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The Global Behavior of a Certain Difference Polynomial Equation

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Abstract. In this paper we will present the Julia set and the global behavior of a polynomial second-order difference equation of type

$$x_{n+1} = ax_n^m x_{n-1} + ax_{n-1}^{m+1} + bx_{n-1}$$

where $m \in \mathbb{N}$, $a > 0$ and $b \geq 0$ with non-negative initial conditions.

1. Introduction

In general, polynomial difference equations and polynomial maps in the plane have been studied in both the real and complex domains (see [10, 11]). First results on quadratic polynomial difference equation have been obtained in [1, 2] but these results gave us only a part of the basins of attraction of equilibrium points and period-two solutions. In [4], the general second-order difference equation is completely investigated and described the regions of initial conditions in the first quadrant for which all solutions tend to equilibrium points, period-two solutions, or the point at infinity, except for the case of infinitely many period-two solutions. In [3], case of infinitely many period-two solutions is completely investigated and corresponding difference equation is special case of equation $x_{n+1} = ax_n^m x_{n-1} + ax_{n-1}^{m+1} + bx_{n-1}$ for $m = 1$. Our principal tool is the theory of monotone maps, and in particular cooperative maps, applied to the system

$$\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= f(v_n, u_n), \end{aligned}$$

where f is a continuous and increasing function in both variables, which guarantee the existence and uniqueness of the stable and unstable manifolds for the fixed points and periodic points (see [7]). If we set $u_n = x_{n-1}$ and $v_n = x_n$ for $n = 0, 1, 2, \ldots$, we obtain the results that are based on the theorems which hold for monotone difference equations. Hence, the method we discussed in this paper is applicable to some special types of difference systems. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{1}$$

where f is a continuous and increasing function in both variables. The following result has been obtained in [1]:

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Theorem 1.1. *Let $I \subseteq R$ and let $f \in C[I \times I, I]$ be a function which increases in both variables. Then for every solution of Eq.*(1) *the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:*

(*i*) *Eventually they are both monotonically increasing.*

(*ii*) *Eventually they are both monotonically decreasing.*

(*iii*) *One of them is monotonically increasing and the other is monotonically decreasing.*

As a consequence of Theorem 1.1 every bounded solution of Eq.(1) approaches either an equilibrium solution or period-two solution and every unbounded solution is asymptotic to the point at infinity in a monotonic way. Thus the major problem in dynamics of Eq.(1) is the problem of determinig the basins of attraction of three different types of attractors: the equilibrium solutions, period-two solution(s) and the point(s) at infinity. The following result can be proved by using the techniques of proof of Theorem 11 in [7].

Theorem 1.2. *Consider Eq.*(1) *where f is increasing function in its arguments and assume that there is no minimal period-two solution. Assume that $E_1(x_1, y_1)$ and $E_2(x_2, y_2)$ are two consecutive equilibrium points in North-East ordering that satisfy*

$$(x_1, y_1) \preceq_{ne} (x_2, y_2)$$

and that E_1 is a local attractor and E_2 is a saddle point or a non-hyperbolic point with second characteristic root in interval $(-1, 1)$, with the neighborhoods where f is strictly increasing. Then the basin of attraction $\mathcal{B}(E_1)$ of E_1 is the region below the global stable manifold $\mathcal{W}^s(E_2)$. More precisely

$$\mathcal{B}(E_1) = \{(x, y) : \exists y_u : \ y < y_u, \quad (x, y_u) \in \mathcal{W}^s(E_2)\}.$$

The basin of attraction $\mathcal{B}(E_2) = \mathcal{W}^s(E_2)$ is exactly the global stable manifold of E_2. The global stable manifold extend to the boundary of the domain of Eq.(1)*. If there exists a period-two solution, then the end points of the global stable manifold are exactly the period two solution.*

Now, the theorems that are applied in [7] provided the two continuous curves $\mathcal{W}^s(E_2)$ (stable manifold) i $\mathcal{W}^u(E_2)$ (unstable manifold), both passing through the point $E_2(x_2, y_2)$ from Theorem 1.2, such that $\mathcal{W}^s(E_2)$ is a graph of decreasing function and $\mathcal{W}^u(E_2)$ is a graph of an increasing function. The curve $\mathcal{W}^s(E_2)$ splits the first quadrant of initial conditions into two disjoint regions, but we do not know the explicit form of the curve $\mathcal{W}^u(E_2)$. In this paper we investigate the following difference equation

