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η-Ricci Solitons on $N(k)$-Contact Metric Manifolds

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Abstract. In this paper, we study η-Ricci solitons on $N(k)$-contact metric manifolds. At first we consider η-Ricci solitons on $N(k)$-contact metric manifolds with harmonic curvature tensor. Then we study η-Ricci solitons on $N(k)$-contact metric manifolds with harmonic Weyl tensor. Moreover, we consider η-Ricci soliton on $N(k)$-contact metric manifolds with η-parallel Ricci tensor. Also η-Ricci soliton on $N(k)$-contact metric manifolds satisfying some curvature restrictions under projective curvature tensor have been considered. Finally, the existence of an η-Ricci soliton on a 3-dimensional $N(k)$-contact metric manifold is ensured by a proper example.

1. Introduction

In 1982, the notion of Ricci flow was introduced by Hamilton[21] to find the canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold M defined as follows:

$$\frac{\partial}{\partial t}g = -2S, \tag{1.1}$$

where S denotes the Ricci tensor and g is the Riemannian metric. Ricci solitons are special solutions of the Ricci flow equation (1.1) of the form $g = \sigma(t)f_t^*g$ with the initial condition $g(0) = g_0$, where $f_t, t \in \mathbb{R}$ is a family of diffeomorphisms on M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci solitons according to [10]. On the manifold M, a Ricci soliton is a triplet (g, V, λ) with g a Riemannian metric, V a vector field(called the soliton vector field) and λ a real scalar such that

$$£_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where £ is the Lie derivative. Metrics satisfying (1.2) are interesting and useful in physics and are often referred to as quasi-Einstein metrics([11],[12]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$, projected from the space of metrics onto its quotient modulo diffeomorphisms and scaling and often arise as blow-up limits for the Ricci flow in compact manifolds. Theoretical physicists have also been

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looking into the equation of Ricci solitons in relation with string theory. The initial contribution in this direction is due to Fridean[19], who analyzed some of its aspects.

The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive respectively. Ricci soliton have been studied by several authors such as ([15],[16],[17],[27],[28],[29]) and many others.

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by Cho and Kimura[13]. This notion has also been studied in [13], for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, ψ), where V is a vector field on M, λ and ψ are real scalars and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\psi\eta \otimes \eta = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g. In this connection, we mention the works of Blaga ([2],[3],[4]), Prakasha et. al. [25], De and De[14], De and Sardar[26], De et. al.[23], Sarkar et. al. [30], Eyasmin et. al.[18] and many others on η-Ricci solitons. In particular, if $\psi = 0$, then η-Ricci soliton (g, V, λ, ψ) reduces to Ricci soliton (g, V, λ). If $\psi \neq 0$, then the η-Ricci soliton is named as proper η-Ricci soliton.

The curvature tensor R is said to be harmonic if $divR = 0$, which implies

$$(\nabla_U S)(V, W) = (\nabla_V S)(U, W), \tag{1.4}$$

where '*div*' denotes divergence. This means that the Levi-Civita connection ∇ of such metric is a Yang-Mills connection while keeping the metric on the manifold fixed. Equation (1.4) means that the Ricci tensor S is of Coddazi type.

Also the Weyl tensor C is said to be harmonic if $divC = 0$, where '*div*' denotes divergence.

If the Weyl tensor is harmonic, then we get

$$(\nabla_U S)(V, W) - (\nabla_V S)(U, W) = \frac{1}{2(n-1)}[(Ur)g(V, W) - (Vr)g(U, W)], \tag{1.5}$$

where r is the scalar curvature.

The projective curvature tensor P[33] in a manifold (M, g) is defined by

$$P(U, V)W = R(U, V)W - \frac{1}{(n-1)}[g(V, W)QU - g(U, W)QV], \tag{1.6}$$

where Q is the Ricci tensor operator defined by $S(U, V) = g(QU, V)$ and $U, V, W \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M.

