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Initial Number of Lucas' Type Series for the Generalized Fibonacci Sequence

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Abstract. Initial numbers for Lucas' type series have so far been established only for Fibonacci $(2, 1)$ and Tribonacci $(3, 1, 3)$ sequences. Characteristics of stated series is their asymptotic relation with the exponent of the series constant. By using a simple procedure based on asymptotic relations of exponents of a sequences constant and Lucas' type series with the application of Nearest Integer Function - NIF, a general rule for initial numbers of Lucas' type series of Generalized Fibonacci sequence has been established, for the first time. All the gained initial numbers are integers, first initial number is always equal to the order of the sequence $F_n(0) = n$ and remaining are functionally dependent on order of the number and are equal to $F_n(k) = 2^{k-1} - 1$. This is premiere presentation of Prim-nacci sequence, too. Determinants of initial numbers of the Lucas' type series for the generalized Fibonacci sequences are a proven factorial function.

1. Introduction

Edouard Anatole Lucas (1842 - 1891) was a mathematician of the highest contribution to the study of Fibonacci series. In his works [1] he established a special type of Fibonacci sequence, denoted by $L(k)$. Universal rule of the Fibonacci sequences:

$$F(n) + F(n + 1) = F(n + 2), \tag{1}$$

with initial numbers for Lucas' series $L_2(0) = 2$ and $L_2(1) = 1$ gives:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ... \tag{2}$$

Binet formula can be easily adapted for calculating the numbers of Lucas' series [11], [7]. As in every Fibonacci sequences with arbitrary initial numbers, the quotient of two successive numbers of the series converges to the constant:

$$\lim_{k\to\infty} \frac{L_2(k + 1)}{L_2(k)} = \varphi_2 = 1,6180339... \tag{3}$$

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Key characteristic of ”golden ratio” constant for $k \in \mathbb{N}$ (4) analogue to basic rule of the sequence (1)

$$\varphi_2^k + \varphi_2^{k+1} = \varphi_2^{k+2} \tag{4}$$

Lucas' series and the exponent of the ”golden ratio” constant have asymptotic relation, which makes Lucas' series a main characteristic series of Fibonacci sequence:

$$\lim_{k\to\infty} \left(L_2(k) - \varphi_2^k\right) = 0 \tag{5}$$

Numerically, asymptotic relation is quickly noticed. Already at $n = 20$ the difference is less than 10^{-4}, and for $n = 40$ is less than 10^{-6}, etc.

Fibonacci and Lucas sequences [2], as well as Jacobshtal sequence [3] are investigated in numerous papers from the time of their discovery in 1843. to the present day [9]. On the other hand, research papers dealing with Tribonacci, Quatronacci and other sequences of higher degree, which are also based on Newton identities, are extremely rare [12]. It is interesting to note that each research of sequences, no matter of its degree, leads us to the most important function in Analytic Number Theory, the Riemann Zeta function [4, 8]. Due to that fact, it is understandable that each such research begins with initial numbers. Tribonacci sequence is determined with rule:

$$T(n) + T(n+1) + T(n+2) = T(n+3) \tag{6}$$

has its own Lucas' type series with determined initial numbers $L(0) = 3$, $L(1) = 1$, $L(2) = 3$ [6]:

$$3, 1, 3, 7, 11, 21, 39, 71, 131, 241, 443, 815, 1499, ... \tag{7}$$

Constant φ_3 is determined with convergence of sequential numbers of Tribonacci sequence,

$$\lim_{k\to\infty} \frac{L_3(k+1)}{L_3(k)} = \varphi_3 = 1.839286... = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right) \tag{8}$$

analogly fulfills the basic rule of Tribonacci sequence for constant φ_3

$$\varphi_3^k + \varphi_3^{k+1} + \varphi_3^{k+2} = \varphi_3^{k+3} \tag{9}$$

Like Fibonacci sequence, exponents of Tribonacci constant and Lucas' type series have characteristic asymptotic relation:

$$\lim_{k\to\infty} \left(L_3(k) - \varphi_3^k\right) = 0 \tag{10}$$

From Quatronacci sequence onwards ($n \geq 4$), the initial numbers of the Lucas' type series are not known. Rules for Lucas' type series for the Fibonacci sequence and Tribonacci sequence lead us to the rules for $n-$nacci sequence: Lucas' type series $n-$nacci sequences and exponents of $n-$nacci constant have asymptotic relation:

$$\lim_{k\to\infty} \left(L_4(k) - \varphi_4^k\right) = 0,\ \lim_{k\to\infty} \left(L_5(k) - \varphi_5^k\right) = 0,\ ... \lim_{k\to\infty} \left(L_n(k) - \varphi_n^k\right) = 0,\ ... \tag{11}$$