$$x_{n+1} = ax_n^m x_{n-1} + ax_{n-1}^{m+1} + bx_{n-1} \tag{2}$$

where $m \in \mathbb{N}$, $a > 0$ and $b \geq 0$, that has infinitely many period-two solutions and we expose the explicit form of the curve that separates the first quadrant into two basins of attraction of a locally stable equilibrium point and of the point at infinity. In complex domain, if $f(z) = \frac{P(z)}{Q(z)}$, where $z \in \mathbb{C} \cup \{\infty\}$ and P and Q are polynomials without common divisors, then Julia set J_f is the set of points z which do not approach infinity after $f(z)$ is repeatedly applied (corresponding to a attractor). At the same way, in real domain, corresponding Julia set J_f is connected and it is boundary of set of initial conditions for which the orbit of $f(n)$ does not tend to infinity. One of the major problems in the dynamics of polynomial maps in real domain is determining the basin of attractions of the point at infinity and in particular the boundary of the that basin known as the Julia set. We precisely determined the Julia set of Eq.(2) (boundary of set of initial conditions in the first quadrant for which the solutions of Eq.(2) does not tend to infinity) and we obtained the global dynamics in the interior of the Julia set, which includes all the points for which solutions are not asymptotic to the point at infinity. It turned out that the Julia set for Eq.(2) is the union of the stable manifolds of some saddle equilibrium points, nonhyperbolic equilibrium points or period-two points. In general, it is very important to mention that there is no explicit form of stable and unstable manifolds for the fixed points and periodic points of any difference equation (or system of difference equations), so the disadvantage of all results is that these manifolds are continuous decreasing (increasing) functions of which the parametrization is uncomfortable and we can only obtain their asymptotic formulas by using the method of undetermined coefficients. So the advantage of our results is that we obtain the exact formula of our Julia set of Eq.(2). We first list some results needed for the proofs of our theorems. The main result for studying local stability of equilibria is linearized stability theorem (see Theorem 1.1 in [9]).

Theorem 1.3. *(linearized stability): Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}) \tag{3}$$

and let $\bar{x}$ *be an equilibrium point of difference equation* (3). *Let* $p = \frac{\partial f(\bar{x},\bar{x})}{\partial u}$ *and* $q = \frac{\partial f(\bar{x},\bar{x})}{\partial v}$ *denote the partial derivatives of* $f(u,v)$ *evaluated at the equilibrium* $\bar{x}$. *Let* λ_1 *and* λ_2 *roots of the quadratic equation* $\lambda^2 - p\lambda - q = 0$.

a) If $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$, *then the equilibrium* $\bar{x}$ *is locally asymptotically stable (sink).*

b) If $|\lambda_1| > 1$ *or* $|\lambda_2| > 1$, *then the equilibrium* $\bar{x}$ *is unstable.*

c) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1 \Leftrightarrow |p| < 1 - q < 2$. *Equilibrium* $\bar{x}$ *is a sink.*

d) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1 \Leftrightarrow |q| > 1$ *and* $|p| < |1 - q|$. *Equilibrium* $\bar{x}$ *is a repeller.*

e) $|\lambda_1| > 1$ *and* $|\lambda_2| < 1 \Leftrightarrow |p| > |1 - q|$. *Equilibrium* $\bar{x}$ *is a saddle point.*

f) $|\lambda_1| = 1$ *or* $|\lambda_2| = 1 \Leftrightarrow |p| = |1 - q|$ *or* $q = -1$ *and* $|p| \leq 2$. *Equilibrium* $\bar{x}$ *is called a non-hyperbolic point.*

The next theorem (Theorem 1.4.1. in [8]) is a very useful tool in establishing bounds for the solutions of nonlinear equations in terms of the solutions of equations with known behaviour.

Theorem 1.4. *Let I be an interval of real numbers, let k be a positive integer, and let* $F : I^{k+1} \to I$ *be a function which is increasing in all its arguments. Assume that* $\{x_n\}_{n=-k}^{\infty}, \{y_n\}_{n=-k}^{\infty}$ *and* $\{z_n\}_{n=-k}^{\infty}$ *are sequences of real numbers such that*

$$x_{n+1} \leq F(x_n, \ldots, x_{n-k}), \quad n = 0, 1, \ldots$$

$$y_{n+1} = F(y_n, \ldots, y_{n-k}), \quad n = 0, 1, \ldots$$

$$z_{n+1} \geq F(z_n, \ldots, z_{n-k}), \quad n = 0, 1, \ldots$$

and

$$x_n \leq y_n \leq z_n, \quad for \quad all \quad -k \leq n \leq 0.$$

Then

$$x_n \leq y_n \leq z_n, \quad for \quad all \quad n > 0.$$

2. Main results

By using the Theorem 1.3, we obtained the following result on local stability of the zero equilibrium of Eq.(2):

Proposition 2.1. *The zero equilibrium of Eq.(2) is one of the following:*

a) locally asymptotically stable if $b < 1$, *b) non-hyperbolic and locally stable if* $b = 1$,

c) unstable if $b > 1$.