In 1988, Tanno [32] introduced the notion of k-nullity distribution of a contact metric manifold as a distribution such that the characteristic vector field ξ of the contact metric manifold belongs to the distribution. The contact metric manifold with ξ belonging to the k-nullity distribution is called $N(k)$-contact metric manifold and such a manifold is also studied by various authors. Generalizing this notion in 1995, Blair, Koufogiorgos and Papantoniou [6] introduced the notion of contact manifold with ξ belonging to the (k, μ)-nullity distribution, where k and μ are real constants. In particular, if $\mu = 0$, then the notion of (k, μ)-nullity distribution reduces to the notion of k-nullity distribution.

The above mentioned works motivate us to study η-Ricci soliton in the frame work of $N(k)$-contact metric manifolds.

The paper is organized as follows:

After preliminaries in Section 2, we consider η-Ricci solitons on $N(k)$-contact metric manifolds whose curvature tensor is harmonic in Section 3. Section 4 is devoted to study η-Ricci solitons on $N(k)$-contact metric manifolds with harmonic Weyl tensor. Next, in Section 5 we study η-Ricci solitons on $N(k)$-contact metric manifolds with η-parallel Ricci tensor. In Section 6, we investigate η-Ricci solitons on $N(k)$-contact metric manifolds with $P.\phi = 0$. Section 7 deals with the study of η-Ricci solitons on $N(k)$-contact metric manifolds with $Q.P = 0$. Finally, in Section 8 an example is constructed to prove the existence of a proper η-Ricci soliton on a 3-dimensional $N(k)$-contact metric manifold.

2. $N(k)$-contact metric manifolds

An n-dimensional manifold $M^n (n = odd)$ is said to admit an almost contact metric structure if it admits a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η satisfying

$$\phi^2 U = -U + \eta(U)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0 \qquad and \qquad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^n \times \mathbb{R}$ defined by

$$J(U, f\tfrac{d}{dt}) = (\phi U - f\xi, \eta(U)\tfrac{d}{dt})$$

is integrable, where U is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), i.e.,

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V). \tag{2.2}$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.1) it can be easily seen that

$$g(U, \phi V) = -g(\phi U, V), \qquad g(U, \xi) = \eta(U), \tag{2.3}$$

for all vector fields $U, V \in \chi(M)$. An almost contact metric structure becomes a contact metric structure if

$$g(U, \phi V) = d\eta(U, V), \tag{2.4}$$

for all vector fields $U, V \in \chi(M)$. The 1-form η is then called a contact form and ξ is its characteristic vector field. We define a (1,1) tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie derivative. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also

$$\nabla_U \xi = -\phi U - \phi h U \tag{2.5}$$

holds in contact metric manifolds. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_U \phi)V = g(U, V)\xi - \eta(V)U, \qquad U, V \in \chi(M), \tag{2.6}$$

where ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is Killing is said to be a K-contact manifold. A Sasakian manifold is K-contact, but not conversely. However a 3-dimensional K-contact manifold is Sasakian[22]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(U, V)\xi = 0$ [4]. On the other hand on a Sasakian manifold the following relation holds:

$$R(U, V)\xi = \eta(V)U - \eta(U)V. \tag{2.7}$$

As a generalization of both $R(U, V)\xi = 0$ and the Sasakian case: Blair, Koufogiorgos and Papantoniou [6] introduced the (k, μ)- nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k, μ)-nullity distribution $N(k, \mu)$ ([6],[24]) of a contact metric manifold M is defined by

$$\begin{aligned} N(k, \mu) : p &\to N_p(k, \mu) \\ &= \{W \in T_p M : R(U, V)W = (kI + \mu h)(g(V, W)U - g(U, W)V)\}, \end{aligned}$$

for all $U, V \in \chi(M)$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k, \mu)$ is called a (k, μ)-contact manifold. In particular on a (k, μ)-contact manifold, we have

$$R(U, V)\xi = k[\eta(V)U - \eta(U)V] + \mu[\eta(V)hU - \eta(U)hV].$$

On a (k, μ)-contact manifold $k \leqslant 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, then the (k, μ)-nullity condition determines the curvature of M^{2n+1}completely [6]. In fact, for a (k, μ)-manifold, the condition of being Sasakian, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent.