In addition to the rules for forming sequences, names, constants, and the main features of constants, Lucas' type series of arbitrary $n-$nacci sequence are the main characteristic of the Generalized Fibonacci sequence. $\varphi_n(k)$ is the main exponential ”string” of $n-$nacci sequence. Therefore, it is a particular challenge in determining the rules which are used for calculation of initial numbers of all Lucas' type of Generalized Fibonacci sequence.

2. Significance of number 2 for generalized Fibonacci sequence

Basic equation for calculation of $n-$naci sequence constant is:

$$\varphi_1 + \frac{1}{\varphi_1^1} = \varphi_2 + \frac{1}{\varphi_2^2} = \varphi_3 + \frac{1}{\varphi_1^3} = ... = \lim_{n\to\infty}\left(\varphi_n + \frac{1}{\varphi_n^n}\right) = 2,\ 1 = \varphi_1 < \varphi_2 < \varphi_3 < ... < \varphi_\infty = 2 \tag{12}$$

From the known rule the sum of exponents of the number 2 is:

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1 \Leftrightarrow 1 + \sum_{k=0}^{n-1} 2^k = 2^n \tag{13}$$

come the numbers of $n-$nacci sequence when $n \to \infty$:

$$\underbrace{0,\ 0,\ 0, ...,\ 0,\ 1}_{\mathrm{n}\to\infty},\ 1,\ 2,\ 4,\ 8,\ 16, ...,\ 2^k,\ 2^{k+1}...,\tag{14}$$

Fibonacci sequence ($n = 2$) has the famous constant of "golden ratio", $\varphi_2 = 1.61803...$ Tribonacci sequence ($n = 3$) with the constant $\varphi_3 = 1.83929...$ is known for its application in geometry (snub cube and pentagonal icositetrahedron). Constants of higher order ($n \geq 4$) have no explicit use in the literature. As fundamental property of each constant, it is known that sum of n consecutive degrees of the $n-$nacci constant is equal to the value of $n-$th degree of the constant (Table 1). Based on these sequences and constants systematization we can formulate the following theorem.

Theorem 2.1. *A series* $\{\varphi_n\}$ *of* $n-$*nacci constants is monotonically increasing sequence from the interval* $[1, 2]$, *which converges towards 2.*

Proof: For proof of this theorem we use the following lemma.

Lemma 2.2. *If the functions* $f, g : [a, b] \to \mathbb{R}$ *are monotonically increasing continuous functions with* $f(a)\cdot f(b) < 0$, $g(a) \cdot g(b) < 0$ *and* $f(x) < g(x)$, $\forall x \in [a, b]$, *then* f *and* g *have unique zeros* $x_f \in (a, b)$ *and* $x_g \in (a, b)$ *and (15) holds:*

$$x_g < x_f \tag{15}$$

The existence of unique zero follows from well-known theorems of mathematical analysis. Suppose the opposite $x_f \leq x_g$. However, from the monotony of the function f and condition $f(x) < g(x)$, we obtain,

$$0 = f(x_f) \leq f(x_g) < g(x_g) = 0 \tag{16}$$

which is an obvious contradiction.

A series $\{\varphi_n\}$ of $n-$nacci constants of $n-$nacci sequences is one of the real solutions of the equation (17) from the interval $[1, 2]$:

$$\varphi^n - \varphi^{n-1} - \varphi^{n-2} - ... - \varphi - 1 = 0 \quad\Rightarrow\quad \varphi^n = \frac{\varphi^n - 1}{\varphi - 1} \quad\Rightarrow\quad \varphi^{n+1} = 2\varphi^n - 1. \tag{17}$$

