Filomat 35:11 (2021), 3855–3877
https://doi.org/10.2298/FIL2111855S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Gauss's Binomial Formula and Additive Property of Exponential Functions on $\mathbb{T}_{(q,h)}$

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Abstract. In this article, we focus our attention on (q,h)-Gauss's binomial formula from which we discover the additive property of (q,h)-exponential functions. We state the (q,h)-analogue of Gauss's binomial formula in terms of proper polynomials on $\mathbb{T}_{(q,h)}$ which own essential properties similar to ordinary polynomials. We present (q,h)-Taylor series and analyze the conditions for its convergence. We introduce a new (q,h)-analytic exponential function which admits the additive property. As consequences, we study (q,h)-hyperbolic functions, (q,h)-trigonometric functions and their significant properties such as (q,h)-Pythagorean Theorem and double-angle formulas. Finally, we illustrate our results by a first order (q,h)-difference equation, (q,h)-analogues of dynamic diffusion equation and Burger's equation. Introducing (q,h)-Hopf-Cole transformation, we obtain (q,h)-shock soliton solutions of Burger's equation.

1. Introduction

In the literature, the discretization of continuous equations has been studied in two main discrete sets: h-lattice and q-numbers respectively:

$$h\mathbb{Z} := \{hx : x \in \mathbb{Z},\ h > 0\}, \quad \mathbb{K}_q := \{q^n : \quad n \in \mathbb{Z}, \quad q \in \mathbb{R}, \quad q \neq 1\} \cup \{0\},$$

where the parameter h is devoted to the Planck's constant in quantum mechanics while the parameter q refers to the number of elements in finite fields. Both discrete sets recover $\mathbb{R}$, as $h \to 0$ and $q \to 1$, respectively. Stefan Hilger introduced the notion of time scales $\mathbb{T}$, as an arbitrary nonempty closed subset of real numbers [10]. The concept of time scales allows to unify and extend not only such discrete sets but also any type of continuous and discrete sets. The development of time scales extends the study on differential and difference equations to the so-called dynamic equations on time scales [4]. This epochal invention has a tremendous potential for applications in economics [1], in biomathematics [5] and in mathematical physics [3, 9, 19]. However the study on such a general time scale may have discrepancies and deficiencies even in some elementary subjects. For instance, the polynomials, Taylor's formula and exponential functions have implicit and inapplicable expressions requiring very restrictive conditions. For that reason, a special discrete time scale $\mathbb{T}_{(q,h)}$ which unifies and extends h- and q-analysis, is presented in

2020 *Mathematics Subject Classification*. Primary 39A06, 39A13, 39A14, 34NA05.
Keywords. (q,h)-Gauss's binomial formula, (q,h)-integral, (q,h)-analytic functions, additive property of (q,h)-exponential functions, (q,h)-trigonometric functions, (q,h)-diffusion equation, (q,h)-Burger's equation.
Received: 25 August 2020; Revised: 14 April 2021; Accepted: 01 June 2021
Communicated by Miodrag Spalević
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[6] in order to study (q, h)-fractional calculus. Since then a variety of mathematicians have contributed to the (q, h)-analysis such as (q, h)-analogue of Laplace transform [16], (q, h)-analogue of quantum splines [8] and (q, h)-analogue of binomials and the wave equation [18].

The main objective of the current article is to overcome some of the deficiencies in front of the applicability of time scales. In Section 2, we improve (q, h)-calculus that we presented in [18]. The emphasize is placed on (q, h)-analogue of integration. We construct (q, h)-integral as a series explicitly from which we develop fundamental theorems (existence and uniqueness of antiderivatives, indefinite and definite integral, fundamental theorem of calculus and integration by parts formula) and their proofs on $\mathbb{T}_{(q,h)}$ which can be regarded as alternative proofs to the ones on an arbitrary $\mathbb{T}$.

Section 3 is devoted to present (q, h)-analogue of Gauss's binomial formula. Such a binomial formula needs to be constructed in terms of proper polynomials on $\mathbb{T}_{(q,h)}$ rather than the ordinary ones. For this purpose, we introduce generalized quantum binomial $(\gamma x - \delta x_0)^n_{q,h}$, whose behavior on $\mathbb{T}_{(q,h)}$ is as significants as the behavior of the ordinary polynomial $(\gamma x - \delta x_0)^n$ in $\mathbb{R}$. We establish (q, h)-analogue of Taylor's formula in terms of $(\gamma x - \delta x_0)^n_{q,h}$. We finalize this section by related Leibniz rules and additive property of degrees for $(\gamma x - \delta x_0)^n_{q,h}$, $n \in \mathbb{Z}$.

In the literature, unlike the real case, the additive property of exponential functions on $\mathbb{K}_q$ requires a very restrictive condition, i.e. the variables must be q-commuting variables [17]. This handicap constitutes some drawbacks in analysis and in the theory of difference equations on $\mathbb{K}_q$ and therefore on an arbitrary time scale. The significant contribution of Section 4 is to introduce a new exponential function which satisfies the additive property on $\mathbb{T}_{(q,h)}$. For this purpose, we first present (q, h)-Taylor series, analyze the conditions for its convergence and introduce (q, h)-analytic functions. We express such an exponential function in terms of convergent (q, h)-Taylor series and show that it recovers ordinary exponential function, h-exponential function, Jackson's q-exponential function [12] and Euler's second q-exponential function [15]. We prove the additive property of exponential functions on $\mathbb{T}_{(q,h)}$ by utilizing (q, h)-Gauss's binomial formula which is expressed in terms of the polynomials $(\gamma x - \delta x_0)^n_{q,h}$. Furthermore, this property provides the multiplicative inverse of exponential function from which we introduce new (q, h)-trigonometric functions and state their important properties such as (q, h)-analogue of Pythagorean theorem and double-angle formulas.

As an application of additive property of exponential function, in Section 5, we suggest a first order non-homogeneous linear (q, h)-difference equation whose solution is given by means of variation of parameters formula. In addition, we present a dynamic diffusion equation on $\mathbb{T}_{(q,h)} \times \mathbb{T}_{(\bar{q},\bar{h})}$ with its solutions. This diffusion equation is a generic equation since it provides various kinds of partial difference/differential equations. Finally, in the light of (q, h)-analogue of Hopf-Cole transformation, we offer a (q, h)-Burger's equation and its multi (q, h)-shock soliton solutions.

2. Preliminaries

In this section, we first briefly summarize the calculus on (q, h)-time scales that we presented in [18]. In [6], a two-parameter time scale $\mathbb{T}_{(q,h)}$ is defined by

$$\mathbb{T}_{(q,h)} := \{q^n x + [n]h : x \in \mathbb{R}^+ \cup \{0\}, \quad n \in \mathbb{Z}, \quad h, q \in \mathbb{R}^+, \quad q \neq 1\} \cup \left\{ \frac{h}{1-q} \right\}, \tag{1}$$

where $[n] := \dfrac{q^n - 1}{q - 1}$. If $0 < q < 1$, we note that $\dfrac{h}{1-q}$ is an accumulation point since

$$\lim_{n\to\infty} (q^n x + [n]h) = \lim_{n\to\infty} (q^n x + (1 + q + .. + q^{n-1})h) = \frac{h}{1-q}.$$

Due to the *forward jump operator* σ and the *backward jump operator* ρ, defined for any time scales $\mathbb{T}$,

$$\sigma(x) := \inf\{s \in \mathbb{T} : s > x\}, \qquad \rho(x) := \sup\{s \in \mathbb{T} : s < x\},$$

on every $x \in \mathbb{T}_{(q,h)}$, we have

$$\sigma^n(x) = q^n x + [n]h, \qquad \rho^n(x) = q^{-n}(x - [n]h), \quad n \in \mathbb{N}, \tag{2}$$

which obey the following property

$$(\sigma \circ \rho)(x) = (\rho \circ \sigma)(x) = x, \quad x \in \mathbb{T}_{(q,h)},$$

indicating that $\mathbb{T}_{(q,h)}$ is a *regular discrete time scale* [9]. In the literature, h-derivative, q-derivative, symmetric h-derivative or symmetric q-derivative approximate the classical derivative in the proper limits of q and h. Moreover, it is also possible to comprise and extend h- and q-derivatives in a unified framework.
In order not to repeat expressions every time, throughout this paper we assume that $f(x)$ is any real valued function defined on $\mathbb{T}_{(q,h)}$.

Definition 2.1. [6] *Let $x \neq \frac{h}{1-q}$. The delta (q,h)-derivative of any function $f(x)$, denoted by $D_{(q,h)}f$, is introduced as*

$$D_{(q,h)}f(x) := \frac{f(\sigma(x)) - f(x)}{\sigma(x) - x} = \frac{f(qx+h) - f(x)}{(q-1)x + h}, \tag{3}$$

while the nabla (q,h)-derivative of f, denoted by $\tilde{D}_{(q,h)}$, is defined by

$$\tilde{D}_{(q,h)}f(x) := \frac{f(x) - f(\rho(x))}{x - \rho(x)} = \frac{f(x) - f(\frac{x-h}{q})}{x - (\frac{x-h}{q})}. \tag{4}$$

Since the accumulation point $x = \frac{h}{1-q}$ is right dense, the *delta and nabla (q,h)-derivatives* of f at this point are defined by

$$D_{(q,h)}f(\frac{h}{1-q}) := \lim_{s \to \left(\frac{h}{1-q}\right)^+} \frac{f(s) - f\left(\frac{h}{1-q}\right)}{s - \frac{h}{1-q}} = f'(\frac{h}{1-q}), \tag{5}$$

$$\tilde{D}_{(q,h)}f(\frac{h}{1-q}) := \lim_{s \to \left(\frac{h}{1-q}\right)^+} \frac{f(s) - f\left(\frac{h}{1-q}\right)}{s - \frac{h}{1-q}} = f'(\frac{h}{1-q}), \tag{6}$$

provided that the limits exist. (see [4, Theorem 1.16 i])
Notice that delta (q,h)-derivative reduces to h, q and ordinary derivatives while nabla (q,h)-derivative recovers nabla h, nabla q and ordinary derivatives in the appropriate limits of q and h.

Proposition 2.2. [18] *For the arbitrary functions f and g, the product and the quotient rules are given by*

$$\begin{aligned}
D_{(q,h)}\left(f(x)g(x)\right) &= f(x)D_{(q,h)}g(x) + g(qx+h)D_{(q,h)}f(x) \\
&= g(x)D_{(q,h)}f(x) + f(qx+h)D_{(q,h)}g(x), \\
\tilde{D}_{(q,h)}\left(f(x)g(x)\right) &= f(x)\tilde{D}_{(q,h)}g(x) + g(\frac{x-h}{q})\tilde{D}_{(q,h)}f(x) \\
&= g(x)\tilde{D}_{(q,h)}f(x) + f(\frac{x-h}{q})\tilde{D}_{(q,h)}g(x), \\
D_{(q,h)}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(x)D_{(q,h)}f(x) - f(x)D_{(q,h)}g(x)}{g(x)g(qx+h)} \\
&= \frac{g(qx+h)D_{(q,h)}f(x) - f(qx+h)D_{(q,h)}g(x)}{g(x)g(qx+h)}, \\
\tilde{D}_{(q,h)}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(x)\tilde{D}_{(q,h)}f(x) - f(x)\tilde{D}_{(q,h)}g(x)}{g(x)g(\frac{x-h}{q})} \\
&= \frac{g(\frac{x-h}{q})\tilde{D}_{(q,h)}f(x) - f(\frac{x-h}{q})\tilde{D}_{(q,h)}g(x)}{g(x)g(\frac{x-h}{q})}.
\end{aligned}$$