Set $f(x, y) = ax^m y + ay^{m+1} + by$ and let $p = \frac{\partial f(\bar{x},\bar{x})}{\partial x}$ and $q = \frac{\partial f(\bar{x},\bar{x})}{\partial y}$ denote the partial derivatives of $f(x, y)$ evaluated at the equilibrium $\bar{x}$. The linearized equation at the positive equilibrium $\bar{x}$ is

$$\begin{aligned} z_{n+1} &= pz_n + qz_{n-1}, \\ p &= am\bar{x}^m, \\ q &= a(m+2)\bar{x}^m + b. \end{aligned}$$

Now, in view of Theorem 1.3 we obtain the following results on local stability of the positive equilibrium of Eq.(2):

Proposition 2.2. *The positive equilibrium of Eq.(2) is one of the following:*

a) locally asymptotically stable if $p + q < 1$,

b) non-hyperbolic and locally stable if $p + q = 1$,

c) unstable if $p + q > 1$,

d) saddle point if $p > |q - 1|$,

e) repeller if $1 - q < p < q - 1$.

Theorem 2.3. *If* $b \geq 1$ *then every solution* $\{x_n\}$ *of Eq.(2) satisfies* $\lim_{n\to\infty} x_n = \infty$.

Proof. *If* $\{x_n\}$ *is a solution of Eq.(2) then* $\{x_n\}$ *satisfies the inequality*

$$x_{n+1} \geq bx_{n-1}, \quad n = 0, 1, \ldots$$

which in view of the result on difference inequalities, see Theorem 1.4, implies that $x_n \geq y_n$, $n \geq 1$ *where* $\{y_n\}$ *is a solution of the initial value problem*

$$y_{n+1} = by_{n-1}, \quad y_{-1} = x_{-1} \text{ and } y_0 = x_0 \quad n = 0, 1, \ldots$$

Consequently, $x_0, x_{-1} > 0$ *then* $y_0, y_{-1} > 0$, $y_n \geq 0$ *for all n, and*

$$x_n \geq y_n = \lambda_1 \sqrt{b}^n + \lambda_2 \left(-\sqrt{b}\right)^n, \quad n = 1, 2, \ldots$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ *such that* $y_n \geq 0$ *for all n, which implies* $\lim_{n\to\infty} x_n = \infty$. □

Theorem 2.4. *Consider the difference equation (2) in the first quadrant of initial conditions, where* $m \in \mathbb{N}$, $a > 0$ *and* $0 \leq b < 1$. *Then Eq.(2) has a zero equilibrium and a unique positive equilibrium* $\bar{x}_+ = \sqrt[m]{\frac{1-b}{2a}}$. *The curve* $a(x^m + y^m) = 1 - b$ *is the Julia set and separates the first quadrant into two regions: the region below the given curve is the basin of attraction of point* $E_0(0, 0)$, *the region above the curve is the basin of attraction of the point at infinity and every point on the curve except* $E_+(\bar{x}_+, \bar{x}_+)$ *is a period-two solution of Eq.(2)*

Proof. The equilibrium points of Eq. (2) are the solutions of equation $x(ax^m + ax^m + b) = x$ that is equivalent to

$$x(2ax^m + b - 1) = 0, \tag{4}$$

which implies that Eq. (4) has two equilibria: zero equilibrium and unique positive equilibrium $\bar{x}_+$. Since $b \geq 0$ and $b < 1$, then by applying Proposition (2.1) the zero equilibrium is locally asymptotically stable.

