - \psi \in \mathcal{B}_{\nu,0}$ *and* $\varphi^2 - \psi^2 \in \mathcal{B}_{\nu,0}$*,*

(c) $\varphi - \psi \in \mathcal{B}_{\nu,0}$ *and*

$$\lim_{|z|\to 1^-} \nu(z)|\varphi(z) - \psi(z)| \max\{|\varphi'(z)|, |\psi'(z)|\} = 0.$$

Proof. The direction (a) ⇒ (b) is verified by test functions $p_1(z) = z$ and $p_2(z) = z^2$ easily. Hence it is enough to prove directions (b) ⇒ (c) and (c) ⇒ (a). Now we will prove (b) ⇒ (c). Since $\varphi - \psi \in \mathcal{B}_{\nu,0}$ implies $\nu(z)|\varphi'(z) - \psi'(z)| \to 0$ as $|z| \to 1^-$ and $\varphi^2 - \psi^2 \in \mathcal{B}_{\nu,0}$ implies $\nu(z)|\varphi(z)\varphi'(z) - \psi(z)\psi'(z)| \to 0$ as $|z| \to 1^-$, we obtain that

$$\begin{aligned}
&\nu(z)|\varphi(z) - \psi(z)||\varphi'(z)| \\
&\le \nu(z)|\varphi(z)\varphi'(z) - \psi(z)\psi'(z)| + \nu(z)|\varphi'(z) - \psi'(z)||\psi(z)| \\
&\le \nu(z)|\varphi(z)\varphi'(z) - \psi(z)\psi'(z)| + \nu(z)|\varphi'(z) - \psi'(z)| \to 0,
\end{aligned}$$

as $|z| \to 1^-$. Similarly it is proved that $\nu(z)|\varphi(z) - \psi(z)||\psi'(z)| \to 0$ as $|z| \to 1^-$, and so the condition (c) is true. In order to prove (c) ⇒ (a), we assume (c). For each $n \ge 1$, we put $p_n(z) = z^n$. Then

$$(C_\varphi - C_\psi)p_n(z) = \varphi^n(z) - \psi^n(z) = (\varphi(z) - \psi(z)) \sum_{k=0}^{n-1} \varphi^{n-1-k}(z)\psi^k(z).$$

We will claim that $(C_\varphi - C_\psi)p_n \in \mathcal{B}_{\nu,0}$. Since

$$\begin{aligned}
&\left(\sum_{k=0}^{n-1} \varphi^{n-1-k}(z)\psi^k(z)\right)' \\
&= (n-1)\varphi^{n-2}(z)\varphi'(z) + (n-2)\varphi^{n-3}(z)\varphi'(z)\psi(z) + \varphi^{n-2}(z)\psi'(z) \\
&\qquad + \cdots + \varphi'(z)\psi^{n-2}(z) + (n-2)\varphi(z)\psi^{n-3}(z)\psi'(z) + (n-1)\psi^{n-2}(z)\psi'(z),
\end{aligned}$$

we have that

$$\left|\left(\sum_{k=0}^{n-1} \varphi^{n-1-k}(z)\psi^k(z)\right)'\right| \le \frac{n(n-1)}{2}(|\varphi'(z)| + |\psi'(z)|).$$

Hence this inequality gives that

$$\begin{aligned}
&\nu(z)|((C_\varphi - C_\psi)p_n)'(z)| \\
&\le n\nu(z)|\varphi'(z) - \psi'(z)| + \frac{n(n-1)}{2}\nu(z)|\varphi(z) - \psi(z)|(|\varphi'(z)| + |\psi'(z)|).
\end{aligned}$$

Combining this estimate with the condition (c), we see that $(C_\varphi - C_\psi)p_n \in \mathcal{B}_{\nu,0}$, and so $(C_\varphi - C_\psi)p \in \mathcal{B}_{\nu,0}$ for each polynomial p. Since the polynomial set is dense in X, $C_\varphi - C_\psi : X \to \mathcal{B}_\nu$ is bounded and $\mathcal{B}_{\nu,0}$ is closed in $\mathcal{B}_\nu$, we also see $(C_\varphi - C_\psi)f \in \mathcal{B}_{\nu,0}$ for $f \in X$. This implies the boundedness of $C_\varphi - C_\psi : X \to \mathcal{B}_{\nu,0}$. □

The following lemma is also folklore. We include a standard proof of it for the benefit of the reader.