Following the pioneering article of Hilger [10], on general time scales the notion of Δ-integral and its reductions to well-known special forms such as $\mathbb{R}, \mathbb{Z}, h\mathbb{Z}$ and $\mathbb{K}_q$ are considered in many papers (see [4]). Although within the concept of time scales the Δ-integral results are of course valid for nontrivial structure $\mathbb{T}_{(q,h)}$, the notion of Δ-integral on this specific structure is not analyzed in details in the literature (only except the integration formula on $\mathbb{T}_{(q,h)}$ given as a finite sum by Čermák [6]). For that reason, we aim to construct Δ-integral as a series explicitly on $\mathbb{T}_{(q,h)}$. We develop fundamental theorems of (q,h)-integral and their proofs which can be regarded as alternative proofs to the ones for general time scales. Consequently, we employ these results in Subsection 5.1 for solving a (q,h)-difference equation.
Let $f(x)$ be any continuous function at the accumulation point $x = \frac{h}{1-q}$. Our aim is to construct a (q,h)-antiderivative $F(x)$ of $f(x)$ i.e. $D_{(q,h)}F(x) = f(x)$, for all $x \in \mathbb{T}_{(q,h)}$. Let E denote the forward shift operator, i.e.,

$$E(F(x)) = F(\sigma(x)) = F(qx+h).$$

Then by the definition of (q,h)-derivative (3), we obtain

$$(E-1)F(x) = ((q-1)x+h)f(x),$$

which implies that

$$F(x) = (1-E)^{-1}\left(((1-q)x-h)f(x)\right) = \sum_{i=0}^{\infty} E^i\left(((1-q)x-h)f(x)\right). \tag{7}$$

The forward shifts of $f(x)$ and $xf(x)$ can be derived recursively for $i = 0, 1, 2, \dots$

$$E^i(f(x)) = f(q^i x + [i]h), \quad E^i(xf(x)) = (q^i x + [i]h)f(q^i x + [i]h). \tag{8}$$

Replacing (8) in (7), we have a series for $F(x)$

$$F(x) = ((1-q)x-h)\sum_{i=0}^{\infty} q^i f(q^i x + [i]h). \tag{9}$$

Since f is given to be continuous at the accumulation point $x = \frac{h}{1-q}$, the series (9) is well-defined. The following theorem states the convergence of the series (9), namely the existence of (q,h)-antiderivative $F(x)$ of $f(x)$.

Theorem 2.3. *Let $\mid x^r f(x) \mid < K$ on the interval $(\frac{h}{1-q}, a] \cap \mathbb{T}_{(q,h)}$, for some $K > 0$, $0 < r < 1$ and $0 < q < 1$. Then the series (9) is absolutely and uniformly convergent on $(\frac{h}{1-q}, a] \cap \mathbb{T}_{(q,h)}$. In addition, (q,h)-antiderivative $F(x)$ is continuous at the accumulation point $x = \frac{h}{1-q}$ with $F(\frac{h}{1-q}) = 0$.*

Proof. *Suppose that $\mid x^r f(x) \mid < K$ on the interval $(\frac{h}{1-q}, a] \cap \mathbb{T}_{(q,h)}$, then $\mid (q^i x + [i]h)^r f(q^i x + [i]h) \mid < K$, which implies that for $\frac{h}{1-q} < x \le a$ and $0 < r < 1$,*

$$\mid f(q^i x + [i]h) \mid < K(q^i x + [i]h)^{-r} < K(q^i x)^{-r}.$$

Then the general term of the series (9) reads as

$$\mid q^i f(q^i x + [i]h) \mid < Kq^i (q^i x)^{-r} = Kx^{-r}(q^{1-r})^i.$$

Since $0 < q < 1$ and $1 - r \in (0,1)$, the series $\sum_{i=0}^{\infty}(q^{1-r})^i$ is a convergent geometric series. Thus, by Weierstrass M-test the series (9) is absolutely and uniformly convergent on $(\frac{h}{1-q}, a] \cap \mathbb{T}_{(q,h)}$. Clearly by (9), $F(\frac{h}{1-q}) = 0$ and the continuity *of $F(x)$ at $x = \frac{h}{1-q}$ can be seen from*

$$\left| ((1-q)x - h) \sum_{i=0}^{\infty} q^i f(q^i x + [i]h) \right| < \frac{Kx^{-r}((1-q)x-h)}{1-q^{1-r}},$$

which vanishes as $x \to \frac{h}{1-q}$. Finally, we show that $D_{(q,h)}F(x) = f(x)$ by

$$\begin{aligned} D_{(q,h)}F(x) &= \frac{F(qx+h) - F(x)}{(q-1)x+h} \\ &= \frac{((1-q)(qx+h)-h)\sum_{i=0}^{\infty} q^i f(q^i(qx+h)+[i]h) - ((1-q)x-h)\sum_{i=0}^{\infty} q^i f(q^i x + [i]h)}{(q-1)x+h} \\ &= \sum_{i=0}^{\infty} q^i f(q^i x + [i]h) - \sum_{i=0}^{\infty} q^{i+1} f(q^{i+1}x + [i+1]h) \\ &= \sum_{i=0}^{\infty} q^i f(q^i x + [i]h) - \sum_{i=1}^{\infty} q^i f(q^i x + [i]h) = f(x). \end{aligned}$$

□

Theorem 2.4. *If $F_1(x)$ and $F_2(x)$ are (q,h)-antiderivatives of $f(x)$, then $F_1(x) = F_2(x) + c$, for some constant c.*

The proof is a direct consequence of [20, Corollary 4.2].

Since the existence of the (q,h)-antiderivative requires the condition $0 < q < 1$, throughout this section we stick to that.

Definition 2.5. *Let $f(x)$ be a continuous function at $x = \frac{h}{1-q}$. We introduce the indefinite (q,h)-integral of f by*

$$\int f(x) d_{(q,h)}x := ((1-q)x - h) \sum_{i=0}^{\infty} q^i f(q^i x + [i]h) + c, \tag{10}$$

where c is an arbitrary constant.

Definition 2.6. *Let $f(x)$ be a continuous function at $x = \frac{h}{1-q}$. We introduce the definite (q,h)-integral of f, for $a, b \in \mathbb{T}_{(q,h)}$ with $a < b$*

$$\int_a^b f(x) d_{(q,h)}x := \int_{\frac{h}{1-q}}^b f(x) d_{(q,h)}x - \int_{\frac{h}{1-q}}^a f(x) d_{(q,h)}x, \tag{11}$$

where

$$\int_{\frac{h}{1-q}}^b f(x) d_{(q,h)}x := ((1-q)b - h) \sum_{i=0}^{\infty} q^i f(q^i b + [i]h). \tag{12}$$

We emphasize that the definite (indefinite) (q,h)-integral recovers the ordinary definite (indefinite) integral, definite (indefinite) q- and h-integrals. Indeed, as $h \to 0$, we have $\mathbb{T}_{(q,0)} = \mathbb{K}_q$ and definite (q,h)-integral (11) reduces to definite q-integral

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_{(q,h)}x - \int_0^a f(x) d_{(q,h)}x = ((1-q)b) \sum_{i=0}^{\infty} q^i f(q^i b) - ((1-q)a) \sum_{i=0}^{\infty} q^i f(q^i a).$$

As $q \to 1$, we have $\mathbb{T}_{(1,h)} = h\mathbb{Z}$ in which there is no accumulation point. In this case, definite (q,h)-integral (11) reduces to definite h-integral which is a Riemann sum of $f(x)$ on the interval $[a,b]$

$$\int_a^b f(x)d_h x = -h\sum_{i=0}^{\infty} f(b+ih) + h\sum_{i=0}^{\infty} f(a+ih) = h(f(a)+f(a+h)+\ldots+f(b-h)).$$

Theorem 2.7. *Let $F(x)$ be (q,h)-antiderivative of $f(x)$. Let also $f(x)$ be continuous at $x = \frac{h}{1-q}$. Then we have*

$$\int_a^b f(x)d_{(q,h)}x = F(b) - F(a), \qquad \frac{h}{1-q} < a < b.$$

Proof. The continuity of $F(x)$ at $x = \frac{h}{1-q}$ (see Theorem 2.3) guarantees the existence of $F(\frac{h}{1-q})$. By Theorem 2.4, (q,h)-antiderivative $F(x)$ of $f(x)$ can be written of the form

$$F(x) = ((1-q)x-h)\sum_{i=0}^{\infty} q^i f(q^i x + [i]h) + F(\frac{h}{1-q}),$$

up to the additive constant $F(\frac{h}{1-q})$. Also by Definition 2.6, we have

$$\int_{\frac{h}{1-q}}^b f(x)d_{(q,h)}x = ((1-q)b-h)\sum_{i=0}^{\infty} q^i f(q^i b + [i]h) = F(b) - F(\frac{h}{1-q}),$$

which implies that

$$\int_a^b f(x)d_{(q,h)}x = F(b) - F(\frac{h}{1-q}) - (F(a) - F(\frac{h}{1-q})) = F(b) - F(a).$$

□

Theorem 2.8. *Let $D_{(q,h)}f(\frac{h}{1-q})$ exists. Then (q,h)-analogue of Fundamental Theorem of Calculus can be presented as*

$$\int_a^b D_{(q,h)}f(x)d_{(q,h)}x = f(b) - f(a), \qquad \frac{h}{1-q} < a < b.$$

Proof. Let $D_{(q,h)}f(\frac{h}{1-q})$ exists. Then by equation (5), $D_{(q,h)}f(\frac{h}{1-q}) = f'(\frac{h}{1-q})$. The continuity of $D_{(q,h)}f(x)$ at $x = \frac{h}{1-q}$ follows by

$$\begin{aligned} \lim_{x\to\frac{h}{1-q}} D_{(q,h)}f(x) &= \lim_{x\to\frac{h}{1-q}} \frac{f(qx+h)-f(x)}{(q-1)x+h} = \lim_{x\to\frac{h}{1-q}} \frac{qf'(qx+h)-f'(x)}{q-1} \\ &= \frac{qf'(\frac{h}{1-q})-f'(\frac{h}{1-q})}{q-1} = f'(\frac{h}{1-q}) = D_{(q,h)}f(\frac{h}{1-q}), \end{aligned}$$

where we used L'Hopital's rule. Hence the proof finishes by Theorem 2.7. □

Theorem 2.9. *If $f(x)$ is continuous at $x = \frac{h}{1-q}$, then the second version of Fundamental Theorem of (q,h)-Calculus is as follows*

$$D_{(q,h)}\left(\int_{\frac{h}{1-q}}^x f(s)d_{(q,h)}s\right) = f(x).$$

Proof. By the definition of (q,h)-derivative, we have

$$D_{(q,h)}\left(\int_{\frac{h}{1-q}}^{x} f(s)d_{(q,h)}s\right) = \frac{\int_{\frac{h}{1-q}}^{qx+h} f(s)d_{(q,h)}s - \int_{\frac{h}{1-q}}^{x} f(s)d_{(q,h)}s}{(q-1)x+h} = f(x).$$

The result is a direct consequence of the definite integral (12). □

Theorem 2.10. *The integration by parts formula for indefinite (q,h)-integral can be presented as*

$$\int f(x)D_{(q,h)}g(x)d_{(q,h)}x = f(x)g(x) - \int g(qx+h)D_{(q,h)}f(x)d_{(q,h)}x.$$

Proof. The proof directly follows from the product rule of (q,h)-derivative (see Proposition 2.2). □

3. Generalized Gauss's Binomial Formula

In this section, our primary goal is to present Gauss's binomial formula on $\mathbb{T}_{(q,h)}$, which provides many contributions such as additive property of exponential functions on $\mathbb{T}_{(q,h)}$. We present Gauss's binomial formula expressed in terms of proper polynomials on $\mathbb{T}_{(q,h)}$, rather than the ordinary ones, in a way that they obey the nature of the time scales $\mathbb{T}_{(q,h)}$ without requiring commutation restrictions. To be more precise, the role of such proper polynomials on $\mathbb{T}_{(q,h)}$ needs to be similar to the role of the ordinary polynomials in $\mathbb{R}$.

We introduce a (q,h)-analogue of the polynomial $(\gamma x - \delta x_0)^n$ as follows.