Denote by $f(x,y) = ax^m y + ay^{m+1} + by$ and let p and q denote the partial derivatives of function $f(x,y)$ at point E_+. By straightforward calculation we obtaine that the following hold:

$$\begin{aligned} p+q &= (1-b)m+1, \\ q-p &= 1. \end{aligned}$$

Hence, by applying Proposition (2.2) the positive equilibrium is an unstable non-hyperbolic point. Period-two solution u, v satisfies the system

$$\begin{aligned} u &= (au^m + av^m + b)u \\ v &= (au^m + av^m + b)v. \end{aligned}$$

Obviously, the point $(0,0)$ is solution of the system above, but it is not period two solution. Hence, it has to be $v > 0$ which implies $au^m + av^m + b = 1$. Therefore every point of the set $\{(x,y) : ax^m + ay^m + b = 1\}$ is a period-two solution of Eq.(2) except point E_+. Now, we have to show that the curve $g(x,y) = ax^m + ay^m + b = 1$ is a graph of the decreasing function in the first quadrant. Let for some $x > 0$ there are y_1 and y_2 $(0 < y_1 < y_2)$ such that $g(x,y_1) = g(x,y_2) = 1$. As $g(x,y)$ is increasing in both variables then

$$1 = g(x,y_1) < g(x,y_2) = 1,$$

which is impossible. Thus the curve $g(x,y) = 1$ is the graph of function in the first quadrant. Further over $g(x,y) = 1$ then

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} y' = 0.$$

By applying the fact that is $g(x,y)$ is increasing in both variables we obtain $y' < 0$ in the first quadrant. Hence, $g(x,y) = 1$ is the graph of the decreasing function in the first quadrant. Let $\{x_n\}$ be a solution of Eq.(2) for initial condition (x_0, x_{-1}) which lies below the curve $g(x,y) = 1$. Then

$$\begin{aligned} g(x_0, x_{-1}) &= ax_0^m + ax_{-1}^m + b < 1, \\ x_{n+1} &= g(x_n, x_{n-1})\, x_{n-1}, \end{aligned}$$

and

$$\begin{aligned} x_1 &= g(x_0, x_{-1})\, x_{-1} < x_{-1}, \\ x_2 &= g(x_1, x_0)\, x_0 < g(x_{-1}, x_0)\, x_0 = g(x_0, x_{-1})\, x_0 < x_0. \end{aligned}$$

Thus (x_2, x_1) and (x_0, x_{-1}) are two points in North-East ordering $(x_2, x_1) \leq_{ne} (x_0, x_{-1})$ which means that the point (x_2, x_1) is also below the curve $g(x,y) = 1$ and also holds

$$g(x_2, x_1) < 1.$$

Similarly we find

$$\begin{aligned} x_3 &= g(x_2, x_1)\, x_1 < x_1, \\ x_4 &= g(x_3, x_2)\, x_2 < g(x_1, x_2)\, x_2 = g(x_2, x_1)\, x_2 < x_2. \end{aligned}$$

Continuing on this way we get

$$(0,0) \leq_{ne} \ldots \leq_{ne} (x_4, x_3) \leq_{ne} (x_2, x_1) \leq_{ne} (x_0, x_{-1})$$

which implies that both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically decreasing and bounded below by 0. Since below the curve $g(x,y) = 1$ there are no period-two solutions it must be $x_{2n} \to 0$ and $x_{2n+1} \to 0$. On the other hand, if we consider solution $\{x_n\}$ of Eq.(2) for initial condition (x_0, x_{-1}) which lies above the curve $g(x,y) = 1$ then $g(x_0, x_{-1}) > 1$ and by applying the method shown above we obtain the following condition:

$$(x_{-1}, x_0) \leq_{ne} (x_1, x_2) \leq_{ne} (x_3, x_4) \leq_{ne} \ldots$$

Therefore both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically increasing, hence $x_{2n} \to \infty$ and $x_{2n+1} \to \infty$ as $n \to \infty$. □

The figure 1 is visual illustration of Theorem 2.4 obtained by using Mathematica 9.0, with the boundaries of the basins of attraction obtained by using the software package Dynamica [6].