Lemma 3.2. *For $1 \le p < \infty$, the polynomial set is dense in B^p.*

Proof. Each dilated function f_r is analytic in the closed unit disk $\overline{\mathbb{D}}$, and so it belongs to B^p. Since f_r is approximated by polynomials in B^p, it is enough to prove that every $f \in B^p$ satisfies $\|f - f_r\|_p \to 0$ as $r \to 1^-$. First we will consider the case $p > 1$. Fix $\varepsilon > 0$. Since $f \in B^p$, there exists an $R \in (0,1)$ such that $\int_{\mathbb{D}\setminus R\overline{\mathbb{D}}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z) < \varepsilon$. Noting that $|f'|^p$ is subharmonic in $\mathbb{D}$, we have

$$\begin{aligned}
\int_{\mathbb{D}\setminus R\overline{\mathbb{D}}} |f'(rz)|^p(1-|z|^2)^{p-2}dA(z) &= 2\int_R^1 t(1-t^2)^{p-2}dt \int_0^{2\pi} |f'(rte^{i\theta})|^p \frac{d\theta}{2\pi} \\
&\le 2\int_R^1 t(1-t^2)^{p-2}dt \int_0^{2\pi} |f'(te^{i\theta})|^p \frac{d\theta}{2\pi} \\
&= \int_{\mathbb{D}\setminus R\overline{\mathbb{D}}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z) < \varepsilon
\end{aligned}$$

for any $r \in (0,1)$. Hence we obtain

$$\begin{aligned}\|f - f_r\|_p^p &= \int_{\mathbb{D}} |f'(z) - rf'(rz)|^p (1-|z|^2)^{p-2} dA(z)\\ &\lesssim (1-r)^p \int_{\mathbb{D}} |f'(rz)|^p (1-|z|^2)^{p-2} dA(z)\\ &\qquad + \int_{\mathbb{D}} |f'(z) - f'(rz)|^p (1-|z|^2)^{p-2} dA(z)\\ &\lesssim (1-r)^p \|f\|_p^p + \varepsilon + \int_{R\overline{\mathbb{D}}} |f'(z) - f'(rz)|^p (1-|z|^2)^{p-2} dA(z).\end{aligned}$$

Since f' is uniformly continuous on $R\overline{\mathbb{D}}$, it follows from this estimate that $\|f - f_r\|_p \to 0$ as $r \to 1^-$. For the case $p = 1$, we obtain

$$\begin{aligned}\|f - f_r\|_1 &\lesssim \int_{\mathbb{D}} |f''(z) - r^2 f''(rz)| dA(z)\\ &\leq (1-r^2) \int_{\mathbb{D}} |f''(rz)| dA(z) + \int_{\mathbb{D}} |f''(rz) - f''(z)| dA(z).\end{aligned}$$

By the same argument as in the case $p > 1$, these inequalities also show that $\|f - f_r\|_1 \to 0$ as $r \to 1^-$. □

Corollary 3.3. *Let $1 \leq p < \infty$ and $\{\varphi, \psi\}$ a pair of analytic self-maps of $\mathbb{D}$ which induces the bounded operator $C_\varphi - C_\psi : B^p \to \mathcal{B}_\nu$. Then the following conditions are equivalent:*

(a) $C_\varphi - C_\psi : B^p \to \mathcal{B}_{\nu,0}$ is bounded,
(b) $\varphi - \psi \in \mathcal{B}_{\nu,0}$ and $\varphi^2 - \psi^2 \in \mathcal{B}_{\nu,0}$,
(c) $\varphi - \psi \in \mathcal{B}_{\nu,0}$ and

$$\lim_{|z|\to 1^-} \nu(z)|\varphi(z) - \psi(z)| \max\{|\varphi'(z)|, |\psi'(z)|\} = 0.$$

4. Compactness of $C_\varphi - C_\psi$

Theorem 4.1. *Let $1 \leq p < \infty$. For each pair $\{\varphi, \psi\}$ of analytic self-maps of $\mathbb{D}$, $C_\varphi - C_\psi : B^p \to \mathcal{B}_{\nu,0}$ is compact if and only if φ and ψ satisfy the following two condtions:*

(a) $\displaystyle\lim_{|z|\to 1^-} \max\left\{\frac{|\varphi'(z)|}{1-|\varphi(z)|^2}, \frac{|\psi'(z)|}{1-|\psi(z)|^2}\right\} \nu(z)\rho(\varphi(z), \psi(z)) = 0,$

(b) $\displaystyle\lim_{|z|\to 1^-} \left|\frac{\nu(z)\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\nu(z)\psi'(z)}{1-|\psi(z)|^2}\right| = 0.$

Proof. First, we assume that conditions (a) and (b) hold. Let $K = \{f \in B^p : \|f\|_p \leq 1\}$ be the closed unit ball in B^p. In order to prove the compactness of $C_\varphi - C_\psi : B^p \to \mathcal{B}_{\nu,0}$, by Lemma 2.3, we may prove that $C_\varphi - C_\psi : B^p \to \mathcal{B}_{\nu,0}$ is bounded and

$$\lim_{|z|\to 1} \sup_{f\in K} \nu(z)|((C_\varphi - C_\psi)f)'(z)| = 0. \tag{2}$$