Definition 3.1. *Let $x_0 \in \mathbb{R}$ and $\gamma, \delta \in \{-1, 1\}$. We define the generalized quantum binomial as the polynomial*

$$(\gamma x - \delta x_0)^n_{q,h} := \begin{cases} 1 & \text{if} \quad n = 0, \\ \prod\limits_{i=1}^{n} \gamma\left(x - \gamma\delta q^{i-1}x_0 - [i-1]h\right) & \text{if} \quad n > 0. \end{cases} \tag{13}$$

We need to improve the binomial $(x - x_0)^n_{q,h}$, presented in [18, Definition 3.3], to the form (13) in order to generalize and analyze the polynomials $(x - x_0)^n_{q,h}$, $(x + x_0)^n_{q,h}$, $(-x - x_0)^n_{q,h}$ and $(-x + x_0)^n_{q,h}$ at one hand. The reason of such description can be observed in the forthcoming sections. By the definition (13), one may observe the following relations

$$(-x + x_0)^n_{q,h} = (-1)^n(x - x_0)^n_{q,h}, \tag{14}$$

$$(-x - x_0)^n_{q,h} = (-1)^n(x + x_0)^n_{q,h}. \tag{15}$$

Note also that, the generalized quantum binomial $(\gamma x - \delta x_0)^n_{q,h}$ introduced on (13) recovers the polynomial $(\gamma x - \delta x_0)^n$ as $(q,h) \to (1,0)$ and satisfies the similar properties in $\mathbb{T}_{(q,h)}$, as $(\gamma x - \delta x_0)^n$ does in ordinary calculus. Unless otherwise stated throughout this article, we suppose that $\gamma, \delta \in \{-1, 1\}$.

Proposition 3.2. *The generalized quantum binomial* (13) *satisfies the Leibniz rules*

(i) $D_{(q,h)}(\gamma x - \delta x_0)^n_{q,h} = \gamma[n](\gamma x - \delta x_0)^{n-1}_{q,h}, \quad n = 1, 2, \ldots.$

(ii) $D^k_{(q,h)}(\gamma x - \delta x_0)^n_{q,h} = \gamma^k \frac{[n]!}{[n-k]!}(\gamma x - \delta x_0)^{n-k}_{q,h}, \quad 0 \le k \le n.$

Proof. (i) We apply delta (q,h)-derivative (3) on (13)

$$\begin{aligned} D_{(q,h)}(\gamma x - \delta x_0)^n_{q,h} &= \frac{(\gamma(qx+h) - \delta x_0)^n_{q,h} - (\gamma x - \delta x_0)^n_{q,h}}{(q-1)x+h} \\ &= \frac{q^{n-1}(qx + h - \gamma\delta x_0) - (x - \gamma\delta q^{n-1}x_0 - (1 + q + \ldots + q^{n-2})h)}{(q-1)x+h}\, \gamma(\gamma x - \delta x_0)^{n-1}_{q,h} \\ &= \frac{(q^n - 1)x + (1 + q + \ldots q^{n-1})h}{(q-1)x+h}\, \gamma(\gamma x - \delta x_0)^{n-1}_{q,h} = \gamma[n](\gamma x - \delta x_0)^{n-1}_{q,h}. \end{aligned}$$

Part (ii) yields by applying delta (q,h)-derivative k-times successively on (13). □

We are ready to present (q,h)-analogue of Taylor's formula.

Theorem 3.3. *Let $\{P_i\}_{i\geq 0}$ be a sequence of polynomials where*

$$P_i(x) := \frac{(\gamma x - \delta x_0)^i_{q,h}}{[i]!}, \tag{16}$$

and $[i]! := [i].[i-1]...[2].[1]$ *with* $[0]! := 1$. *Then*
(a) *The polynomials $P_i(x)$ satisfy the following criteria simultaneously*

(i) $P_0(\gamma\delta x_0) = 1, \qquad P_i(\gamma\delta x_0) = 0, \qquad i = 1, 2, 3, \ldots$

(ii) $degP_i = i, \qquad i = 0, 1, 2\ldots$

(iii) $D_{(q,h)}P_i = \gamma P_{i-1}, \quad i = 1, 2, 3\ldots$

(b) *Any polynomial $Q(x)$ of degree n has the following Taylor's Formula*

$$Q(x) = \sum_{i=0}^{n} \gamma^i D^i_{(q,h)} Q(\gamma\delta x_0) \frac{(\gamma x - \delta x_0)^i_{q,h}}{[i]!}, \tag{17}$$

where $D^i_{(q,h)}$ refers to the delta (q,h)-derivative of order i.

Proof. (a) By (16), it is straightforward to observe that the conditions (i) and (ii) are automatically satisfied. By the use of Proposition 3.2, the condition (iii)

$$D_{(q,h)}P_i(x) = D_{(q,h)}\left(\frac{(\gamma x - \delta x_0)^i_{q,h}}{[i]!}\right) = \frac{\gamma[i](\gamma x - \delta x_0)^{i-1}_{q,h}}{[i]!} = \frac{\gamma(\gamma x - \delta x_0)^{i-1}_{q,h}}{[i-1]!} = \gamma P_{i-1}, \tag{18}$$

is also verified.
(b) Let W be $(n+1)$-dimensional vector space of polynomials and $B := \{P_0(x), P_1(x), ...P_n(x)\}$. The set B is linearly independent which follows from (ii), i.e., $degP_i = i$, for each i. Since $|B| = n+1$, B spans W and therefore becomes a basis for W. In other words, any polynomial $Q(x) \in W$ can be written as a linear combination of polynomials in B

$$Q(x) = \sum_{i=0}^{n} a_i P_i(x). \tag{19}$$

Using the condition (i), we obtain

$$Q(\gamma\delta x_0) = \sum_{i=0}^{n} a_i P_i(\gamma\delta x_0) = a_0 P_0(\gamma\delta x_0) = a_0.$$

The linearity of $D_{(q,h)}$ and the condition (iii) provide that

$$D_{(q,h)}Q(x) = \sum_{i=0}^{n} a_i D_{(q,h)} P_i(x) = \sum_{i=1}^{n} a_i \gamma P_{i-1}(x),$$

which implies $D_{(q,h)}Q(\gamma\delta x_0) = \gamma a_1$. It also means that $a_1 = \gamma D_{(q,h)}Q(\gamma\delta x_0)$ since $\gamma = \mp 1$. Applying $D_{(q,h)}$, k times to $Q(x)$, we deduce that

$$D^k_{(q,h)}Q(x) = \sum_{i=k}^{n} a_i D^k_{(q,h)} P_i(x) = \sum_{i=k}^{n} a_i \gamma^k P_{i-k}(x),$$

which allows us to derive $D^k_{(q,h)}Q(\gamma\delta x_0) = a_k\gamma^k$, i.e., $a_k = \gamma^k D^k_{(q,h)}Q(\gamma\delta x_0)$, $\quad 0 \le k \le n$. Therefore

$$Q(x) = \sum_{i=0}^{n} \gamma^i D^i_{(q,h)}Q(\gamma\delta x_0)P_i(x) = \sum_{i=0}^{n} \gamma^i D^i_{(q,h)}Q(\gamma\delta x_0)\frac{(\gamma x-\delta x_0)^i_{q,h}}{[i]!}. \tag{20}$$

□

We present additional properties of the generalized quantum binomial (13) in the following propositions.

Proposition 3.4. *The generalized quantum binomial* (13) *satisfies the identity*

$$(\gamma x-\delta x_0)^{m+n}_{q,h} = (\gamma x-\delta x_0)^m_{q,h}\cdot(\gamma x-\delta(q^m x_0+\gamma\delta[m]h))^n_{q,h}, \qquad m,n=0,1,2\ldots$$

Proof. If $m = 0$ or $n = 0$ or both the proof of the identity trivially follows. Assume that both m and n are positive. By the definition of the generalized quantum binomial (13), we can write

$$\begin{aligned}(\gamma x-\delta x_0)^{m+n}_{q,h} =&\gamma^{m+n}(x-\gamma\delta x_0)(x-\gamma\delta q x_0-h)\cdots(x-\gamma\delta q^{m-1}x_0-[m-1]h)(x-\gamma\delta q^m x_0-[m]h)\\ &\cdots(x-\gamma\delta q^{m+n-1}x_0-[m+n-1]h)\\ =&(\gamma x-\delta x_0)^m_{q,h}\cdot g(x;x_0),\end{aligned}$$

where $g(x;x_0) = \gamma^n\cdot(x-\gamma\delta q^m x_0-[m]h)\cdots(x-\gamma\delta q^{m+n-1}x_0-[m+n-1]h)$. Note that, if we replace x_0 by $q^m x_0+\gamma\delta[m]h$ in $g(x;x_0)$, we finish the proof. □

Inspired by Proposition 3.4, we can extend the generalized quantum binomial (13) to all integers. Using $(\gamma x-\delta x_0)^0_{q,h} = 1$, we define

$$(\gamma x-\delta x_0)^{-n}_{q,h} := \frac{1}{(\gamma x-\delta q^{-n}(x_0-\gamma\delta[n]h))^n_{q,h}}, \tag{21}$$

where we used the relation $[-n] = -q^{-n}[n]$. Equation (21) allows us to generalize the Proposition 3.4 to all integers:

Proposition 3.5. *The following identity holds for $m,n\in\mathbb{Z}$*

$$(\gamma x-\delta x_0)^{m+n}_{q,h} = (\gamma x-\delta x_0)^m_{q,h}\cdot(\gamma x-\delta(q^m x_0+\gamma\delta[m]h))^n_{q,h}.$$

Proof. The all possible cases for $m,n = 0,1,2\ldots$ are considered in Proposition 3.4. Assume that $m = -m' < 0$ and $n > 0$. Using (21), we obtain

$$\begin{aligned}(\gamma x-\delta x_0)^{-m'}_{q,h}\cdot(\gamma x-\delta(q^{-m'}x_0+\gamma\delta[-m']h))^n_{q,h} &= \frac{(\gamma x-\delta q^{-m'}(x_0-\gamma\delta[m']))^n_{q,h}}{(\gamma x-\delta q^{-m'}(x_0-\gamma\delta[m']h))^{m'}_{q,h}}\\ &= \begin{cases}\gamma^{n-m'}(x-\gamma\delta x_0)(x-\gamma\delta q x_0-h)\cdots(x-\gamma\delta q^{n-m'-1}x_0-[n-m'-1]h), & n\ge m'>0;\\ \dfrac{1}{(\gamma x-\delta q^{-(m'-n)}(x_0-\gamma\delta[m'-n]h))^{m'-n}_{q,h}}, & m'>n>0\end{cases}\\ &= (\gamma x-\delta x_0)^{n-m'}_{q,h} = (\gamma x-\delta x_0)^{n+m}_{q,h}.\end{aligned}$$

The proofs of the cases $m > 0$, $n < 0$ and $m < 0$, $n < 0$ are similar to the above proof by using (21) and the definition of generalized quantum binomial (13). □

Proposition 3.6. *The following Leibniz rules hold:*

(i) $D_{(q,h)}(\gamma x-\delta x_0)^{-n}_{q,h} = \gamma[-n](\gamma x-\delta x_0)^{-n-1}_{q,h}, \qquad n\in\mathbb{Z}^+.$

(ii) $D^k_{(q,h)}(\gamma x-\delta x_0)^{-n}_{q,h} = \dfrac{(-\gamma)^k}{q^n q^{n+1}\cdots q^{n+k-1}} \dfrac{[n+k-1]!}{[n-1]!}(\gamma x-\delta x_0)^{-n-k}_{q,h}, \quad n,k\in\mathbb{Z}^+.$

Proof. (i) The quotient rule (see Proposition 2.2) and the relation $[-n]=-q^{-n}[n]$ lead us to have

$$\begin{aligned}
D_{(q,h)}(\gamma x-\delta x_0)^{-n}_{q,h} =& D_{(q,h)}\left(\frac{1}{(\gamma x-\delta(q^{-n}x_0+\gamma\delta[-n]h))^n_{q,h}}\right)\\
=&\frac{-\gamma[n]\cdot(\gamma x-\delta(q^{-n}x_0+\gamma\delta[-n]h))^{n-1}_{q,h}}{(\gamma x-\delta(q^{-n}x_0+\gamma\delta[-n]h))^n_{q,h}\cdot(\gamma(qx+h)-\delta(q^{-n}x_0+\gamma\delta[-n]h))^n_{q,h}}\\
=&\frac{-\gamma[n]\cdot(\gamma x-\delta(q^{-n}x_0+\gamma\delta[-n]h))^{n-1}_{q,h}}{\gamma\cdot(\gamma x-\delta(q^{-n}x_0+\gamma\delta[-n]h))^{n-1}_{q,h}\cdot(x-\gamma\delta q^{-1}x_0-[-1]h)\cdot(\gamma(qx+h)-\delta(q^{-n}x_0+\gamma\delta[-n]h))^n_{q,h}}\\
=&\frac{-\gamma[n]}{\gamma^{n+1}q^n(x-\gamma\delta q^{-n-1}x_0-[-n-1]h)\cdot(x-\gamma\delta q^{-n}x_0-[-n]h)\cdots(x-\gamma\delta q^{-1}x_0-[-1]h)}\\
=&\frac{\gamma[-n]}{(\gamma x-\delta q^{-n-1}(x_0-\gamma\delta[n+1]h))^{n+1}_{q,h}}=\gamma[-n](\gamma x-\delta x_0)^{-n-1}_{q,h}.
\end{aligned}$$

The proof of (ii) directly follows from (i). □

In the light of (q,h)-analogue of Taylor's formula (Theorem 3.3), we state and prove (q,h)-analogue of Gauss's Binomial formula.