Functions $f_n(x) = x + \frac{1}{x^n} - 2$ and $f_{n+1}(x) = x + \frac{1}{x^{n+1}} - 2$, $x \in [1, 2]$ are continuous and holds:

$$x > 1 \Rightarrow \frac{1}{x^n} > \frac{1}{x^{n+1}} \Rightarrow f_n(x) > f_{n+1}(x),\ \forall n \in \mathbb{N} \tag{18}$$

It follows:

$$x \in [1, n^{\frac{1}{n+1}}] \Rightarrow f'_n(x) = 1 - \frac{n}{x^{n+1}} < 0 \Rightarrow f_n \downarrow \quad\text{and}\quad x \in [n^{\frac{1}{n+1}}, 2] \Rightarrow f'_n(x) > 0 \Rightarrow f_n \uparrow . \tag{19}$$

Function $f_n(x)$ reaches the minimum in $x_n^* = n^{\frac{1}{n+1}} \in [1,\ 2]$ that is negative because of:

$$0 = f_n(1) > f_n(x_n^*) = n^{\frac{1}{n+1}} + n^{-\frac{n}{n+1}} - 2 = n^{\frac{1}{n+1}}(1 + \frac{1}{n}) - 2 \tag{20}$$

As x_4^* is the maximal value of $n^{\frac{1}{n+1}}$ it follows that:

$$0 > f_4(x_4^*) = f_4(4^{\frac{1}{5}}) > f_n(4^{\frac{1}{5}}) \text{ and } f_n(2) = \frac{1}{2^n} > 0, \text{ i.e. } f_n(4^{\frac{1}{5}}) f_n(2) < 0. \tag{21}$$

Further it follows that all the functions $f_n(x)$, $n \geq 4$ are monotonically increasing for $x \in [n^{1/(n+1)}, 2]$, and thus for $x \in [4^{1/5}, 2] \subset [n^{1/(n+1)}, 2]$. Terms of Lemma 1 are satisfied for arbitrary functions:

$$f_n : [4^{\frac{1}{5}}, 2] \to \mathbb{R},\ f_n(x) = x + \frac{1}{x^n} - 2 \text{ and } f_{n+1} : [4^{\frac{1}{5}}, 2] \to \mathbb{R},\ f_{n+1}(x) = x + \frac{1}{x^{n+1}} - 2,\ n \geq 4 \tag{22}$$

and based on it follows the existence and uniqueness of their zeros $\varphi_n \in [4^{1/5}, 2]$ and $\varphi_{n+1} \in [4^{1/5}, 2]$, with $\varphi_n < \varphi_{n+1}$, $n \geq 4$. Trivially it follows that $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4$. Graphs of the functions $f_n(x)$ whose zeros are in the interval $[1, 2]$ are constants of sequences are shown in Figure 1.

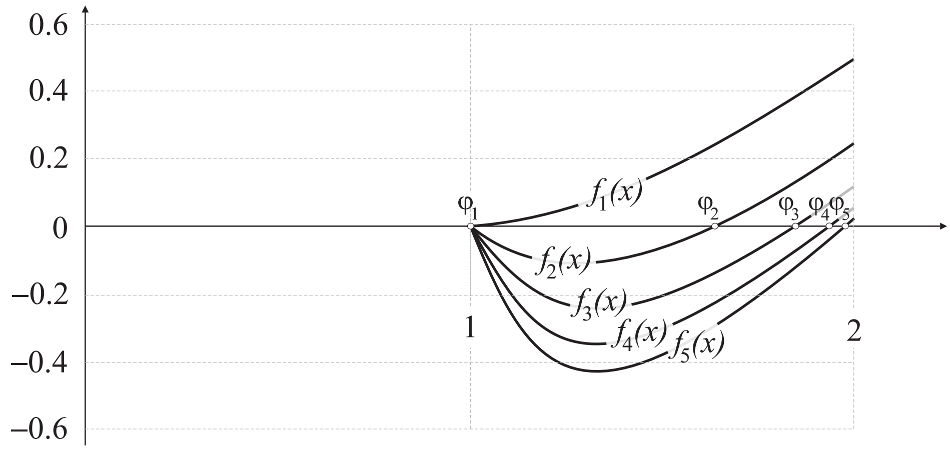

Figure 1. Graphs of functions $f_n(x)$ and constants of sequences

So, we get that the series $\{\varphi_n\}$ is monotonically increasing for $n \geq 4$ and bounded $(1 \leq \varphi_n < 2)$. Every monotonically increasing sequence which is bounded on the upper side, converges to its supremum denoted by ℓ. Further from $0 < \frac{1}{\varphi_n^n} \leq (\frac{1}{\varphi_2})^n$ follows:

$$\lim_{n\to\infty} \frac{1}{\varphi_n^n} = 0 \tag{23}$$

From equation 12 we obtain:

$$\lim_{n\to\infty} \left(\varphi_n + \frac{1}{\varphi_n^n}\right) = \lim_{n\to\infty} \varphi_n + \lim_{n\to\infty} \frac{1}{\varphi_n^n} = \ell + 0 = 2 \quad \Rightarrow \quad \ell = \lim_{n\to\infty} \varphi_n = 2 \tag{24}$$

This confirms crucial role of the number 2 in generalized Fibonacci sequence. Zero $\varphi_1 = 1$ of the function $f_1(x)$ gives us the basis to establish a new sequence - Primnacci sequence. Value of the Primnacci constant $\varphi_1 = 1$ satisfies the equation 4. Formation of this sequence can be described by the initial number 1 and the value of each additional member is equal to the previous.

$$1,\ 1,\ 1,\ 1,\ 1, \ldots,\ 2^0, \ldots \tag{25}$$

3. Initial number of Lucas's series n-nacci sequence

If for the each $n-$nacci sequence, for initial numbers are chosen degrees of 0 to $(n-1)$ of the constant n, members of the $n-$nacci sequences are degrees of $n-$nacci constant n:

$$\underbrace{\varphi_n^0, \varphi_n^1, \varphi_n^2, ...\varphi_n^{n-1}}_{\text{initial numbers}}, \quad \underbrace{\varphi_n^n, \varphi_n^{n+1}, \varphi_n^{n+2}, ...}_{\text{members of } n-\text{nacci series}} \tag{26}$$

The process of determining Lucas' type series is simple. Therefore, it is enough to develop $n-$nacci series of constants, "go to infinity", i.e. for the sufficiently large n in favorable moment Nearest Integer Function [5] is applied on n successive members of $n-$nacci series. It is known that n successive members of $n-$nacci series satisfy the equation (27), one of the basic characteristics listed in Table 1:

Table 1: The systematization of sequences and constants

Fibonacci:	$\varphi_2 = 1.618033...$	$\varphi_2^{n-2} + \varphi_2^{n-1} = \varphi_2^n$
Tribonacci:	$\varphi_3 = 1.839286...$	$\varphi_3^{n-3} + \varphi_3^{n-2} + \varphi_3^{n-1} = \varphi_3^n$
Quatronacci:	$\varphi_4 = 1.927561...$	$\varphi_4^{n-4} + \varphi_4^{n-3} + \varphi_4^{n-2} + \varphi_4^{n-1} = \varphi_4^n$
Pentanacci:	$\varphi_5 = 1.965948...$	$\varphi_5^{n-5} + \varphi_5^{n-4} + \varphi_5^{n-3} + \varphi_5^{n-2} + \varphi_5^{n-1} = \varphi_5^n$
Hexanacci:	$\varphi_6 = 1.983582...$	$\varphi_6^{n-6} + \varphi_6^{n-5} + \varphi_6^{n-4} + \varphi_6^{n-3} + \varphi_6^{n-2} + \varphi_6^{n-1} = \varphi_6^n$
Septanacci:	$\varphi_7 = 1.991964...$	$\varphi_7^{n-7} + \varphi_7^{n-6} + \varphi_7^{n-5} + \varphi_7^{n-4} + \varphi_7^{n-3} + \varphi_7^{n-2} + \varphi_7^{n-1} = \varphi_7^n$
...	...	...

$$\varphi_n^k + \underbrace{\varphi_n^{k+1} + ... + \varphi_n^{k+n-1} = \varphi_n^{k+n}}_{\text{Nearest Integer Function}} \tag{27}$$