The k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by [32]

$$N(k) : p \longrightarrow Np(k) = \{W \in T_pM : R(U, V)W = k[g(V, W)U - g(U, W)V]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call the manifold an $N(k)$-contact metric manifold [9]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [5]. In a (k, μ)-contact manifold if $\mu = 0$, then the manifold becomes an $N(k)$-contact manifold.

In [1], $N(k)$-contact metric manifolds were studied in details. For more details we refer to ([7],[8]).

In $N(k)$-contact metric manifolds the following relations hold:

$$h^2 = (k-1)\phi^2, k \leq 1, \tag{2.8}$$

$$(\nabla_U \phi)V = g(U + hU, V)\xi - \eta(V)(U + hU), \tag{2.9}$$

$$R(\xi, U)V = k[g(U, V)\xi - \eta(V)U], \tag{2.10}$$

$$S(U, \xi) = (n-1)k\eta(U), \tag{2.11}$$

$$(\nabla_U \eta)V = g(U + hU, \phi V), \tag{2.12}$$

$$R(U, V)\xi = k[\eta(V)U - \eta(U)V], \tag{2.13}$$

where $U, V, W \in \chi(M)$, R is the Riemannian curvature tensor and S is the Ricci tensor of the manifold.

In a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(U, V)W &= g(V, W)QU - g(U, W)QV + S(V, W)U - S(U, W)V \\ &-\frac{r}{2}[g(V, W)U - g(U, W)V], \end{aligned} \tag{2.14}$$

where Q is the Ricci operator and r is the scalar curvature of the manifold. Substituting $W = \xi$ in (2.14) and using (2.11) and (2.13), we get for $n = 3$

$$\begin{aligned} k[\eta(V)U - \eta(U)V] &= \eta(V)QU - \eta(U)QV \\ &+(2k - \frac{r}{2})[\eta(V)U - \eta(U)V]. \end{aligned} \tag{2.15}$$

Replacing V by ξ in (2.15), we get

$$QU = [\frac{r}{2} - k]U + [3k - \frac{r}{2}]\eta(U)\xi. \tag{2.16}$$

Therefore

$$S(U, V) = [\frac{r}{2} - k]g(U, V) + [3k - \frac{r}{2}]\eta(U)\eta(V). \tag{2.17}$$

Equation (2.17) implies that a 3-dimensional $N(k)$-contact metric manifold is an η-Einstein manifold.

Definition 2.1. *The Ricci tensor of a $N(k)$-contact metric manifold is said to be η-parallel [20] if*

$$g((\nabla_U Q)V, W) = 0,$$

for arbitrary vector fields U, V, W.

Proposition 2.1. *For an η-Ricci soliton on a $N(k)$-contact metric manifold, the Ricci tensor S is of the form*

$$S(U, V) = -g(hU, \phi V) - \lambda g(U, V) - \psi\eta(U)\eta(V), \tag{2.18}$$

$$QU = \phi hU - \lambda U - \psi\eta(U)\xi \tag{2.19}$$

and

$$\lambda + \psi = -k(n-1). \tag{2.20}$$

Remark 2.1. *The above form of the Ricci tensor is also deduced by De et al[23].*

Comparing the above equation (2.17) with (2.18), we get in a 3-dimensional $N(k)$-contact metric manifold

$$\lambda = -(\frac{r}{2} - k) \quad and \quad \psi = -(3k - \frac{r}{2}). \tag{2.21}$$

3. η-Ricci solitons on $N(k)$-contact metric manifolds with harmonic curvature tensor