Theorem 3.7. *The Gauss's Binomial formula on $\mathbb{T}_{(q,h)}$ can be expressed as*

$$(\gamma x-\delta x_0)^n_{q,h}=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(0-\delta x_0)^{n-k}_{q,h}\cdot(\gamma x-0)^k_{q,h}, \tag{22}$$

where $\begin{bmatrix} n\\ k\end{bmatrix}=\dfrac{[n]!}{[n-k]![k]!}$.

Proof. Let $f(x)=(\gamma x-\delta x_0)^n_{q,h}$. We utilize Theorem 3.3, about $x_0=0$ which implies that

$$(\gamma x-\delta x_0)^n_{q,h}=\sum_{k=0}^{n}\gamma^k D^k_{(q,h)}f(0)\frac{(\gamma x-0)^k_{q,h}}{[k]!}. \tag{23}$$

Clearly $f(0)=(0-\delta x_0)^n_{q,h}$. By Proposition 3.2, we derive

$$D^k_{(q,h)}f(x)=\gamma^k\frac{[n]!}{[n-k]!}(\gamma x-\delta x_0)^{n-k}_{q,h},$$

which leads to

$$D^k_{(q,h)}f(0)=\gamma^k\frac{[n]!}{[n-k]!}(0-\delta x_0)^{n-k}_{q,h}, \quad 0\le k\le n.$$

Therefore the equation (23) can be written as

$$(\gamma x-\delta x_0)^n_{q,h}=\sum_{k=0}^{n}\gamma^k\gamma^k\frac{[n]!}{[n-k]![k]!}(0-\delta x_0)^{n-k}_{q,h}\cdot(\gamma x-0)^k_{q,h}=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(0-\delta x_0)^{n-k}_{q,h}\cdot(\gamma x-0)^k_{q,h},$$

since $\gamma^{2k}=1$ and $\begin{bmatrix} n\\ k\end{bmatrix}=\dfrac{[n]!}{[n-k]![k]!}$. □

Remark 3.8. *As $h \to 0$, the formula* (22) *becomes*

$$(x + x_0)^n_{q,0} = (x + x_0)^n_q = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{(n-k)(n-k-1)}{2}} x_0^{n-k} x^k, \tag{24}$$

where $(0 + x_0)^{n-k}_q = x_0.(qx_0)(q^2x_0)\cdots(q^{n-k-1}x_0) = q^{\frac{(n-k)(n-k-1)}{2}} x_0^{n-k}$ *and* $(x - 0)^k_q = x^k$ *for* $\gamma = 1,\ \delta = -1$. *Note that, the formula* (24) *is nothing but the celebrated Gauss's Binomial formula.*

It is essential to emphasize that the formula (22) is a generalization of the classical Newton's Binomial formula, because as $(q, h) \to (1, 0)$, (22) reduces to

$$(x + x_0)^n = \sum_{k=0}^{n} \binom{n}{k} x_0^{n-k} x^k,$$

under the setting $\gamma = 1$ and $\delta = -1$. For that reason, the binomial formula (22) can be also regarded as (q, h)-analogue of Newton's Binomial formula. In [2], another (q, h)-Newton's binomial formula was presented. But in that work, the formula is expressed on ordinary polynomials and the coordinates need to satisfy some commutation relations.

4. Additive property of exponential functions on $\mathbb{T}_{(q,h)}$

In the literature, the additive property of exponential functions is very crucial not only in the field of analysis but also in the theory of differential equations. The lack of the additive identity even in $\mathbb{K}_q$ (it exists only for q- commuting coordinates), therefore on arbitrary time scales, restricts the applicability of exponential functions. For that reason, this section is devoted to present the additive property of exponential functions on $\mathbb{T}_{(q,h)}$.

We begin this section by introducing a power series written in terms of $(\gamma x - \delta x_0)^n_{q,h}$ as follows:

$$\sum_{n=0}^{\infty} c_n\, (\gamma x - \delta x_0)^n_{q,h}.$$

Clearly this series converges to c_0 at the point $x = \gamma\delta x_0$. We start with Nth partial sum of such power series and find the form of the coefficients.

Lemma 4.1. *If* $f(x) = \sum_{n=0}^{N} c_n\, (\gamma x - \delta x_0)^n_{q,h}$, *then*

$$c_n = \frac{\gamma^n D^n_{(q,h)} f(\gamma\delta x_0)}{[n]!}. \tag{25}$$

Proof. Let $f(x) = \sum_{n=0}^{N} c_n\, (\gamma x - \delta x_0)^n_{q,h}$, then $f(\gamma\delta x_0) = c_0$. Since $D_{(q,h)}$ is a linear operator, using Proposition 3.2 we obtain

$$D_{(q,h)} f(x) = \sum_{n=0}^{N} c_n D_{(q,h)}\, (\gamma x - \delta x_0)^n_{q,h} = \sum_{n=1}^{N} c_n \gamma [n]\, (\gamma x - \delta x_0)^{n-1}_{q,h},$$

which implies that $D_{(q,h)} f(\gamma\delta x_0) = c_1\gamma[1]$, i.e., $c_1 = \gamma D_{(q,h)} f(\gamma\delta x_0)$ since $\gamma^2 = 1$. Continuing in the same way, we derive

$$D^k_{(q,h)} f(x) = \sum_{n=0}^{N} c_n D^k_{(q,h)}\, (\gamma x - \delta x_0)^n_{q,h} = \sum_{n=k}^{N} c_n \gamma^k \frac{[n]!}{[n-k]!}\, (\gamma x - \delta x_0)^{n-k}_{q,h}.$$

Hence for $0 \le k \le N$, we have $D^k_{(q,h)} f(\gamma\delta x_0) = c_k \gamma^k \dfrac{[k]!}{[k-k]!}$, i.e., $c_k = \dfrac{\gamma^k D^k_{(q,h)} f(\gamma\delta x_0)}{[k]!}$. □

Definition 4.2. *We introduce (q,h)-Taylor series of f at $\gamma\delta x_0$ as the series*

$$\sum_{n=0}^{\infty} \frac{\gamma^n D^n_{(q,h)} f(\gamma\delta x_0)\, (\gamma x - \delta x_0)^n_{q,h}}{[n]!}. \tag{26}$$

We aim to discuss the conditions under which (q,h)-Taylor series (26) is convergent. For this purpose, we need the following auxiliary lemma.

Lemma 4.3. *For any $q < 1$ and $0 \le x_0 < x$, the following inequality*

$$\frac{|\,(\gamma x - \delta x_0)^n_{q,h}\,|}{[n]!} \le \frac{(x + x_0)^n}{[n]!}$$

holds.

Proof. We investigate the proof for different choices of γ and δ.
(a) Let $\gamma = \delta = 1$. In this case generalized quantum binomial (13) becomes

$$(\gamma x - \delta x_0)^n_{q,h} = (x - x_0)^n_{q,h} = (x - x_0)(x - qx_0 - h)\cdots(x - q^{n-1}x_0 - [n-1]h). \tag{27}$$

The inequality $0 < x - q^i x_0 - [i]h \le x - q^i x_0 \le x + q^i x_0 \le x + x_0$ holds for all $0 \le i \le n-1$ and for $x > x_0 > 0$. Furthermore, since $\lim\limits_{i\to\infty}(x - q^i x_0 - [i]h) = \lim\limits_{i\to\infty}(x - [i]h) = x - \dfrac{h}{1-q} > 0$, even the smallest distance in (27) is positive in the case $x_0 = 0$. Therefore we acquire

$$\frac{|\,(x - x_0)^n_{q,h}\,|}{[n]!} = \frac{(x - x_0)(x - qx_0 - h)\cdots(x - q^{n-1}x_0 - [n-1]h)}{[n]!} \le \frac{(x + x_0)(x + qx_0)\cdots(x + q^{n-1}x_0)}{[n]!} \le \frac{(x + x_0)^n}{[n]!}.$$

(b) Let $\gamma = \delta = -1$. The relation (14) and part (a) allow us to obtain

$$\frac{|\,(\gamma x - \delta x_0)^n_{q,h}\,|}{[n]!} = \frac{|\,(-x + x_0)^n_{q,h}\,|}{[n]!} = \frac{|\,(-1)^n (x - x_0)^n_{q,h}\,|}{[n]!} = \frac{|(x - x_0)^n_{q,h}|}{[n]!} \le \frac{(x + x_0)^n}{[n]!}.$$

(c) Let $\gamma = 1, \delta = -1$. In this case generalized quantum binomial (13) becomes

$$(\gamma x - \delta x_0)^n_{q,h} = (x + x_0)^n_{q,h} = (x + x_0)(x + qx_0 - h)\cdots(x + q^{n-1}x_0 - [n-1]h).$$

Since $\lim\limits_{i\to\infty}(x + q^i x_0 - [i]h) = x - \dfrac{h}{1-q} > 0$, then the inequality $0 < x + q^i x_0 - [i]h \le x + q^i x_0 \le x + x_0$ holds for all $0 \le i \le n-1$. Then we have

$$\frac{|\,(x + x_0)^n_{q,h}\,|}{[n]!} = \frac{(x + x_0)(x + qx_0 - h)\cdots(x + q^{n-1}x_0 - [n-1]h)}{[n]!} \le \frac{(x + x_0)(x + qx_0)\cdots(x + q^{n-1}x_0)}{[n]!} \le \frac{(x + x_0)^n}{[n]!}.$$

(d) Finally let $\gamma = -1, \delta = 1$. By the relation (15) and part (c), we derive

$$\frac{|\,(\gamma x - \delta x_0)^n_{q,h}\,|}{[n]!} = \frac{|\,(-x - x_0)^n_{q,h}\,|}{[n]!} = \frac{|\,(-1)^n (x + x_0)^n_{q,h}\,|}{[n]!} = \frac{|\,(x + x_0)^n_{q,h}\,|}{[n]!} \le \frac{(x + x_0)^n}{[n]!}.$$

□

Theorem 4.4. *If $q < 1$, $0 \le x_0 < x$ and $| D^n_{(q,h)} f(\gamma\delta x_0) | < K^n$ for some $K > 0$ and $n \in \mathbb{N}_0$, then the series* (26) *is absolutely and uniformly convergent on* $\{x : x + x_0 < \frac{1}{(1-q)K}\} \cap \mathbb{T}_{(q,h)}$.