Table 2: Procedure of determination of the initial numbers Lucas' type series of Fibonacci, Tribonacci and Quatronacci sequences

n	φ_2^n	Fibonacci feedback sequence	φ_3^n	Tribonacci feedback sequence	φ_4^n	Quatronacci feedback sequence
0	1.000↓	**$L_2(0) = 2$**	1.000↓	**$L_3(0) = 3$**	1.000↓	**$L_4(0) = 4$**
1	1.618↓	**$L_2(1) = 1$**	1.839↓	**$L_3(1) = 1$**	1.928↓	**$L_4(1) = 1$**
2	2.618↓	3↑	3.383↓	**$L_3(2) = 3$**	3.715↓	**$L_4(2) = 3$**
3	4.236↓	4↑	6.222↓	7↑	7.162↓	**$L_4(3) = 7$**
4	6.854↓	7↑	11.445↓	11↑	13.805↓	15↑
5	11.090↓	11↑	21.050↓	21↑	26.610↓	26↑
6	17.944↓	18↑	38.717↓	39↑	51.292↓	51↑
7	29.034↓	29↑	71.211↓	71↑	98.869↓	99↑
8	46.978↓	47↑	130.977↓	131↑	190.575↓	191↑
9	76.013↓	76↑	240.905↓	241↑	367.346↓	367↑
10	122.991↓	123↑	443.093↓	443↑	708.082↓	708↑
11	199.005↓	199↑	814.974↓	815↑	1364.872↓	1365↑
12	321.996↓	322↑	1498.971↓	1499↑	2630.875↓	2631↑
13	521.001↓	521↑	2757.038↓	2757↑	5071.175↓	5071↑
14	842.998↓	843↑	5070.984↓	5071↑	9775.003↓	9775↑
15	1364.000↓	1364↑	9326.993↓	9327↑	18841.924↓	18842↑
16	2206.999↓	2207↑	17155.015↓	17155↑	36318.977↓	36319↑
17	3571.000↓	3571↑	31552.991↓	31553↑	70007.079↓	70007↑
18	5777.999↓	5778↑	58034.999↓	58035↑	134942.984↓	134943↑
19	9349.000↓	9349↑	106743.005↓	106743↑	260110.965↓	260111↑
20	15126.999↓	15127↑	196330.996↓	196331↑	501380.005↓	501380↑
21	24476.000↓	24476↑	361109.000↓	361109↑	966441.032↓	966441↑
22	39602.999↓	39603↑	664183.002↓	664183↑	**⌊1862874.985⌉**	**=1862875**
23	64079.000↓	64079↑	**⌊1221622.998⌉**	**=1221623**	**⌊3590806.986⌉**	**=3590807**
24	**⌊103681.999⌉**	**=103682**	**⌊2246915.001⌉**	**=2246915**	**⌊6921503.008⌉**	**=6921503**
25	**⌊167761.000⌉**	**=167761**	**⌊4132721.001⌉**	**=4132721**	**⌊13341626.011⌉**	**=13341626**

Remark. Direction of arrows denote the way of calculation of the coefficients.

By using the equation (11) k^{th} exponent of the n constant asymptotically converges to k^{th} Lucas' number of n nacci sequence. Value of Nearest Integer Function, denoted as $NIF(x) = \lfloor x \rceil$, is the integer number closest to x. Thus, we get k^{th} Lucas' number of $n-$nacci sequence, denoted as $L_n(k)$ (28):

$$\mathrm{NIF}\,(\,\varphi_{\mathrm{n}}^{\mathrm{k+n}}) - NIF(\,\varphi_{\mathrm{n}}^{\mathrm{k+n\text{-}1}}) - \ldots - NIF(\,\varphi_{\mathrm{n}}^{\mathrm{k+1}}) = L_n(k) \neq \ \varphi_{\mathrm{n}}^{\mathrm{k}} \tag{28}$$

Successively, we determine all the values of Lucas series $L_n(k-1), L_n(k-2),..., L_n(2), L_n(1), L_n(0)$ of $n-$nacci sequences by the following system of equations:

$NIF(\varphi_n^{k+n-1}\ ) - NIF(\varphi_n^{k+n-2}) - ... - NIF(\varphi_n^{k+1}) - L_n(k) = L_n(k-1),$
$NIF(\varphi_n^{k+n-2}) - NIF(\varphi_n^{k+n-3}) - ... - L_n(k) - L_n(k-1) = L_n(k-2),$
$\vdots$
$L_n(n+1) - L_n(n) - ... - L_n(3) - L_n(2) = L_n(1),$
$L_n(n) - L_n(n-1) - ... - L_n(2) - L_n(1) = L_n(0).$

For specific numerical application on NIF(x) is not necessary to go deep in to "infinity". From the Fibonacci sequence to Heptanacci, the series of 25 members is sufficient. In Table 2, methods of determination of the initial numbers are presented and members of Lucas' type series for the Fibonacci sequence (known initial numbers: 2,1), and Lucas' type series for Tribonacci (known initial numbers 3, 1, 3), and Quatronacci sequences (first time established initial numbers: 4, 1, 3, 7) are given [10].