Theorem 3.1. *There does not exist a proper η-Ricci soliton in a $N(k)$-contact metric manifold whose curvature tensor is harmonic.*

Proof. Taking covariant differentiation of (2.18) with respect to W, we obtain

$$\begin{aligned}(\nabla_W S)(U, V) \quad = \quad & -g((\nabla_W h)U, \phi V) - g(hU, (\nabla_W \phi)V) \\ & -\psi\{(\nabla_W \eta)U\eta(V) + (\nabla_W \eta)V\eta(U)\}.\end{aligned} \tag{3.1}$$

Using (2.9) and (2.12) in (3.1), we get

$$\begin{aligned}(\nabla_W S)(U, V) \quad = \quad & -g((\nabla_W h)U, \phi V) + \eta(V)g(hU, W + hW) \\ & -\psi[\{g(W, \phi U) + g(hW, \phi U)\}\eta(V) \\ & +\{g(W, \phi V) + g(hW, \phi V)\}\eta(U)].\end{aligned} \tag{3.2}$$

By hypothesis,

$$(\nabla_W S)(U, V) - (\nabla_U S)(W, V) = 0. \tag{3.3}$$

In view of (3.2) and (3.3) we get

$$\begin{aligned}& -g((\nabla_W h)U - (\nabla_U h)W, \phi V) - \psi[2g(\phi U, W)\eta(V) + \{g(W, \phi V) + g(hW, \phi V)\}\eta(U) \\ & -\{g(U, \phi V) + g(hU, \phi V)\}\eta(W)] = 0.\end{aligned} \tag{3.4}$$

Putting $V = \xi$ in the above equation we infer

$$\psi g(\phi U, W) = 0.$$

It follows that $\psi = 0$. This completes the proof. □

Corollary 3.1. *If a 3-dimensional $N(k)$-contact metric manifold admits an η-Ricci soliton with harmonic curvature tensor, then the manifold is of constant sectional curvature k.*

Proof. Since $\psi = 0$, therefore from (2.21) we get $r = 6k$ in a 3-dimensional $N(k)$-contact metric manifold. Hence (2.16) gives

$$QU = 2kU.$$

Thus from (2.14) it follows that the manifold is of constant sectional curvature k. This finishes the proof. □

4. η-Ricci solitons on $N(k)$-contact metric manifolds with harmonic Weyl tensor

Theorem 4.1. *If a $N(k)$-contact metric manifold admits an η-Ricci soliton, then harmonic curvature tensor and harmonic Weyl tensor are equivalent.*

Proof. Let the $N(k)$-contact metric manifold M be of harmonic Weyl tensor. Then (1.5) gives

$$(\nabla_W S)(U,V) - (\nabla_U S)(W,V) = \frac{1}{2(n-1)}[(Wr)g(U,V) - (Ur)g(W,V)]. \tag{4.1}$$

Making use of (3.2) in (4.1), we get

$$-g((\nabla_W h)U - (\nabla_U h)W, \phi V) - \psi[2g(\phi U, W)\eta(V) + \{g(W,\phi V) + g(hW,\phi V)\}\eta(U)$$
$$-\{g(U,\phi V) + g(hU,\phi V)\}\eta(W)] = \frac{1}{2(n-1)}[(Wr)g(U,V) - (Ur)g(W,V)]. \tag{4.2}$$

Putting $V = \xi$ in the forgoing equation gives

$$-2\psi g(\phi U, W) = \frac{1}{2(n-1)}[(Wr)\eta(U) - (Ur)\eta(W)]. \tag{4.3}$$

Replacing U by ϕU in (4.3), we obtain

$$2\psi[-g(U,W) + \eta(U)\eta(W)] = \frac{1}{2(n-1)}((\phi U)r)\eta(W). \tag{4.4}$$