Proof. Let $q < 1$. By Lemma 4.3, we have

$$\frac{\left|\gamma^n D^n_{(q,h)} f(\gamma\delta x_0)(\gamma x - \delta x_0)^n_{q,h}\right|}{[n]!} \le K^n \frac{(x + x_0)^n}{[n]!}.$$

For the bounding series, we utilize Ratio test and observe that the below limit

$$\lim_{n\to\infty} \left| \frac{K^{n+1}(x + x_0)^{n+1}}{[n+1]!} \frac{[n]!}{K^n(x + x_0)^n} \right| = K \lim_{n\to\infty} \frac{x + x_0}{[n+1]} = K(x + x_0)(1 - q) < 1,$$

provided that $x + x_0 < \frac{1}{(1-q)K}$. Thus, by Weierstrass M-test the series (26) is absolutely and uniformly convergent on $\{x : x + x_0 < \frac{1}{(1-q)K}\} \cap \mathbb{T}_{(q,h)}$.
□

Definition 4.5. *A function $f : \mathbb{T}_{(q,h)} \to \mathbb{R}$ is called as (q, h)-analytic at $\gamma\delta x_0$ if and only if there exists a power series centered at $\gamma\delta x_0$ that converges to f in the neighborhood of $\gamma\delta x_0$.*

Therefore Theorem 4.4 provides sufficient conditions for a function f to be (q, h)-analytic.
Now we aim to introduce (q, h)-analogue of exponential function in a way that it is expressed in Taylor series as in (26) and its delta (q, h)-derivative is proportional to itself.

Definition 4.6. *For an arbitrary nonzero constant $\alpha \in \mathbb{R}$, we define an exponential function on $\mathbb{T}_{(q,h)}$ by the series*

$$Exp_{(q,h)}(\alpha(\gamma x - \delta x_0)) := \sum_{j=0}^{\infty} \frac{\alpha^j (\gamma x - \delta x_0)^j_{q,h}}{[j]!}, \tag{28}$$

provided that the series is convergent.

Proposition 4.7. *For a nonzero constant α, (q, h)-exponential function* (28) *satisfies the chain rule*

$$D_{(q,h)} Exp_{(q,h)}(\alpha(\gamma x - \delta x_0)) = \alpha\gamma Exp_{(q,h)}(\alpha(\gamma x - \delta x_0)).$$

Proof. Using Proposition 3.2, one can compute the delta (q, h)-derivative of such exponential function as

$$D_{(q,h)} Exp_{(q,h)}(\alpha(\gamma x - \delta x_0)) = \sum_{j=0}^{\infty} \frac{\alpha^j}{[j]!} D_{(q,h)}(\gamma x - \delta x_0)^j_{q,h} = \alpha\gamma \sum_{j=1}^{\infty} \frac{\alpha^{j-1}}{[j-1]!}(\gamma x - \delta x_0)^{j-1}_{q,h} = \alpha\gamma Exp_{(q,h)}(\alpha(\gamma x - \delta x_0)).$$

□

It is straightforward that $Exp_{(q,h)}(0) = 1$ and $|D^n_{(q,h)} Exp_{(q,h)}(\alpha(\gamma^2\delta x_0 - \delta x_0))| =| \gamma^n\alpha^n |=| \alpha^n |$. Theorem 4.4 assures that (28) is absolutely and uniformly convergent on $\{x : x + x_0 < \dfrac{1}{(1-q) \mid \alpha \mid}\} \cap \mathbb{T}_{(q,h)}$ for $q < 1$. Therefore, (28) is a (q, h)-analytic function. This consequence is consistent with q-exponential function. [13].

Remark 4.8. *Exponential function* (28) *recovers many exponential functions studied in the literature.*
(i) For $\gamma = 1$ and $x_0 = 0$, (28) *reduces to the (q, h)-exponential function* $\exp_{(q,h)}(\alpha x)$*, introduced in [18]*

$$Exp_{(q,h)}(\alpha(x - 0)) = \sum_{j=0}^{\infty} \frac{\alpha^j (x - 0)^j_{q,h}}{[j]!} = \exp_{(q,h)}(\alpha x),$$

whose convergence follows from Theorem 4.4. Furthermore, as $(q,h) \to (1,0)$, (28) *becomes the ordinary exponential function* $e^{\alpha x} = \sum_{j=0}^{\infty} \frac{(\alpha x)^j}{j!}$.

For $q<1$, *if we consider*

$$Exp_{(q,h)}(\alpha(0-\delta x)) = \sum_{n=0}^{\infty} \frac{\alpha^n (0-\delta x)^n_{q,h}}{[n]!} = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n (\delta x)(\delta qx + \gamma h)\cdots(\delta q^{n-1}x + \gamma[n-1]h)}{[n]!}, \tag{29}$$

its convergence yields by Ratio test

$$\lim_{n\to\infty}\left|\frac{(-\alpha)^{n+1}(\delta x)(\delta qx+\gamma h)\cdots(\delta q^n x+\gamma[n]h)}{[n+1]!}\frac{[n]!}{(-\alpha)^n(\delta x)(\delta qx+\gamma h)\cdots(\delta q^{n-1}x+\gamma[n-1]h)}\right|$$
$$= \lim_{n\to\infty}\frac{|\alpha(\delta q^n x+\gamma[n]h)|}{[n+1]} \le |\alpha||x| \lim_{n\to\infty}\frac{q^n}{[n+1]} + |\alpha|h \lim_{n\to\infty}\frac{[n]}{[n+1]} = |\alpha|h < 1,$$

provided that $h < \frac{1}{|\alpha|}$.

(ii) In addition, as $h \to 0$, (q,h)*-exponential function* (28) *recovers q-exponential functions* e_q *and* E_q. *Indeed, if* $\gamma = \alpha = 1$ *and* $x_0 = 0$, (q,h)*-exponential function* (28) *reduces to* e_q

$$Exp_{(q,0)}(x-0) = \sum_{j=0}^{\infty}\frac{(x-0)^j_q}{[j]!} = \sum_{j=0}^{\infty}\frac{x^j}{[j]!} = e^x_q,$$

since $(x-0)^j_q = x^j$. *If* $x=0$, $x_0=y$, *then we derive*

$$(0-\delta y)^j_q = \gamma^j(-\gamma\delta y)(-\gamma\delta qy)...(-\gamma\delta q^{j-1}y) = (-1)^j\delta^j y^j q^{\frac{j(j-1)}{2}},$$

which implies

$$Exp_{(q,0)}(\alpha(0-\delta y)) = \sum_{j=0}^{\infty}\frac{\alpha^j(0-\delta y)^j_q}{[j]!} = \sum_{j=0}^{\infty}\frac{(-\alpha\delta y)^j q^{\frac{j(j-1)}{2}}}{[j]!}. \tag{30}$$

Now (30) *reduces to Euler's second q-exponential functions for the choices* $\alpha=1$, $\delta=-1$ *and* $\delta=\alpha=1$

$$\begin{aligned} Exp_{(q,0)}(0+y) &= \sum_{j=0}^{\infty}\frac{y^j q^{\frac{j(j-1)}{2}}}{[j]!} = E^y_q,\\ Exp_{(q,0)}(0-y) &= \sum_{j=0}^{\infty}\frac{(-1)^j y^j q^{\frac{j(j-1)}{2}}}{[j]!} = E^{-y}_q, \end{aligned}$$

respectively.

(iii) As $q \to 1$, (q,h)*-exponential function* (28) *recovers h-exponential function. Indeed if* $\gamma=1$, $x_0=0$, (q,h)*-exponential function* (28) *reduces to*

$$Exp_{(1,h)}(\alpha(x-0)) = \sum_{j=0}^{\infty}\frac{\alpha^j(x-0)^j_h}{j!} = \sum_{j=0}^{\infty}\left(\frac{x}{h}(\frac{x}{h}-1)\cdots(\frac{x}{h}-(j-1))\right)\frac{(\alpha h)^j}{j!} = \sum_{j=0}^{\infty}\binom{\frac{x}{h}}{j}(\alpha h)^j = (1+\alpha h)^{\frac{x}{h}}.$$

In the limit $q\to 1$ *with* $x=0$, $x_0=y$, *and* $\delta=-1$, $(0+y)^j_h$ *still depends on* γ, *i.e.,* $(0+y)^j_h = \gamma^j(\gamma\, y)(\gamma y - h)\cdots(\gamma y - (j-1)h)$. *When* $\gamma=1$, *we derive a similar function*

$$Exp_{(1,h)}(\alpha(0+y)) = \sum_{j=0}^{\infty}\frac{\alpha^j y(y-h)(y-2h)...(y-(j-1)h)}{j!} = (1+\alpha h)^{\frac{y}{h}}.$$

On the other hand, when $\gamma = -1$, (q,h)-exponential function (28) *reduces to another h-exponential function [21].*

$$
\begin{aligned}
Exp_{(1,h)}(\alpha(0+y)) &= \sum_{j=0}^{\infty} \frac{\alpha^j (0+y)_h^j}{j!} = \sum_{j=0}^{\infty} \frac{\alpha^j y(y+h)(y+2h)...(y+(j-1)h)}{j!} \\
&= \sum_{j=0}^{\infty} \left(\frac{-y}{h} (\frac{-y}{h} - 1) \cdots (\frac{-y}{h} - (j-1)) \right) \frac{(-\alpha h)^j}{j!} = \sum_{j=0}^{\infty} \binom{\frac{-y}{h}}{j} (-\alpha h)^j = (1-\alpha h)^{\frac{-y}{h}}.
\end{aligned}
$$

Similarly, $(0-y)_h^j = \gamma^j(-\gamma\, y)(-\gamma y - h)\cdots(-\gamma y-(j-1)h)$, *produces similar functions:* $Exp_{(1,h)}(\alpha(0-y)) = (1+\alpha h)^{\frac{-y}{h}}$ *and* $Exp_{(1,h)}(\alpha(0-y)) = (1-\alpha h)^{\frac{y}{h}}$ *for $\gamma = 1$ and $\gamma = -1$, respectively.*

Theorem 4.9. *For a nonzero constant α, (q,h)-exponential function* (28) *satisfies the following additive property*

$$Exp_{(q,h)}(\alpha(\gamma x - \delta y)) = Exp_{(q,h)}(\alpha(0-\delta y)) \cdot Exp_{(q,h)}(\alpha(\gamma x - 0)). \tag{31}$$

Proof. Consider

$$Exp_{(q,h)}(\alpha(0-\delta y))Exp_{(q,h)}(\alpha(\gamma x - 0)) = \sum_{j=0}^{\infty} \frac{\alpha^j (0-\delta y)_{q,h}^j}{[j]!} \sum_{k=0}^{\infty} \frac{\alpha^k (\gamma x - 0)_{q,h}^k}{[k]!} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{j+k}(0-\delta y)_{q,h}^j (\gamma x - 0)_{q,h}^k}{[j]![k]!}. \tag{32}$$

We multiply and divide the equation (32) by $[j+k]!$ and then use the substitution $j+k=n$, then we derive

$$
\begin{aligned}
Exp_{(q,h)}(\alpha(0-\delta y))Exp_{(q,h)}(\alpha(\gamma x - 0)) &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{\alpha^n [n]! (0-\delta y)_{q,h}^{n-k} (\gamma x - 0)_{q,h}^k}{[n-k]![k]![n]!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (0-\delta y)_{q,h}^{n-k} (\gamma x - 0)_{q,h}^k \right) \frac{\alpha^n}{[n]!} = \sum_{n=0}^{\infty} \frac{\alpha^n (\gamma x - \delta y)_{q,h}^n}{[n]!} \\
&= Exp_{(q,h)}(\alpha(\gamma x - \delta y)),
\end{aligned}
$$

where we utilized (q,h)-Gauss's binomial formula. □

Remark 4.10. *In order to deal with applications precisely, we need to analyze the reductions of Theorem 4.9. Due to the definition of generalized quantum binomial, $(\gamma x - \delta y)_{q,h}^n = 0$ when $x = \gamma\delta q^{i-1} y + [i-1]h$ for some $1 \le i \le n$. Therefore Theorem 4.9 implies that*

$$Exp(\alpha(\delta(q^{i-1}-1)y + \gamma[i-1]h)) = Exp(\alpha(0-\delta y))\; Exp(\alpha(\delta q^{i-1} y + \gamma[i-1]h - 0)),$$

from which we obtain

$$Exp_{(q,h)}(\alpha(\delta y - \delta y)) = Exp_{(q,h)}(\alpha(0-\delta y))\; Exp_{(q,h)}(\alpha(\delta y - 0)) = 1,$$

as $q \to 1$ and $h \to 0$. For simplicity, we consider the case $i = 1$, i.e., we take $\delta = \gamma$ and $y = x$, which implies

$$Exp_{(q,h)}(\alpha(0-\gamma x)) \cdot Exp_{(q,h)}(\alpha(\gamma x - 0)) = Exp_{(q,h)}(\alpha(\gamma x - \gamma x)) = 1. \tag{33}$$