Table 3: The procedure for determining the initial number Lucas' type series of Pentanacci, Hexanacci and Heptanacci sequences

n	φ_5^n	Pentanacci feedback sequence	φ_6^n	Hexanacci feedback sequence	φ_7^n	Heptanacci feedback sequence
0	1.000↓	$\mathbf{L_5(0) = 5}$	1.000↓	$\mathbf{L_6(0) = 6}$	1.000↓	$\mathbf{L_7(0) = 7}$
1	1.966↓	$\mathbf{L_5(1) = 1}$	1.984↓	$\mathbf{L_6(1) = 1}$	1.992↓	$\mathbf{L_7(1) = 1}$
2	3.865↓	$\mathbf{L_5(2) = 3}$	3.935↓	$\mathbf{L_6(2) = 3}$	3.968↓	$\mathbf{L_7(2) = 3}$
3	7.598↓	$\mathbf{L_5(3) = 7}$	7.805↓	$\mathbf{L_6(3) = 7}$	7.904↓	$\mathbf{L_7(3) = 7}$
4	14.938↓	$\mathbf{L_5(4) = 15}$	15.481↓	$\mathbf{L_6(4) = 15}$	15.744↓	$\mathbf{L_7(4) = 15}$
5	29.367↓	31↑	30.708↓	$\mathbf{L_6(5) = 31}$	31.362↓	$\mathbf{L_7(5) = 31}$
6	57.734↓	57↑	60.912↓	63↑	62.473↓	$\mathbf{L_7(6) = 63}$
7	113.502↓	113↑	120.824↓	120↑	124.443↓	127
8	223.140↓	223↑	239.664↓	239↑	247.886↓	247
9	438.681↓	439↑	475.393↓	475↑	493.780↓	493
10	862.424↓	863↑	942.982↓	943↑	983.593↓	983
11	1695.481↓	1695↑	1870.483↓	1871↑	1959.281↓	1959
12	3333.227↓	3333↑	3710.258↓	3711↑	3902.818↓	3903
13	6552.953↓	6553↑	7359.604↓	7359↑	7774.274↓	7775
14	12882.766↓	12883↑	14598.385↓	14598↑	15486.076↓	15487
15	25326.850↓	25327↑	28957.106↓	28957↑	30847.710↓	30847
16	49791.276↓	49791↑	57438.818↓	57439↑	61447.533↓	61447
17	97887.072↓	97887↑	113934.654↓	113935↑	122401.286↓	122401
18	192440.916↓	192441↑	225998.824↓	225999↑	243818.978↓	243819
19	378328.880↓	378329↑	448287.390↓	448287↑	$\lfloor$**485678.676**$\rceil$	**=485679**
20	743774.995↓	743775↑	889215.176↓	**=889215**	$\lfloor$**967454.533**$\rceil$	**=967455**
21	$\lfloor$**1462223.140**$\rceil$	**=1462223**	$\lfloor$**1763831.967**$\rceil$	**=1763832**	$\lfloor$**1927134.791**$\rceil$	**=1927135**
22	$\lfloor$**2874655.004**$\rceil$	**=2874655**	$\lfloor$**3498706.828**$\rceil$	**=3498707**	$\lfloor$**3838783.506**$\rceil$	**=3838783**
23	$\lfloor$**5651422.935**$\rceil$	**=5651423**	$\lfloor$**6939974.838**$\rceil$	**=6939975**	$\lfloor$**7646719.302**$\rceil$	**=7646719**
24	$\lfloor$**11110404.954**$\rceil$	**=11110405**	$\lfloor$**13766015.023**$\rceil$	**=13766015**	$\lfloor$**15231991.071**$\rceil$	**=15231991**
25	$\lfloor$**21842481.028**$\rceil$	**=21842481**	$\lfloor$**27306031.221**$\rceil$	**=27306031**	$\lfloor$**30341580.857**$\rceil$	**=30341581**

Remark. Direction of arrows denote the way of calculation of the coefficients.