Putting $W = \xi$ in the above equation, we get

$$((\phi U)r) = 0,$$

which implies r = constant. Hence equation (1.5) implies

$$(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = 0.$$

Therefore in a $N(k)$-contact metric manifold admitting an η-Ricci soliton the harmonic weyl tensor implies harmonic curvature tensor.
In a Riemannian manifold, if the curvature tensor is harmonic, then the weyl tensor is also harmonic. But the converse, is not true, in general.
Thus harmonic curvature and harmonic Weyl tensor are equivalent in a $N(k)$-contact metric manifold admitting an η-Ricci soliton. This completes the proof. □

5. η-Ricci solitons on $N(k)$-contact metric manifolds with η-parallel Ricci tensor

Theorem 5.1. *If a $N(k)$-contact metric manifold admits an η-Ricci soliton with η-parallel Ricci tensor, then the manifold becomes a Sasakian manifold.*

Proof. Let the Ricci tensor of a $N(k)$-contact metric manifold be η-parallel. Then

$$g((\nabla_V Q)U, W) = 0, \tag{5.1}$$

for arbitrary vector fields U, V, W.
Taking covariant derivative of (2.19) with respect to an arbitrary vector field V, we get

$$(\nabla_V Q)U = \nabla_V \phi h U - \phi h \nabla_V U - \psi\{((\nabla_V \eta)U)\xi + \eta(U)\nabla_V \xi\}.$$

Using (2.5) and (2.12) in the above equation, we get

$$(\nabla_V Q)U = \nabla_V \phi hU - \phi h \nabla_V U - \psi\{g(V + hV, \phi U)\xi + \eta(U)(-\phi V - \phi hV)\}. \tag{5.2}$$

Using (5.2) in (5.1), we infer

$$g(\nabla_V \phi hU, W) - g(\phi h \nabla_V U, W) - \psi\{g(V + hV, \phi U)\eta(W) - g(\phi V, W)\eta(U)$$
$$-g(\phi hV, W)\eta(U)\} = 0. \tag{5.3}$$

Putting $U = \xi$ and using (2.5) in (5.3) yields

$$g(\phi h\phi V, W) + g(\phi h\phi hV, W) + \psi\{g(\phi V, W) + g(\phi hV, W)\} = 0. \tag{5.4}$$

Using (2.8) in the above equation, we get

$$g(hV, W) - (k - 1)\{g(V, W) - \eta(V)\eta(W)\} + \psi\{g(\phi V, W) + g(\phi hV, W)\} = 0. \tag{5.5}$$

Replacing V by ϕV in (5.5) yields

$$g(h\phi V, W) - (k - 1)g(\phi V, W) + \psi\{-g(V, W) + \eta(V)\eta(W) + g(hV, W)\} = 0. \tag{5.6}$$

Interchanging V and W in the above equation, we get

$$g(h\phi W, V) - (k - 1)g(\phi W, V) + \psi\{-g(W, V) + \eta(V)\eta(W) + g(hW, V)\} = 0. \tag{5.7}$$

Subtracting (5.7) from (5.6) we infer

$$(k - 1)g(\phi V, W) = 0.$$

It follows that $k = 1$. Therefore the manifold is a Sasakian manifold. This finishes the proof. □

6. η-Ricci solitons on $N(k)$-contact metric manifolds with $P.\phi = 0$

Theorem 6.1. *There does not exist a proper η-Ricci soliton in a $N(k)$-contact metric manifold whose projective curvature tensor satisfies the curvature condition $P.\phi = 0$.*

Proof. We assume that the $N(k)$-contact metric manifold M admitting an η-Ricci soliton satisfies the curvature condition

$$P.\phi = 0,$$

where P is the projective curvature tensor. This implies

$$P(U, V)\phi W - \phi(P(U, V)W) = 0. \tag{6.1}$$

Putting $W = \xi$ in (6.1), we get

$$\phi(P(U, V)\xi) = 0. \tag{6.2}$$

Again putting $W = \xi$ in (1.6) and using (2.19), we get

$$P(U, V)\xi = (k + \frac{\lambda}{n-1})[\eta(V)U - \eta(U)V] - \frac{1}{n-1}[\eta(V)\phi hU - \eta(U)\phi hV]. \tag{6.3}$$