(i) If $\gamma = \delta = 1$ as $h \to 0$, the additive property (31) *provides the famous relation among the two q-exponential functions e_q and E_q:*

$$Exp_{(q,0)}(\alpha(0-x)) \cdot Exp_{(q,0)}(\alpha(x-0)) = \sum_{j=0}^{\infty} \frac{\alpha^j (0-x)_q^j}{[j]!} \sum_{k=0}^{\infty} \frac{\alpha^k (x-0)_q^k}{[k]!} = E_q^{-\alpha x} \cdot e_q^{\alpha x} = 1.$$

Similarly when $\gamma = \delta = -1$, we derive

$$Exp_{(q,0)}(\alpha(0+x))Exp_{(q,0)}(\alpha(-x-0)) = E_q^{\alpha x}.e_q^{-\alpha x} = 1.$$

(ii) On the other hand, as $q \to 1$ with $\gamma = \delta = 1$, we deduce that

$$Exp_{(1,h)}(\alpha(0-x)) \cdot Exp_{(1,h)}(\alpha(x-0)) = \sum_{j=0}^{\infty} \frac{\alpha^j (0-x)_h^j}{j!} \sum_{k=0}^{\infty} \frac{\alpha^k (x-0)_h^k}{k!} = (1+\alpha h)^{\frac{-x}{h}} \cdot (1+\alpha h)^{\frac{x}{h}} = 1,$$

while for $\gamma = \delta = -1$, we have

$$Exp_{(1,h)}(\alpha(0+x))Exp_{(1,h)}(\alpha(-x-0)) = (1-\alpha h)^{\frac{-x}{h}}.(1-\alpha h)^{\frac{x}{h}} = 1.$$

Note that other solutions of the equation $(\gamma x - \delta y)_{q,h}^n = 0$ produce similar reductions for Theorem 4.9.

We would like to stress that from the special case (33), it is possible to define the multiplicative inverse of (q,h)-exponential function as

$$(Exp_{(q,h)}(\alpha(\gamma x - 0)))^{-1} := Exp_{(q,h)}(\alpha(0 - \gamma x)), \tag{34}$$

which inspires us to define the (q,h)-analogue of trigonometric functions as follows:

Definition 4.11. *We introduce (q,h)-analogue of sine function*

$$\sin_{(q,h)}(\gamma x - \delta x_0) := \frac{Exp_{(q,h)}(i(\gamma x - \delta x_0)) - Exp_{(q,h)}(i(-\gamma x + \delta x_0))}{2i} \tag{35}$$

and cosine function

$$\cos_{(q,h)}(\gamma x - \delta x_0) := \frac{Exp_{(q,h)}(i(\gamma x - \delta x_0)) + Exp_{(q,h)}(i(-\gamma x + \delta x_0))}{2}. \tag{36}$$

The (q,h)-analogue of trigonometric functions (35), (36) are well-defined since one can show that the linear, homogenous (q,h)-IVP

$$D^2_{(q,h)} u(x) + a u(x) = 0, \tag{37}$$

$$D_{(q,h)} u(\gamma \delta x_0) = \gamma c, \quad u(\gamma \delta x_0) = b, \tag{38}$$

has unique solution $\sin_{(q,h)}(\gamma x - \delta x_0)$, where we take the constants $a = 1, b = 0, c = 1$. When $a = 1, b = 1, c = 0$, the IVP (37)-(38) has unique solution $\cos_{(q,h)}(\gamma x - \delta x_0)$. Similarly, the choices $a = -1, b = 0, c = 1$ and $a = -1, b = 1, c = 0$ imply respectively that the associated IVPs (37)-(38) has unique solutions as generalized (q,h)-hyperbolic functions

$$\sinh_{(q,h)}(\gamma x - \delta x_0) := \frac{Exp_{(q,h)}((\gamma x - \delta x_0)) - Exp_{(q,h)}((-\gamma x + \delta x_0))}{2}, \tag{39}$$

$$\cosh_{(q,h)}(\gamma x - \delta x_0) := \frac{Exp_{(q,h)}((\gamma x - \delta x_0)) + Exp_{(q,h)}((-\gamma x + \delta x_0))}{2}. \tag{40}$$

Note that by using the relation

$$(-\gamma x + \delta x_0)_{q,h}^n = (-1)^n (\gamma x - \delta x_0)_{q,h}^n,$$

the series representations for (q,h)-trigonometric (35)-(36) and hyperbolic functions (39)-(40) can be calculated straightforwardly:

$$\cos_{(q,h)}(\gamma x - \delta x_0) = \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma x - \delta x_0)_{q,h}^{2n}}{[2n]!}, \quad \sin_{(q,h)}(\gamma x - \delta x_0) = \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma x - \delta x_0)_{q,h}^{2n+1}}{[2n+1]!},$$

$$\cosh_{(q,h)}(\gamma x-\delta x_0)=\sum_{n=0}^{\infty}\frac{(\gamma x-\delta x_0)_{q,h}^{2n}}{[2n]!},\quad \sinh_{(q,h)}(\gamma x-\delta x_0)=\sum_{n=0}^{\infty}\frac{(\gamma x-\delta x_0)_{q,h}^{2n+1}}{[2n+1]!}.$$

Furthermore, as a direct consequence of Theorem 4.4, (q,h)-trigonometric and hyperbolic functions are (q,h)-analytic.

Proposition 4.12. *The following results hold for the (q,h)-trigonometric and hyperbolic functions.*

(i) $D_{(q,h)}\sin_{(q,h)}(\gamma x-\delta x_0)=\gamma\cos_{(q,h)}(\gamma x-\delta x_0),\ D_{(q,h)}\cos_{(q,h)}(\gamma x-\delta x_0)=-\gamma\sin_{(q,h)}(\gamma x-\delta x_0).$

(ii) $D_{(q,h)}\sinh_{(q,h)}(\gamma x-\delta x_0)=\gamma\cosh_{(q,h)}(\gamma x-\delta x_0),\ D_{(q,h)}\cosh_{(q,h)}(\gamma x-\delta x_0)=\gamma\sinh_{(q,h)}(\gamma x-\delta x_0).$

(iii) $\int\sin_{(q,h)}(\gamma x-\delta x_0)d_{(q,h)}x=-\gamma\cos_{(q,h)}(\gamma x-\delta x_0)+c,\ \int\cos_{(q,h)}(\gamma x-\delta x_0)d_{(q,h)}x=\gamma\sin_{(q,h)}(\gamma x-\delta x_0)+c.$

(iv) $\int\sinh_{(q,h)}(\gamma x-\delta x_0)d_{(q,h)}x=\gamma\cosh_{(q,h)}(\gamma x-\delta x_0)+c,\ \int\cosh_{(q,h)}(\gamma x-\delta x_0)d_{(q,h)}x=\gamma\sinh_{(q,h)}(\gamma x-\delta x_0)+c.$

Proof. The proofs follow from the definitions of (q,h)-trigonometric and hyperbolic functions. □

We finish this section by presenting consequences of Theorem 4.9 such as (q,h)- analogue of Pythagorean Theorem and related double-angle formulas for (q,h)-trigonometric functions.

Theorem 4.13. *The (q,h)-trigonometric functions satisfy the following identities:*

(i) $\sin_{(q,h)}(0+x)\sin_{(q,h)}(x-0)+\cos_{(q,h)}(0+x)\cos_{(q,h)}(x-0)=1.$

(ii) $\sin_{(q,h)}(x+x)=2\sin_{(q,h)}(x-0)\cos_{(q,h)}(0+x).$

(iii) $\cos_{(q,h)}(x+x)=1-2\sin_{(q,h)}(0+x)\sin_{(q,h)}(x-0)=2\cos_{(q,h)}(0+x)\cos_{(q,h)}(x-0)-1.$

Proof. (i) First of all let us calculate

$$\begin{aligned}\sin_{(q,h)}(0+x).\sin_{(q,h)}(x-0)=\frac{-1}{4}(&Exp_{(q,h)}(i(0+x))Exp_{(q,h)}(i(x-0))-Exp_{(q,h)}(i(0+x))Exp_{(q,h)}(i(-x-0))\\&-Exp_{(q,h)}(i(0-x))Exp_{(q,h)}(i(x-0))+Exp_{(q,h)}(i(0-x))Exp_{(q,h)}(i(-x-0))),\end{aligned}$$

where

$$Exp_{(q,h)}(i(0+x))Exp_{(q,h)}(i(x-0))=\sum_{j=0}^{\infty}\frac{i^j(0+x)_{q,h}^j}{[j]!}\sum_{k=0}^{\infty}\frac{i^k(x-0)_{q,h}^k}{[k]!}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{i^{j+k}(0+x)_{q,h}^j(x-0)_{q,h}^k}{[j]![k]!}.$$

Let us multiply and divide with the term $[j+k]!$ and change the index as $j+k=n$. Then we have

$$Exp_{(q,h)}(i(0+x))Exp_{(q,h)}(i(x-0))=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(0+x)_{q,h}^{n-k}(x-0)_{q,h}^k[n]!}{[k]![n-k]!}\right)\frac{i^n}{[n]!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(0+x)_{q,h}^{n-k}(x-0)_{q,h}^k\right)\frac{i^n}{[n]!}.$$

Utilizing Theorem 3.7 with $\gamma=1,\ \delta=-1,\ x_0=x$, we have

$$(x+x)_{q,h}^n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(0+x)_{q,h}^{n-k}(x-0)_{q,h}^k.$$

Therefore we get

$$Exp_{(q,h)}(i(0+x))Exp_{(q,h)}(i(x-0))=\sum_{n=0}^{\infty}\frac{i^n(x+x)_{q,h}^n}{[n]!}=Exp_{(q,h)}(i(x+x)).$$

With a similar discussion let us present all products of exponentials in a detailed way as follows:

$$\begin{aligned} Exp_{(q,h)}(i(0+x))Exp_{(q,h)}(i(-x-0)) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(0+x)_{q,h}^{n-k}(-x-0)_{q,h}^{k}\right)\frac{i^n}{[n]!} \\ &= \sum_{n=0}^{\infty}\frac{i^n(-x+x)_{q,h}^n}{[n]!} = Exp_{(q,h)}(i(-x+x)) = 1, \end{aligned}$$

where we used Theorem 3.7 with $\gamma = -1,\ \delta = -1,\ x_0 = x$. We deduce

$$\begin{aligned} Exp_{(q,h)}(i(0-x))Exp_{(q,h)}(i(x-0)) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(0-x)_{q,h}^{n-k}(x-0)_{q,h}^{k}\right)\frac{i^n}{[n]!} \\ &= \sum_{n=0}^{\infty}\frac{i^n(x-x)_{q,h}^n}{[n]!} = Exp_{(q,h)}(i(x-x)) = 1, \end{aligned}$$

where Theorem 3.7 with the parameters $\gamma = 1,\ \delta = 1,\ x_0 = x$, is used. Finally,

$$\begin{aligned} Exp_{(q,h)}(i(0-x))Exp_{(q,h)}(i(-x-0)) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(0-x)_{q,h}^{n-k}(-x-0)_{q,h}^{k}\right)\frac{i^n}{[n]!} \\ &= \sum_{n=0}^{\infty}\frac{i^n(-x-x)_{q,h}^n}{[n]!} = Exp_{(q,h)}(i(-x-x)), \end{aligned}$$

arises with $\gamma = -1,\ \delta = 1,\ x_0 = x$. Therefore

$$\sin_{(q,h)}(0+x).\sin_{(q,h)}(x-0) = \frac{-1}{4}\Big(Exp_{(q,h)}(i(x+x)) + Exp_{(q,h)}(i(-x-x)) - 2\Big).$$