In Table 3, methods for determining the initial numbers are presented. Members of Lucas' type series of Pentanacci (first time established initial numbers: 5, 1, 3, 7, 15), Hexanacci (first time established initial numbers: 6, 1, 3, 7, 15, 31) and Heptanacci (first time established initial numbers: 7, 1, 3, 7, 15, 31, 63) the sequence are given, also.

Lucas' series of the Fibonacci sequence starts with the number 2. The number 1 is omitted and it is Lucas' type series of the Primnacci sequence. From conditions of the main equation for calculating constant Generalized Fibonacci sequence (4) for $n = 1$, we get the value of Primnacci constant:

$$\varphi_1 + \frac{1}{\varphi_1^1} = 2 \Leftrightarrow \varphi_1 = 1 \tag{29}$$

Table 4: Initial numbers and members for Lucas' type series of Generalized Fibonacci sequence

Lucas' type of:	Initial numbers	Members of series
Primnacci	1,	1, 1, 1, 1, 1, 1, ...
Fibonacci	2, 1	3, 4, 7, 11, 18, 29,
Tribonacci	3, 1, 3,	7, 11, 21, 39, 71, 131, ...
Quatronacci	4, 1, 3, 7,	15, 26, 51, 99, 191, 367,
Pentanacci	5, 1, 3, 7, 15,	31, 57, 113, 223, 439, 863, ...
Hexanacci	6, 1, 3, 7, 15, 31,	63, 120, 239, 475, 943, 1871, ...
Septanacci	7, 1, 3, 7, 15, 31, 63,	127, 247, 493, 983, 1959, 3903,
...	...	...

Based on the results in Table 4 we can formulate the rule for defining the initial numbers of Lucas series of $n-$nacci sequence:

$$\underbrace{L_n(0) = n,\ L_n(1) = 2^1 - 1,\ L_n(2) = 2^2 - 1, ...,\ L_n(n-1) = 2^{n-1} - 1,}_{\text{initial numbers of } n-\text{nacci sequence}} \tag{30}$$

Each subsequent member of the Primnacci series is identical to the previous one. Therefore, Lucas' series of Primnacci sequence is a series of numbers 1. Lucas' type and the Fibonacci series Primnacci sequences are identical. The systematization of initial numbers Lucas' type series and members of the series is given in Table 4.

The first initial number of Lucas's series n nacci sequence is always equal to the order of the sequence $''n''$ and initial numbers are equal to the sum of the first exponents of the number 2 according to the equation (13).

4. Determinant of Lucas' type series and its factorial function

By using all Lucas' type series a determinant of generalized Fibonacci sequence $\lambda_n = |a_{ij}|_n$ is formed in such a manner that $|a_{ij}|$ is $i-$th member of $j-$nacci series. In accordance with previously determined rule for initial numbers values of some elements of the determinant n are:

$$a_{ij} = j,\ \ (j = 1,\ 2,\ 3, ..., n)$$

$$a_{2j} = 2^1 - 1,\ \ (j = 2,\ 3,\ 4, ..., n)$$

$$a_{3j} = 2^2 - 1,\ \ (j = 3,\ 4, 5, ..., n)$$

$$...$$

$$a_{kj} = 2^{k-1} - 1,\ \ (j = k-1,\ k, k+1, ..., n)$$

$$\lambda_n = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & ... & n \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & ... & 2^1-1 \\ 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & ... & 2^2-1 \\ 1 & 4 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & ... & 2^3-1 \\ 1 & 7 & 11 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & ... & 2^4-1 \\ 1 & 11 & 21 & 26 & 31 & 31 & 31 & 31 & 31 & 31 & ... & 2^5-1 \\ 1 & 18 & 39 & 51 & 57 & 63 & 63 & 63 & 63 & 63 & ... & 2^6-1 \\ 1 & 29 & 71 & 99 & 113 & 120 & 127 & 127 & 127 & 127 & ... & 2^7-1 \\ 1 & 47 & 131 & 191 & 223 & 239 & 247 & 255 & 255 & 255 & ... & 2^8-1 \\ 1 & 76 & 241 & 367 & 439 & 475 & 493 & 502 & 511 & 511 & ... & 2^9-1 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & a_{n7} & a_{n8} & a_{n9} & a_{n10} & \dots & 2^{n-1}-1 \end{vmatrix}$$