Using (6.3) in (6.2), we obtain

$$(k + \frac{\lambda}{n-1})[\eta(V)\phi U - \eta(U)\phi V] + \frac{1}{n-1}[\eta(V)hU - \eta(U)hV] = 0. \tag{6.4}$$

Replacing U by ϕU in the above equation, we get

$$(k+\frac{\lambda}{n-1})\eta(V)(-U+\eta(U)\xi)+\frac{1}{n-1}\eta(V)h\phi U=0. \tag{6.5}$$

Again replacing U by ϕU in (6.5) yields

$$(k+\frac{\lambda}{n-1})\phi U+\frac{1}{n-1}hU=0. \tag{6.6}$$

Taking inner product with X in the above equation, we infer

$$(k+\frac{\lambda}{n-1})g(\phi U,X)+\frac{1}{n-1}g(hU,X)=0. \tag{6.7}$$

Interchanging U and X in (6.7), we get

$$(k+\frac{\lambda}{n-1})g(\phi X,U)+\frac{1}{n-1}g(hX,U)=0. \tag{6.8}$$

Subtracting (6.8) from (6.7), we obtain

$$(k+\frac{\lambda}{n-1})g(\phi U,X)=0.$$

It follows that $\lambda=-k(n-1)$. Hence from (2.20) we get $\psi=0$. This completes the proof. □

7. η-Ricci solitons on $N(k)$-contact metric manifolds with $Q.P=0$

Theorem 7.1. *There does not exist a proper η-Ricci soliton in a $N(k)$-contact metric manifold whose projective curvature tensor satisfies the curvature condition $Q.P=0$.*

Proof. We assume that the $N(k)$-contact metric manifold M admitting an η-Ricci soliton satisfies the curvature condition

$$Q.P=0,$$

where P is the projective curvature tensor and Q is the Ricci operator defined by $S(U,V)=g(QU,V)$. This implies

$$Q(P(U,V)W)-P(QU,V)W-P(U,QV)W-P(U,V)QW=0. \tag{7.1}$$

Using (2.18) in (7.1), we get

$$\phi h(P(U,V)W)+2\lambda P(U,V)W-\psi\eta(P(U,V)W)\xi-P(\phi hU,V)W-P(U,\phi hV)W$$
$$+\psi\eta(U)P(\xi,V)W+\psi\eta(V)P(U,\xi)W+\psi\eta(W)P(U,V)\xi=0. \tag{7.2}$$

Putting $W=\xi$ in the above equation yields

$$\phi h(P(U,V)\xi)+(2\lambda+\psi)P(U,V)\xi-\psi\eta(P(U,V)\xi)\xi-P(\phi hU,V)\xi-P(U,\phi hV)\xi$$
$$+\psi\eta(U)P(\xi,V)\xi+\psi\eta(V)P(U,\xi)\xi=0. \tag{7.3}$$

Using (6.3) in (7.3), we infer

$$\phi h[(k+\frac{\lambda}{n-1})\{\eta(V)U-\eta(U)V\}-\frac{1}{n-1}\{\eta(V)\phi hU-\eta(U)\phi hV\}]$$

$$+(2\lambda+\psi)[(k+\frac{\lambda}{n-1})\{\eta(V)U-\eta(U)V\}-\frac{1}{n-1}\{\eta(V)\phi hU-\eta(U)\phi hV\}]$$
$$-[(k+\frac{\lambda}{n-1}\eta(V)\phi hU-\frac{1}{n-1}\eta(V)\phi h\phi hU]$$
$$-[-(k+\frac{\lambda}{n-1}\eta(U)\phi hV+\frac{1}{n-1}\eta(U)\phi h\phi hV]$$
$$+\psi\eta(U)[(k+\frac{\lambda}{n-1})\{\eta(V)\xi-Y\}+\frac{1}{n-1}\phi hV]$$
$$+\psi\eta(V)[(k+\frac{\lambda}{n-1})\{U-\eta(U)\xi\}-\frac{1}{n-1}\phi hU]=0, \tag{7.4}$$