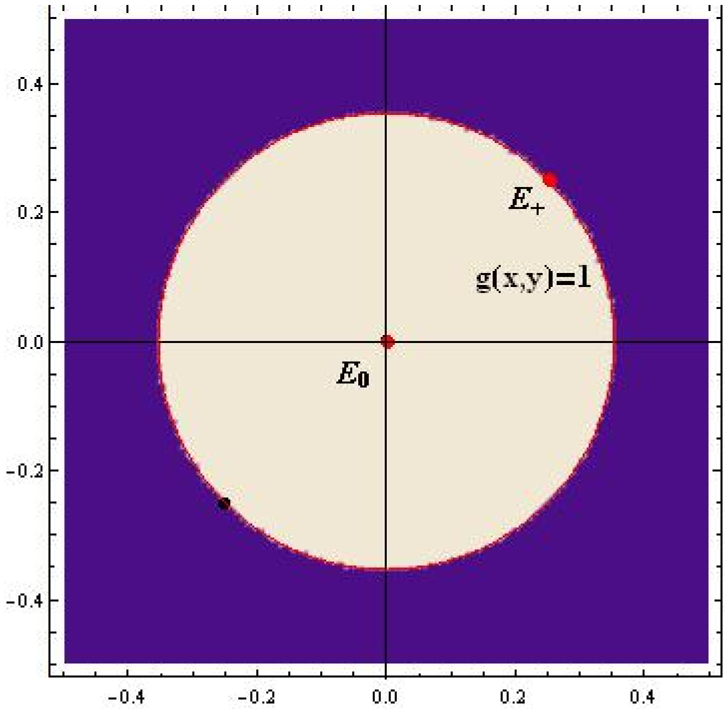

Figure 1

All this leads to the following theorem:

Theorem 2.5. *Consider the difference equation (2), where* $m = 2$, $a > 0$, $0 \leq b < 1$ *and initial conditions* $x_{-1}, x_0 \in \mathbb{R}$. *Then Eq.(2) has a zero equilibrium, positive equilibrium* $\bar{x}_+ = \sqrt{\frac{1-b}{2a}}$ *and negative equilibrium* $\bar{x}_- = -\sqrt{\frac{1-b}{2a}}$. *The curve* $a(x^2 + y^2) = 1 - b$ *is the Julia set and separates the real plane into two regions: the region inside the given curve is the basin of attraction of point* $E_0(0,0)$, *the region outside the curve is the basin of attraction of the point at infinity and every point on the curve except* $E_+(\bar{x}_+, \bar{x}_+)$ *and* $E_-(\bar{x}_-, \bar{x}_-)$ *is a period-two solution of Eq.(2).*

Proof. Since $m = 2$, it is clear that Eq. (2) has three equilibrium points. By applying Theorem 2.4 we obtain the zero equilibrium is locally asymptotically stable and the positive equilibrium $\bar{x}_+$ is an unstable non-hyperbolic point. Similarly, one can show that the negative equilibrium $\bar{x}_-$ is also an unstable non-hyperbolic point and every point on the given curve except E_+ and E_- is a period-two solution of Eq.(2). Set $\mathcal{S} = \left\{(x, y) : a(x^2 + y^2) < 1 - b\right\}$, $M_0(x_{-1}, x_0) \in \mathcal{S}$ and let $r_0 = d(M_0, E_0)$ denotes distance between point M_0 and E_0. Now, $M_0(x_{-1}, x_0) \in \mathcal{S} \Leftrightarrow a(x_{-1}^2 + x_0^2) < 1 - b \Leftrightarrow ar_0^2 < 1 - b$. Consider the sequence of points $\{M_n\}$ where $M_{n+1}(x_n, x_{n+1})$, $x_{n+1} = \left(ax_n^2 + ax_{n-1}^2 + b\right)x_{n-1}$ and $r_{n+1} = d(M_{n+1}, E_0)$ denotes distance between point M_{n+1} and E_0. If $M_n \in \mathcal{S}$ then

$$a(x_{n-1}^2 + x_n^2) < 1 - b \Leftrightarrow ar_n^2 + b < 1.$$

Therefore

$$\begin{aligned} r_{n+1}^2 &= x_n^2 + x_{n+1}^2 = \\ &= x_n^2 + \left(ax_n^2 + ax_{n-1}^2 + b\right)^2 x_{n-1}^2 = \\ &= x_n^2 + \left(ar_n^2 + b\right)^2 x_{n-1}^2 < x_n^2 + x_{n-1}^2 = r_n^2 \end{aligned}$$

which implies $r_{n+1} < r_n$ and $M_{n+1} \in \mathcal{S}$. Obviously, the sequence of positive real numbers $\{r_n\}$ is decreasing which guarantee that $\{r_n\}$ is convergent. In a view of Theorem 2.4, if $M_0\,(x_{-1}, x_0) \in \mathcal{S}$ and x_{-1}, x_0 are arbitrary nonnegative numbers, then every solution $\{x_n\}$ satisfies $\lim_{n\to\infty} x_n = 0$. Clearly, the following holds:

$$|x_{n+1}| = \left(ax_n^2 + ax_{n-1}^2 + b\right)|x_{n-1}| = \left(ar_n^2 + b\right)|x_{n-1}| < |x_{n-1}|,$$