Values of determinant for $n = 1, 2, 3, 4, 5$ and 6 are:

$\lambda_1 = |\,1\,| = +1 = +1 \cdot 0!,\quad \lambda_2 = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 = -1 \cdot 1!,\quad \lambda_3 = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & 3 \end{vmatrix} = +2 = +1 \cdot 2!,$

$\lambda_4 = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 4 & 7 & 7 \end{vmatrix} = -6 = -1 \cdot 3!,\quad \lambda_5 = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 4 & 7 & 7 & 7 \\ 1 & 7 & 11 & 15 & 15 \end{vmatrix} = +24 = +1 \cdot 4!,$

$\lambda_6 = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 & 3 \\ 1 & 4 & 7 & 7 & 7 & 7 \\ 1 & 7 & 11 & 15 & 15 & 15 \\ 1 & 11 & 21 & 26 & 31 & 31 \end{vmatrix} = -120 = -1 \cdot 5!, \text{.... etc.}$

It can be noticed that the values of determinant are equal to factorial of determinant's order with alternative sign so the following is true:

Theorem 4.1. *Value of n determinant of n order Lucas' type series of generalized Fibonacci sequence is equal to factorial of same order with alternative sign:*

$$\lambda_n = (-1)^{n-1}\,(n-1)!,\quad n = 1, 2, 3, ... \tag{31}$$

Proof: By subtracting $(n-1)$ column from $n-$th column, $(n-2)$ from $(n-1)$ column, $(j-2)$ from $(j-1)$ column, ..., first from the second value of determinant n changes, new determinant is obtained for whose elements the following is true:

$$\lambda_n = |b_{ij}|_n,\ \ b_{ij} = a_{ij} - a_{i,j-1} = 0,\ \ (i = 2, 3, ..., j)$$

$$b_{ij} = 1,\ \ (j = 1, 2, ..., n)$$

$$b_{j+1,j} = a_{j+1,j} - a_{j+1,j-1} = (2^j - 1) - (2^j - 1 - j) = j,\ \ (j = j+1, j+2, ..., n)$$

$$a_{j-1,j-1} = 2^{j-2} - 1;\ \ a_{j,j-1} = 2^{j-1} - 1;\ \ a_{j+1,j} = 2^j - 1;\ \ a_{j+1,j-1} = 2^j - 1 - j$$

$$L_n = \begin{vmatrix} 1 & 1 & 1 & 1 & ... & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & ... & 0 & 0 & 0 & 0 \\ 1 & 6 & 4 & 4 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & .... & ... & ... \\ ... & ... & ... & ... & ... & n-3 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & n-2 & n-2 & 0 & 0 \\ ... & ... & ... & ... & ... & 2n-2 & n-1 & n-1 & 0 \end{vmatrix}.$$

By developing this determinant by $n-$th column above main diagonal zeros are obtained, meaning that it is equal to product of elements on the main diagonal which is equal to factorial and its sign is defined by developing by $n-$th column:

$$L_n = (-1)^{n+1} \begin{vmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 1 & 2 & 0 & ... & 0 & 0 \\ 1 & 3 & 3 & ... & 0 & 0 \\ ... & ... & ... & ... & 0 & 0 \\ ... & ... & ... & ... & n-2 & 0 \\ .. & ... & ... & ... & n-1 & n-1 \end{vmatrix} = (-1)^{n+1}(n-1)!.$$

5. Conclusion

$N-$nacci sequence with initial numbers $F_n(0) = F_n(1) = ... = F_n(n-2) = 0$ and $F_n(n-1) = 1$, sequence based on exponents of $n-$nacci constant $F_n(k) = \varphi_n^k$ for $k \in [0, n-1]$ and Lucas' type series of $n-$nacci sequence with initial numbers $L_n(0) = n$ and $L_n(k) = 2^k - 1$ for $k \in [1, n-1]$ are basic three sequences. Rule (30) has been established by application of asymptotic relations and presents significant improvement of generalized Fibonacci sequence.

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