Similarly one can derive

$$\cos_{(q,h)}(0+x).\cos_{(q,h)}(x-0) = \frac{1}{4}\Big(Exp_{(q,h)}(i(x+x)) + Exp_{(q,h)}(i(-x-x)) + 2\Big),$$

which implies that

$$\sin_{(q,h)}(0+x).\sin_{(q,h)}(x-0) + \cos_{(q,h)}(0+x).\cos_{(q,h)}(x-0) = 1.$$

(ii) For the second part, one can calculate

$$\begin{aligned} \cos_{(q,h)}(0+x).\sin_{(q,h)}(x-0) &= \frac{1}{4i}\Big(Exp_{(q,h)}(i(0+x)) + Exp_{(q,h)}(i(0-x))\Big)\Big(Exp_{(q,h)}(i(x-0)) - Exp_{(q,h)}(i(-x-0))\Big) \\ &= \frac{1}{4i}\Big(Exp_{(q,h)}(i(x+x)) - Exp_{(q,h)}(i(-x-x)) + 1 - 1\Big) = \frac{1}{2}\sin_{(q,h)}(x+x). \end{aligned}$$

(iii) Following the part (i), it is straightforward that

$$\sin_{(q,h)}(0+x).\sin_{(q,h)}(x-0) = \frac{-1}{4}\Big(Exp_{(q,h)}(i(x+x)) + Exp_{(q,h)}(i(-x-x)) - 2\Big) = \frac{1-\cos_{(q,h)}(x+x)}{2}$$

and

$$\cos_{(q,h)}(0+x).\cos_{(q,h)}(x-0) = \frac{1}{4}\Big(Exp_{(q,h)}(i(x+x)) + Exp_{(q,h)}(i(-x-x)) + 2\Big) = \frac{1+\cos_{(q,h)}(x+x)}{2}.$$

□

Remark 4.14. *It is clear that* $(x-0)^n_{1,0} = (0+x)^n_{1,0} = x^n$ *and* $(-x-0)^n_{1,0} = (0-x)^n_{1,0} = (-1)^n x^n$ *which imply that* $\sin_{(1,0)}(x-0) = \sin_{(1,0)}(0+x) = \sin(x)$ *and* $\cos_{(1,0)}(x-0) = \cos_{(1,0)}(0+x) = \cos(x)$*. Thus Theorem 4.13 approximates the classical Pythagorean Theorem and double angle formulas. Moreover, by the use of Remark 4.8, as* $h \to 0$ *we have*

$$\sin_{(q,0)}(0+x) = \frac{E_q^{ix} - E_q^{-ix}}{2i} = Sin_q(x), \;\; \cos_{(q,0)}(0+x) = \frac{E_q^{ix} + E_q^{-ix}}{2} = Cos_q(x)$$

and

$$\sin_{(q,0)}(x-0) = \frac{e_q^{ix} - e_q^{-ix}}{2i} = sin_q(x), \;\; \cos_{(q,0)}(x-0) = \frac{e_q^{ix} + e_q^{-ix}}{2} = cos_q(x),$$

which provide the q-Pythagorean theorem

$$Sin_q(x)sin_q(x) + Cos_q(x)cos_q(x) = 1.$$

Because of the lack of the additive identity for q-exponential functions, the literature was lack of q-double-angle formulas, which we are able to fulfill as

$$sin_q(x+x) = 2sin_q(x)Cos_q(x), \; cos_q(x+x) = 1 - 2Sin_q(x)sin_q(x) = 2Cos_q(x)cos_q(x) - 1,$$

as a consequence of the Theorem 4.13.

5. Applications on (q,h)-difference Equations

5.1. The variation of parameters formula for first order (q,h)-difference equations

The additive property of generalized quantum exponential function (31) can be applied to a (q,h)-difference equation. Assume $q < 1$ and consider the first order linear (q,h)-difference equation

$$\begin{aligned} &D_{(q,h)}y(x) + y(qx+h) = f(x), \\ &y(x_0) = y_0, \end{aligned}$$

where $\frac{h}{1-q} < x_0 \in \mathbb{T}_{(q,h)}$. Multiplying the equation by $Exp_{(q,h)}(x-0)$ we obtain

$$Exp_{(q,h)}(x-0)D_{(q,h)}y(x) + Exp_{(q,h)}(x-0)y(qx+h) = Exp_{(q,h)}(x-0)f(x).$$

Using Proposition 2.2, we deduce

$$D_{(q,h)}(Exp_{(q,h)}(x-0)y(x)) = Exp_{(q,h)}(x-0)f(x).$$

Integrating the resulting equation over $[x_0, x] \cap \mathbb{T}_{(q,h)}$, we obtain

$$\int_{x_0}^{x} D_{(q,h)}(Exp_{(q,h)}(s-0)y(s))d_{(q,h)}s = \int_{x_0}^{x} Exp_{(q,h)}(s-0)f(s)d_{(q,h)}s.$$

The fundamental theorem of (q,h)-calculus (see Theorem 2.8) concludes that

$$Exp_{(q,h)}(x-0)y(x) - Exp_{(q,h)}(x_0-0)y(x_0) = \int_{x_0}^{x} Exp_{(q,h)}(s-0)f(s)d_{(q,h)}s,$$

from which we obtain

$$y(x) = \frac{Exp_{(q,h)}(x_0-0)}{Exp_{(q,h)}(x-0)}y_0 + \frac{1}{Exp_{(q,h)}f(x-0)}\int_{x_0}^{x} Exp_{(q,h)}(s-0)f(s)d_{(q,h)}sd_{(q,h)}s.$$

By Theorem 4.9 and its consequence (34), we accomplish the solution of the form

$$y(x) = Exp_{(q,h)}(x_0-x)y_0 + Exp_{(q,h)}(0-x)\int_{x_0}^{x} Exp_{(q,h)}(s-0)f(s)d_{(q,h)}s. \tag{41}$$

Note that the solution (41) is the variation of parameters formula for first order (q,h)-difference equations.

5.2. Dynamic Diffusion Equation

We state a dynamic diffusion equation on $\mathbb{T}_{(q,h)} \times \mathbb{T}_{(\bar{q},\bar{h})}$ as

$$\partial^t_{(q,h)} u(x,t) - v(\partial^x_{(\bar{q},\bar{h})})^2 u(x,t) = 0, \quad v \in \mathbb{R}/\{0\}, \tag{42}$$

where the quantum parameters $q, \bar{q}$ and $h, \bar{h}$ need not to be equal. (See [18] for the definition of partial delta (q,h)-derivative.) We emphasize that (42) is a generic equation producing various kinds of partial differential/ difference type of equations. Using various choices of limits $q \to 1$, $h \to 0$, $\bar{q} \to 1$, $\bar{h} \to 0$ in (42), it is possible to obtain sixteen different kinds of partial differential/ difference equations. For instance, as $h \to 0$ and $\bar{q} \to 1$, the equation (42) produces a q-difference-$\bar{h}$-difference diffusion equation on $\mathbb{K}_q \times \bar{h}\mathbb{Z}$

$$\partial^t_q u(x,t) - c^2(\partial^x_{\bar{h}})^2 u(x,t) = 0.$$

In this case if also $q \to 1$, we derive a differential-$\bar{h}$-difference Heat equation on $\mathbb{R} \times \bar{h}\mathbb{Z}$

$$\partial^t u(x,t) - v(\partial^x_{\bar{h}})^2 u(x,t) = 0.$$

When $\bar{h} = h \to 0$, we obtain q-difference-$\bar{q}$-difference equation on $\mathbb{K}_q \times \mathbb{K}_{\bar{q}}$

$$\partial^t_q u(x,t) - v(\partial^x_{\bar{q}})^2 u(x,t) = 0.$$

In this case if also $q \to 1$, we derive a differential-$\bar{q}$-difference Heat equation on $\mathbb{R} \times \mathbb{K}_{\bar{q}}$

$$\partial^t u(x,t) - v(\partial^x_{\bar{q}})^2 u(x,t) = 0.$$

In order to solve generalized diffusion equation (42), we seek for the solutions of the form $u(x,t) = f(x)g(t)$, where we assume that f is a formal power series written in terms of $(\gamma x - \delta x_0)^i_{\bar{q},\bar{h}}$

$$f(x) = \sum_{i=0}^{\infty} a_i (\gamma x - \delta x_0)^i_{\bar{q},\bar{h}}. \tag{43}$$

We plug u and its delta (q,h)-partial derivatives on the diffusion equation (42) and get

$$\left(\partial^t_{(q,h)} - v(\partial^x_{(\bar{q},\bar{h})})^2\right) u(x,t) = \sum_{i=0}^{\infty} \left(a_i D_{(q,h)} g(t) - v\, a_{i+2}[i+2][i+1] g(t)\right) (\gamma x - \delta x_0)^i_{\bar{q},\bar{h}} = 0.$$

Comparing the coefficients of $(\gamma x - \delta x_0)^i_{\bar{q},\bar{h}}$ for all $i \geq 0$, we have

$$a_{i+2} = \frac{D_{(q,h)} g(t)}{g(t)} \frac{a_i}{v[i+2][i+1]}. \tag{44}$$

Since a_i are constants, the relation (44) leads us to obtain a (q,h)-difference equation

$$D_{(q,h)} g(t) = \alpha g(t), \tag{45}$$

for any nonzero constant α. Using Proposition 4.7, we may obtain the solution for (45) as $g(t) = Exp_{(q,h)}(\alpha(t - \delta t_0))$ with initial condition $g(\delta t_0) = 1$. On the other hand, the coefficients a_i yield as

$$a_i = \begin{cases} (\frac{\alpha}{v})^k \frac{a_0}{[2k]!} & \text{if} \quad i = 2k, \quad k \geq 0, \\ (\frac{\alpha}{v})^k \frac{a_1}{[2k+1]!} & \text{if} \quad i = 2k+1, \quad k \geq 0. \end{cases} \tag{46}$$

In this case one may assume that $a_0 = 1$, $a_1 = \gamma D_{(\bar{q},\bar{h})} f(\gamma\delta x_0)$ and $\frac{\alpha}{v} = \gamma^2 D_{(\bar{q},\bar{h})} f^2(\gamma\delta x_0)$, then (43) becomes the Taylor series of $f(x)$

$$f(x) = \sum_{i=0}^{\infty} \frac{\gamma^i \, D^i_{(\bar{q},\bar{h})} f(\gamma\delta x_0) \, (\gamma x - \delta x_0)^i_{\bar{q},\bar{h}}}{[i]!}, \tag{47}$$

which is convergent under the assumption hypothesis of Theorem 4.4. More specifically, if we set $a_0 = 1$, $\frac{\alpha}{v} = \beta^2$ and $a_1 = \beta$ in (46), then $f(x)$ can be written as

$$f(x) = \sum_{i=0}^{\infty} \frac{\beta^i (\gamma x - \delta x_0)^i_{\bar{q},\bar{h}}}{[i]!} = Exp_{(\bar{q},\bar{h})}(\beta(\gamma x - \delta x_0)).$$

Furthermore, one may choose the constants $a_0 = c_1$, $a_1 = c_2$, $\frac{\alpha}{v} = -1$, then (43) can be expressed in terms of (q,h)-trigonometric functions

$$f(x) = c_1 \cos_{(\bar{q},\bar{h})}(\gamma x - \delta x_0) + c_2 \sin_{(\bar{q},\bar{h})}(\gamma x - \delta x_0).$$

Alternatively if $a_0 = c_1$, $a_1 = c_2$, $\frac{\alpha}{v} = 1$, then (43) can be presented via (q,h)-hyperbolic functions

$$f(x) = c_1 \cosh_{(\bar{q},\bar{h})}(\gamma x - \delta x_0) + c_2 \sinh_{(\bar{q},\bar{h})}(\gamma x - \delta x_0).$$

To sum up, the solution of the IVP

$$\partial^t_{(q,h)} u(x,t) - v(\partial^x_{(\bar{q},\bar{h})})^2 u(x,t) = 0, \quad v \in \mathbb{R}/\{0\}, \tag{48}$$

$$u(x,\delta t_0) = f(x), \tag{49}$$

can be expressed as

$$u(x,t) = Exp_{(q,h)}(v\beta^2(t - \delta t_0)) \, f(x),$$

where f is of the form (47).