which implies that both subsequences $\{|x_{2n}|\}$ and $\{|x_{2n+1}|\}$ are monotonically decreasing. Since there is no period-two solution of Eq. (2) in set $\mathcal{S}$ that leads $\{|x_{2n}|\}$ and $\{|x_{2n+1}|\}$ approache a zero equilibrium. Thus it must be $\lim_{n\to\infty} r_n = 0$. The case when M_0 is outside the curve $a(x^2 + y^2) = 1 - b$ is similar and will be omitted (the sequences $\{r_n\}$, $\{|x_{2n}|\}$ and $\{|x_{2n+1}|\}$ are increasing, since there is no period-two solution of Eq. (2) outside the curve that leads $\{|x_{2n}|\}$ and $\{|x_{2n+1}|\}$ approache the point at infinity). □

In view of Theorem 1.4 which implies results on difference inequalities we get the following:

Proposition 2.6. *Consider the difference equation of type*

$$x_{n+1} = Ax_n^m x_{n-1} + Bx_{n-1}^{m+1} + Cx_{n-1} \tag{5}$$

in the first quadrant of initial conditions, where the given parameters satisfy conditions $m \in \mathbb{N}$, $A > 0$, $B > 0$ *and* $0 \le C < 1$. *Then the global stable manifold of the positive equilibrium is between two curves*

$$p_1 : \min\{A, B\}(x^m + y^m) + C = 1 \tag{6}$$

and

$$p_2 : \max\{A, B\}(x^m + y^m) + C = 1 \tag{7}$$

Proof. It easy to show that Eq. (5) has two equilibria: zero equilibrium and unique positive equilibrium $\bar{x}_+ = \sqrt[m]{\frac{1-C}{A+B}}$. Since $C < 1$ the zero equilibrium is always locally asymptotically stable thus the positive equilibrium must be unstable equilibrium point. The theorems applied in [7] provided the following global behavior. More precisely, if the positive equilibrium is a saddle point or a non-hyperbolic point then there exists a global stable manifold which contains point $E_+(\bar{x}, \bar{x})$, where $\bar{x}$ is the positive equilibrium. In this case global behavior of Eq. (5) is described by Theorem 9 in [4]. If the positive equilibrium is a repeller then there exists a period-two solution and we obtaine that the period-two solution is a saddle point and there are two global stable manifolds which contain points $P_1(u, v)$ and $P_2(v, u)$ whre (u, v) is unique period-two solution of Eq.(5). In this case the global behavior of Eq.(5) is described by Theorem 10 in [4]. Although the Theorems 9 and 10 in [4] have been applied on a polynomial second-order difference equation they are sepcial cases of general Theorems in [7] applied on function f, where f is increasing function in its arguments. So, the global dynamics of Eq.(5) is exactly the same as the global dynamics of equations decribed by Theorems 9 and 10 in [4]. Furthermore

$$x_{n+1} = Ax_n^m x_{n-1} + Bx_{n-1}^{m+1} + Cx_{n-1} \ge \left(\min\{A, B\}(x^m + y^m) + C\right) x_{n-1},$$

and

$$x_{n+1} = Ax_n^m x_{n-1} + Bx_{n-1}^{m+1} + Cx_{n-1} \le \left(\max\{A, B\}(x^m + y^m) + C\right) x_{n-1}$$

for all n, by applying Theorem 1.4 for solution $\{x_n\}$ of Eq.(5) the following inequality holds

$$y_n \le x_n \le z_n,$$

for all n, where $\{y_n\}$ is a solution of the difference equation

$$y_{n+1} = \left(\min\{A, B\}(y_n^m + y_{n-1}^m) + C\right) y_{n-1} \tag{8}$$

and $\{z_n\}$ is a solution of the difference equation

$$z_{n+1} = \left(\max\{A, B\}(z_n^m + z_{n-1}^m) + C\right) z_{n-1} \tag{9}$$

Since Eq. (8) and Eq. (9) satisfy all conditions of Theorem 2.4 this implies that the statement of Proposition 2.6 holds. □

3. Conclusion

In this paper we restrict our attention to certain polynomial m degree second-order difference equation Eq. (2). It is important to mention that we have accurately determined the Julia set of Eq. (2) and the basins of attractions for the zero equilibrium and the positive equilibrium point. In general, all theoretical concepts which are very useful in proving the results of global attractivity of equilibrium points and period-two solutions only give us existence of global stable manifold(s) whose computation leads to very uncomfortable calculus (see [5, 6]).

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