By Theorem 2.4 we see that (a) and (b) imply the boundedness of $C_\varphi - C_\psi : B^p \to \mathcal{B}_\nu$. Thus we will claim that φ and ψ satisfy the condition (c) in Corollary 3.3. Since $|\varphi(z) - \psi(z)| \leq 2\rho(\varphi(z), \psi(z))$ for $z \in \mathbb{D}$, we see that

$$\nu(z)|\varphi(z) - \psi(z)||\varphi'(z)| \leq 2\frac{\nu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\rho(\varphi(z), \psi(z)),$$

and

$$\nu(z)|\varphi(z)-\psi(z)||\psi'(z)| \le 2\frac{\nu(z)|\psi'(z)|}{1-|\psi(z)|^2}\rho(\varphi(z),\psi(z)),$$

and so the condition (a) shows

$$\lim_{|z|\to 1}\nu(z)|\varphi(z)-\psi(z)|\max\{|\varphi'(z)|,|\psi'(z)|\}=0.$$

Moreover we also obtain that

$$\begin{aligned}
&\nu(z)|\varphi'(z)-\psi'(z)|\\
&=\nu(z)\left|\frac{(1-|\varphi(z)|^2)\varphi'(z)}{1-|\varphi(z)|^2}-\frac{(1-|\psi(z)|^2)\psi'(z)}{1-|\psi(z)|^2}\right|\\
&\le\left|\frac{\nu(z)\varphi'(z)}{1-|\varphi(z)|^2}-\frac{\nu(z)\psi'(z)}{1-|\psi(z)|^2}\right|\\
&\qquad+\nu(z)\left|-\frac{|\varphi(z)|^2\varphi'(z)}{1-|\varphi(z)|^2}+\frac{|\psi(z)|^2\varphi'(z)}{1-|\varphi(z)|^2}-\frac{|\psi(z)|^2\varphi'(z)}{1-|\varphi(z)|^2}+\frac{|\psi(z)|^2\psi'(z)}{1-|\psi(z)|^2}\right|\\
&\le 2\left|\frac{\nu(z)\varphi'(z)}{1-|\varphi(z)|^2}-\frac{\nu(z)\psi'(z)}{1-|\psi(z)|^2}\right|+\left||\varphi(z)|^2-|\psi(z)|^2\right|\frac{\nu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\\
&\le 2\left|\frac{\nu(z)\varphi'(z)}{1-|\varphi(z)|^2}-\frac{\nu(z)\psi'(z)}{1-|\psi(z)|^2}\right|+4\rho(\varphi(z),\psi(z))\frac{\nu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}.
\end{aligned}$$

Hence (a) and (b) show that $\nu(z)|\varphi'(z)-\psi'(z)|\to 0$ as $|z|\to 1^-$, that is $\varphi-\psi\in\mathcal{B}_{\nu,0}$. By Corollary 3.3, we see that $C_\varphi - C_\psi : B^p\to\mathcal{B}_{\nu,0}$ is bounded.

Now we prove that (2) holds. Fix $z\in\mathbb{D}$ and $f\in K$. Thus we have

$$\begin{aligned}
&\nu(z)|((C_\varphi-C_\psi)f)'(z)|\\
&=\nu(z)|f'(\varphi(z))\varphi'(z)-f'(\psi(z))\psi'(z)|\\
&=\nu(z)\left|\frac{\varphi'(z)}{1-|\varphi(z)|^2}(1-|\varphi(z)|^2)f'(\varphi(z))-\frac{\psi'(z)}{1-|\psi(z)|^2}(1-|\psi(z)|^2)f'(\psi(z))\right|\\
&\le\left|\frac{\nu(z)\varphi'(z)}{1-|\varphi(z)|^2}-\frac{\nu(z)\psi'(z)}{1-|\psi(z)|^2}\right|(1-|\varphi(z)|^2)|f'(\varphi(z))|\\
&\qquad+\left|(1-|\varphi(z)|^2)f'(\varphi(z))-(1-|\psi(z)|^2)f'(\psi(z))\right|\frac{\nu(z)|\psi'(z)|}{1-|\psi(z)|^2}.
\end{aligned}$$

Combining this with Lemma 2.1 and 2.2, we obtain

$$\nu(z)|((C_\varphi-C_\psi)f)'(z)|\lesssim\left|\frac{\nu(z)\varphi'(z)}{1-|\varphi(z)|^2}-\frac{\nu(z)\psi'(z)}{1-|\psi(z)|^2}\right|+\frac{\nu(z)|\psi'(z)|}{1-|\psi(z)|^2}\rho(\varphi(z),\psi(z))$$

for any $z\in\mathbb{D}$ and $f\in K$. Conditions (a) and (b) imply (2). By Lemma 2.3 we see that $(C_\varphi-C_\psi)(K)$ is a compact subset in $\mathcal{B}_{\nu,0}$. Hence $C_\varphi-C_\psi : B^p\to\mathcal{B}_{\nu,0}$ is compact.