Proposition 5.1. *The following operator representation holds*

$$Exp_{(q,h)}\left(v(t-\delta t_0)(\partial^x_{(\bar{q},\bar{h})})^2\right) \cdot Exp_{(\bar{q},\bar{h})}(\beta(\gamma x - \delta x_0)) = Exp_{(q,h)}(v\beta^2(t-\delta t_0)) \cdot Exp_{(\bar{q},\bar{h})}(\beta(\gamma x - \delta x_0)).$$

Proof. By the definition of the generalized quantum exponential function (28), we write the operator

$$Exp_{(q,h)}\left(v(t-\delta t_0)(\partial^x_{(\bar{q},\bar{h})})^2\right) = \sum_{n=0}^{\infty} \frac{v^n (t-\delta t_0)^n_{q,h} (\partial^x_{(\bar{q},\bar{h})})^{2n}}{[n]!}.$$

Since

$$(\partial^x_{(\bar{q},\bar{h})})^{2n} Exp_{(\bar{q},\bar{h})}(\beta(\gamma x - \delta x_0)) = \beta^{2n} Exp_{(\bar{q},\bar{h})}(\beta(\gamma x - \delta x_0)),$$

we conclude that

$$\begin{aligned} Exp_{(q,h)}\left(v(t-\delta t_0)(\partial^x_{(\bar{q},\bar{h})})^2\right) \cdot Exp_{(\bar{q},\bar{h})}(\beta(\gamma x - \delta x_0)) &= \sum_{n=0}^{\infty} \frac{v^n (t-\delta t_0)^n_{q,h} \beta^{2n}}{[n]!} Exp_{(\bar{q},\bar{h})}(\beta(\gamma x - \delta x_0)) \\ &= Exp_{(q,h)}(v\beta^2(t-\delta t_0)) \cdot Exp_{(\bar{q},\bar{h})}(\beta(\gamma x - \delta x_0)). \end{aligned}$$

□

Now consider a function in formal power series $f(x) = \sum_{j=0}^{\infty} a_j(\gamma x - \delta x_0)^j_{(\bar{q},\bar{h})}$. Then the function

$$h(x,t) = Exp_{(q,h)}\left(v(t-\delta t_0)(\partial^x_{(\bar{q},\bar{h})})^2\right) f(x) = \sum_{j=0}^{\infty} a_j Exp_{(q,h)}\left(v(t-\delta t_0)(\partial^x_{(\bar{q},\bar{h})})^2\right)(\gamma x - \delta x_0)^j_{(\bar{q},\bar{h})}$$

is a solution of (q,h)-Heat equation (42). We may present the evolution operator for (q,h)-Heat equation

$$U(t) = Exp_{(q,h)}\left(v(t-\delta t_0)(\partial^x_{(\bar{q},\bar{h})})^2\right),$$

from which the initial value problem (48)-(49) has a solution

$$u(x,t) = Exp_{(q,h)}\left(v(t-\delta t_0)(\partial^x_{(\bar{q},\bar{h})})^2\right) u(x,\delta t_0) = Exp_{(q,h)}\left(v(t-\delta t_0)(\partial^x_{(\bar{q},\bar{h})})^2\right) f(x).$$

5.3. (q,h)-analogue of shock soliton solutions of generalized Burger's equation

In this section, we aim to present (q,h)-analogue of Burger's equation and its shock soliton solutions. Let us assume that $u(x,t)$ be the solution of the dynamic diffusion equation (42). We introduce the (q,h)-analogue of the *Hopf-Cole transformation* [7, 11] as

$$\psi(x,t) := -2v\frac{\partial^x_{(\bar{q},\bar{h})}u(x,t)}{u(x,t)}. \tag{50}$$

Then by the use of the diffusion equation (42) and the (q,h)-Hopf-Cole transformation (50), the function $\psi(x,t)$ satisfies the following equation

$$\left(\partial^t_{(q,h)} - v(\partial^x_{(\bar{q},\bar{h})})^2\right)\psi = \frac{1}{2}[(E^t\psi\psi E^x)\partial^x_{(\bar{q},\bar{h})}\psi] - \frac{1}{2}\partial^x_{(\bar{q},\bar{h})}[(E^x\psi)\psi] + \frac{1}{4v}[((E^x)^2 - E^t)\psi](E^x\psi)\psi, \tag{51}$$

where E^t and E^x are forward shift operators with respect to t, x respectively, i.e. $E^t\psi(x,t) = \psi(x, qt+h)$ and $E^x\psi(x,t) = \psi(\bar{q}x + \bar{h}, t)$. One can observe that, as $(q,\bar{q},h,\bar{h}) \to (1,1,0,0)$, (51) recovers the classical Burger's equation

$$\psi_t - v\psi_{xx} = -\psi\psi_x. \tag{52}$$

Thus the equation (51) can be regarded as (q,h)-analogue of Burger's equation. In order to compute its solutions, we may start with a solution of (q,h)-diffusion equation (42)

$$u(x,t) = Exp_{(q,h)}(v\beta^2(t-\delta t_0)) \cdot Exp_{(\bar{q},\bar{h})}(\beta(\gamma x - \delta x_0)).$$

By utilizing the Hopf-Cole transformation (50), we compute a constant solution $\psi(x,t) = -2v\beta\gamma$ for (51). Since dynamic diffusion equation (42) is a linear equation, we superpose its two linearly independent solutions as

$$u(x,t) = \sum_{i=1}^{2} Exp_{(q,h)}(v\beta_i^2(t-\delta t_0)) \cdot Exp_{(\bar{q},\bar{h})}(\beta_i(\gamma x - \delta x_0)),$$

from which we derive

$$\psi(x,t) = -2v\gamma\frac{\sum_{i=1}^{2}\beta_i Exp_{(q,h)}(v\beta_i^2(t-\delta t_0)) \cdot Exp_{(\bar{q},\bar{h})}(\beta_i(\gamma x - \delta x_0))}{\sum_{i=1}^{2} Exp_{(q,h)}(v\beta_i^2(t-\delta t_0)) \cdot Exp_{(\bar{q},\bar{h})}(\beta_i(\gamma x - \delta x_0))}. \tag{53}$$

When $\gamma = 1$ and as $(q,h) \to (1,0)$, (53) reduces to 2-shock soliton solutions of classical Burger's equation

$$\psi(x,t) = -2v\frac{\sum_{i=1}^{2}\beta_i e^{v\beta_i^2 t + \beta_i x}}{\sum_{i=1}^{2} e^{v\beta_i^2 t + \beta_i x}}.$$

Then the solutions (53) can be regarded as *two (q,h)-shock soliton solutions*. Notice that, if $\beta_1 = 1, \beta_2 = -1$, (53) yields as *$(q,h)$-stationary shock soliton solutions*

$$\psi(x,t) = -2v\gamma \frac{Exp_{(\bar{q},\bar{h})}(\gamma x - \delta x_0) - Exp_{(\bar{q},\bar{h})}(-\gamma x + \delta x_0)}{Exp_{(\bar{q},\bar{h})}(\gamma x - \delta x_0) + Exp_{(\bar{q},\bar{h})}(-\gamma x + \delta x_0)} = -2v\gamma \tanh_{(\bar{q},\bar{h})}(\gamma x - \delta x_0).$$

By superposing N linearly independent solutions of (42)

$$u(x,t) = \sum_{i=1}^{N} Exp_{(q,h)}(v\beta_i^2(t-\delta t_0)) \cdot Exp_{(\bar{q},\bar{h})}(\beta_i(\gamma x - \delta x_0)),$$

we derive multi (q,h)-shock soliton solutions of (51) as

$$\psi(x,t) = -2v\gamma \frac{\sum_{i=1}^{N} \beta_i Exp_{(q,h)}(v\beta_i^2(t-\delta t_0)) \cdot Exp_{(\bar{q},\bar{h})}(\beta_i(\gamma x - \delta x_0))}{\sum_{i=1}^{N} Exp_{(q,h)}(v\beta_i^2(t-\delta t_0)) \cdot Exp_{(\bar{q},\bar{h})}(\beta_i(\gamma x - \delta x_0))}. \tag{54}$$

Notice that as $h \to 0$, (q,h)-shock soliton solutions (54) recover the q-shock soliton solutions presented in [14]. As $(q,h) \to (1,0)$ the solutions (54) generalize multi shock soliton solutions of classical Burger's equation (52).

Acknowledgement

The authors would like to thank the anonymous reviewer for her/his valuable suggestions and comments.

This article is dedicated to the medical staff who put their life into danger during Covid-19 pandemi all over the world.

References

[1] F. M. Atici, D. C. Biles, A. Lebedinsky An application of time scales to economics, Math. Comput. Modelling 43 (2006) 718—726.
[2] H. B. Benaoum, (q,h)-analogue of Newton's binomial Formula, J. Phys. A: Math.Gen. 32 (1999) 2037-2040.
[3] M. Blaszak, B. Silindir, B.M. Szablikowski, The R-matrix approach to integrable systems on time scales, J. Phys. A: Math. Theor. 41 (2008) Article id: 385203.
[4] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, Birkhauser, Boston, 2001.
[5] M. Bohner, S. Streipert, D.F.M Torres Exact Solution to a Dynamic SIR Model, Nonlinear Analysis: Hybrid Systems 32 (2019) 228-238.
[6] J. Čermák, L. Nechvátal, On (q,h)-analogue of fractional calculus, Journal of Nonlinear Mathematical Physics 17:1 (2010) 51-68.
[7] J. D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics. Quart. Appl. Math. 9 (1951) 225-236.
[8] R. Goldman and P. Siemonov, Generalized quantum splines, Computer Aided Geometric Design 47 (2016) 29-54.
[9] M. Gürses, S.G. Guseinov and B. Silindir, Integrable equations on time scales, J. Math. Phys. 46 (2005) 1-22.
[10] S. Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18-56.
[11] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl. Math. 3 (1950) 201-230.
[12] F. H. Jackson, A basic-sine and cosine with sybolical solution of certain differential equations, Proc. Edinb. Math. Scoc. 22 (1904) 28-39.
[13] D. S. McAnally, q-exponential and q-gamma functions. II. q-gamma functions $^{a)}$, J. Math. Phys. 36 (1995) 574.
[14] S. Nalci, O. Pashaev, q-Analog of shock soliton solution, J. Phys. A: Math. Theor. 43 (2010) Article id: 445205.
[15] V. Kac and P. Cheung *Quantum Calculus*, Springer, 2002.
[16] M. R. S. Rahmat, The (q,h)-Laplace transform on discrete time scales, Computers and Mathematics with Applications 62 (2011) 272-281.
[17] M.P. Schützenberger, Une interprétation de certaines solutions de l'équation fonctionnelle: $F(x+y) = F(x)F(y)$, C. R. Acad. Sci. Paris 236 (1953) 352-353.
[18] B. Silindir and A Yantir, Generalized quantum exponential function and its applications, Filomat 33:15 (2019) 4907-4922.
[19] B. M. Szablikowski, M. Blaszak, B. Silindir, Bi-Hamiltonian structures for integrable systems on regular time scales, J. Math. Phys. 50 (2009) Article id: 073502.
[20] S. G. Topal, Rolle's and generalized mean value theorems on time scales, Journal of difference equations and applications 8:4 (2010) 333-344.
[21] Z. Tuncer, Lattice, quantum analysis and applications to discrete D'Alembert problems, M.Sc. Thesis, Graduate School of Natural and Applied Sciences, Dokuz Eylül University, İzmir, 2018.