which implies

$$(\lambda+\psi)(k+\frac{\lambda}{n-1})[\eta(V)U-\eta(U)V]-\frac{(\lambda+\psi)}{n-1}[\eta(V)\phi hU-\eta(U)\phi hV]=0. \tag{7.5}$$

Putting $V=\xi$ and taking inner product with X, we get

$$(\lambda+\psi)(k+\frac{\lambda}{n-1})[g(U,X)-\eta(U)\eta(X)]-\frac{(\lambda+\psi)}{n-1}g(\phi hU,X)=0. \tag{7.6}$$

Substituting $U=X=e_i$ in (7.6), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i(1\leqslant i\leqslant n)$, we get

$$(\lambda+\psi)\{\lambda+k(n-1)\}=0.$$

This implies either $\lambda+\psi=0$ or $\lambda=-k(n-1)$.
Here $\lambda+\psi=0$ contradicts (2.20). Therefore $\lambda=-k(n-1)$. Hence from (2.20) we get $\psi=0$. This finishes the proof. □

Remark 7.1. *If we consider 3-dimensional $N(k)$-contact metric manifolds admitting η-Ricci soliton whose projective curvature tensor satisfies the curvature conditions $P.\phi=0$ and $Q.P=0$, then in a similar way as in Corollary 1.1, we may prove that the manifold under consideration is a manifold of constant sectional curvature k.*

8. Example

We consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3,(x,y,z)\neq(0,0,0)\}$, where (x,y,z) are the standard coordinate in $\mathbb{R}^3$. Let e_1,e_2,e_3 be three linearly independent vector fields in $\mathbb{R}^3$ such that [31]

$$[e_1,e_2]=(1+\alpha)e_3,\ \ [e_2,e_3]=2e_1\ \ \textit{and}\ \ [e_3,e_1]=(1-\alpha)e_2,$$

where $\alpha=\pm\sqrt{1-k}$.
Let g be the Riemannian metric defined by

$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0,\ \ g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1.$$

Let η be the 1-form defined by

$$\eta(U)=g(U,e_1)$$

for any $U\in\chi(M)$. Let ϕ be the (1,1)-tensor field defined by

$$\phi e_1=0,\qquad \phi e_2=e_3,\qquad \phi e_3=-e_2.$$

Using the linearity of ϕ and g we have

$$\eta(e_1) = 1,$$

$$\phi^2(U) = -U + \eta(U)e_1$$

and

$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Moreover

$$he_1 = 0, \quad he_2 = \alpha e_2 \quad \text{and} \quad he_3 = -\alpha e_3.$$

In[23] the authors obtained the expression of the curvature tensor and the Ricci tensor as follows:

$$R(e_1, e_2)e_2 = (1-\alpha^2)e_1, R(e_3, e_2)e_2 = -(1-\alpha^2)e_3, R(e_1, e_3)e_3 = (1-\alpha^2)e_1,$$

$$R(e_2, e_3)e_3 = -(1-\alpha^2)e_2, \quad R(e_2, e_3)e_1 = 0,$$

$$R(e_1, e_2)e_1 = -(1-\alpha^2)e_2, \quad R(e_3, e_1)e_1 = (1-\alpha^2)e_3$$

and

$$S(e_1, e_1) = 2(1-\alpha^2),$$

$$S(e_2, e_2) = 0,$$

$$S(e_3, e_3) = 0.$$

Hence the manifold defines $N(k)$-contact η-Ricci soliton of dimension three for $\lambda = 0$ and $\psi = -2k$.

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