To prove that the compactness of $C_\varphi-C_\psi$ gives conditions (a) and (b), we take an arbitrary sequence $(z_n)_{n\in\mathbb{N}}$ of $\mathbb{D}$ with $|z_n|\to 1^-$ as $n\to\infty$. Moreover we may assume that $|\varphi(z_n)|>1/2$ and $|\psi(z_n)|>1/2$ for sufficiently large n. Put

$$f_n(z)=\frac{\varphi(z_n)-z}{1-\overline{\varphi(z_n)}z},\quad\text{and}\quad g_n(z)=\left(\frac{\varphi(z_n)-z}{1-\overline{\varphi(z_n)}z}\right)^2$$

for $n \geq 1$ and $z \in \mathbb{D}$. By Forelli-Rudin estimate (cf. [19, Lemma 3.10]), we see $\{f_n, g_n\} \subset B^p$ and can choose a positive constant C which is independent of n, φ and ψ such that $\|f_n\|_p \leq C$ and $\|g_n\|_p \leq C$. Let $K_C = \{f \in B^p : \|f\|_p \leq C\}$. By Lemma 2.3, the compactness of $C_\varphi - C_\psi$ implies

$$\lim_{n\to\infty} \sup_{f\in K_C} \nu(z_n)|((C_\varphi - C_\psi)f)'(z_n)| = 0. \tag{3}$$

By the definition of f_n, we have

$$\begin{aligned}
&\nu(z_n)|((C_\varphi - C_\psi)f_n)'(z_n)| \\
&= \nu(z_n)\left|\frac{-\varphi'(z_n)}{1-|\varphi(z_n)|^2} + \frac{\psi'(z_n)(1-|\varphi(z_n)|^2)}{(1-\overline{\varphi(z_n)}\psi(z_n))^2}\right| \\
&\geq \left|\frac{\nu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} - \frac{\nu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2}(1-\rho(\varphi(z_n),\psi(z_n))^2)\right|.
\end{aligned} \tag{4}$$

Since $0 \leq \rho(\varphi(z_n), \psi(z_n)) < 1$, (3) and (4) give

$$\begin{aligned}
\lim_{n\to\infty} \bigg| &\frac{\nu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2}\rho(\varphi(z_n),\psi(z_n)) \\
&- \frac{\nu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2}\rho(\varphi(z_n),\psi(z_n))(1-\rho(\varphi(z_n),\psi(z_n))^2)\bigg| = 0.
\end{aligned} \tag{5}$$

Since $g_n'(\varphi(z_n)) = 0$, we have

$$\begin{aligned}
&\nu(z_n)|((C_\varphi - C_\psi)g_n)'(z_n)| \\
&= 2\nu(z_n)\frac{|\varphi(z_n)-\psi(z_n)||\psi'(z_n)|}{|1-\overline{\varphi(z_n)}\psi(z_n)|^3}(1-|\varphi(z_n)|^2) \\
&= 2\frac{\nu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2}\rho(\varphi(z_n),\psi(z_n))(1-\rho(\varphi(z_n),\psi(z_n))^2).
\end{aligned}$$

The equation (3) also gives

$$\lim_{n\to\infty} \frac{\nu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2}\rho(\varphi(z_n),\psi(z_n))(1-\rho(\varphi(z_n),\psi(z_n))^2) = 0. \tag{6}$$

(5) and (6) show that

$$\lim_{n\to\infty} \frac{\nu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2}\rho(\varphi(z_n),\psi(z_n)) = 0. \tag{7}$$

By replacing the role of φ and ψ in definitions of $\{f_n, g_n\}$ and the above argument, we also have that

$$\lim_{n\to\infty} \frac{\nu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2}\rho(\varphi(z_n),\psi(z_n)) = 0. \tag{8}$$

Since $(z_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ with $|z_n| \to 1$ as $n \to \infty$ was arbitrary, (7) and (8) imply the condition (a) holds. Furthemore, the estimate (4) gives

$$\begin{aligned}
&\nu(z_n)|((C_\varphi - C_\psi)f_n)'(z_n)| \\
&\geq \left|\frac{\nu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} - \frac{\nu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2}\right| - \frac{\nu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2}\rho(\varphi(z_n),\psi(z_n)).
\end{aligned}$$

By (3) and (8), we obtain

$$\lim_{n\to\infty}\left|\frac{\nu(z_n)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2}-\frac{\nu(z_n)|\psi'(z_n)|}{1-|\psi(z_n)|^2}\right|=0,$$

and so this indicates the condition (b). □

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