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Generalized Midpoint Fractional Integral Inequalities via h-Convexity

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Abstract. In this research, generalizations of midpoint type inequalities are established. h-convexity is used as a tool. These inequalities are for differentiable functions which involve Riemann-Liouville fractional integrals. Also, some consequences of these established inequalities are obtained.

1. Introduction

Fractional calculus plays an important role in many fields like engineering, economics, physics, and many disciplines of mathematics. For more information about fraction calculus please refer to ([11] , [16], [19], [20], [24]). Similarly, It is well known that the convexity of a function plays a vital role in the field of inequalities. Here, first we define a generalized convexity namely h-convexity.

Definition 1.1. *[33] Let I, J be intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. A non-negative function $f : I \to \mathbb{R}$ is called h-convex if for all $x, y \in I$, $\alpha \in (0,1)$, we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y).$$

Next, the following inequality is known as Hermite-Hadamard inequality for convex functions:
If $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave.

In [18], U. S. Kırmacı give the following identity and using this identiy, obtain some bounds for the left hand side of the inequality (1)

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Lemma 1.2. *Let $f : I^* \to \mathbb{R}$ be differentiable function on I^*, $a, b \in I^*$ (I^* is interior of I) with $a < b$. If $f' \in L[a, b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \tag{2}$$

$$= (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (1-t) f'(ta+(1-t)b)dt\right].$$

One can see ([1], [3], [5], [8], [9], [23], [25], [26], [31]) to study the new bound for left-hand side and right-hand side of the inequality (1). Here we give the well-known Riemann-Liouville fractional integral operators which will be helpful to obtain our main results.

Definition 1.3. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

In the following, Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals are obtained in [28] and [27].

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{3}$$

with $\alpha > 0$.

Theorem 1.5. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} f(b) + J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

We will use the following lemmas to find our results.

Lemma 1.6. *[4] Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then for all $x \in [a, b]$ the following equality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x)\right] - f(a+b-x) \tag{5}$$

$$= \frac{(x-a)^2}{b-a}\int_0^1 (1-t^{\alpha}) f'(tb+(1-t)(a+b-x))\,dt + \frac{(b-x)^2}{b-a}\int_0^1 (t^{\alpha}-1) f'(ta+(1-t)(a+b-x))\,dt.$$

Lemma 1.7. *[6] Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L^1[a,b]$, then we have the following equality for fractional integrals*

$$\frac{\Gamma(\alpha+1)}{b-a}\left((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\right)-f(a+b-x) \tag{6}$$

$$=\frac{(x-a)^2}{b-a}\int_0^1 t^{\alpha}f'(t(a+b-x)+(1-t)b)\,dt-\frac{(b-x)^2}{b-a}\int_0^1 t^{\alpha}f'(t(a+b-x)+(1-t)a)\,dt,$$

for all $x\in[a,b]$.

Many authors generalized Hermite-Hadamard inequality for many fractional and conformable integral operators. One can see ([2], [7], [10], [12]-[15], [17], [21], [22], [29], [30], [32]-[36]) for more information. In the upcoming section, we established some new generalized midpoint type inequalities for Riemann-Liouville fractional integrals by the mean of h-convexity. Some important consequences are also given in the upcoming section.

2. Main Results

In this Section, by help of Lemma 1.6 and Lemma 1.7, we establish some generalized midpoint type inequalities for h-convex functions.

Theorem 2.1. *$f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\le a<b$. If $|f'|^q$ is h-convex on $[a,b]$ for some fixed $q>1$, then for all $x\in[a,b]$ the following fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J^{\alpha}_{b-}f(a+b-x)+(b-x)^{1-\alpha}J^{\alpha}_{a+}f(a+b-x)\right]-f(a+b-x)\right| \tag{7}$$

$$\le\frac{1}{b-a}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^2\left[|f'(b)|^q+|f'(a+b-x)|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+(b-x)^2\left[|f'(a)|^q+|f'(a+b-x)|^q\right]^{\frac{1}{q}}\right]\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By the Lemma 1.6, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J^{\alpha}_{b-}f(a+b-x)+(b-x)^{1-\alpha}J^{\alpha}_{a+}f(a+b-x)\right]-f(a+b-x)\right| \tag{8}$$

$$\le\frac{(x-a)^2}{b-a}\int_0^1|1-t^{\alpha}|\left|f'(tb+(1-t)(a+b-x))\right|dt$$

$$+\frac{(b-x)^2}{b-a}\int_0^1|t^{\alpha}-1|\left|f'(ta+(1-t)(a+b-x))\right|dt.$$

Using the Hölder's inequality and h-convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\int_0^1 |1-t^\alpha| \left|f'(tb+(1-t)(a+b-x))\right| dt \qquad (9)\\
\leq\ & \left(\int_0^1 |1-t^\alpha|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ & \left(\int_0^1 (1-t^{p\alpha}) dt\right)^{\frac{1}{p}} \left(\left|f'(b)\right|^q \int_0^1 h(t)dt + \left|f'(a+b-x)\right|^q \int_0^1 h(1-t)dt\right)^{\frac{1}{q}}\\
=\ & \left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}} \left(\left|f'(b)\right|^q + \left|f'(a+b-x)\right|^q\right)^{\frac{1}{q}} \left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Here we use

$$(X-Y)^q \leq X^q - Y^q,$$

for any $X > Y \geq 0$ and $q \geq 1$.

Similarly, we have

$$\begin{aligned}
&\int_0^1 |t^\alpha - 1| \left|f'(ta+(1-t)(a+b-x))\right| dt \qquad (10)\\
\leq\ & \left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}} \left(\left|f'(a)\right|^q + \left|f'(a+b-x)\right|^q\right)^{\frac{1}{q}} \left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Combining (8), (9)and (10), inequality (7) is obtained. □

Corollary 2.2. *Under assumption of Theorem 2.1 with $x = \frac{a+b}{2}$, the following inequality hold:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right] - f\left(\frac{a+b}{2}\right)\right| \qquad (11)\\
\leq\ & \frac{b-a}{4}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\left|f'(a)\right|^q\left(1+h\left(\frac{1}{2}\right)\right) + \left|f'(b)\right|^q h\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}\right.\\
+\ & \left.\left(\left|f'(b)\right|^q\left(1+h\left(\frac{1}{2}\right)\right) + \left|f'(a)\right|^q h\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}\right]\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.3. *By taking $h(t) = t^s$ in (7), the following inequality holds for s-convexity:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha} J_{b-}^\alpha f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^\alpha f(a+b-x)\right] - f(a+b-x)\right|\\
\leq\ & \frac{1}{b-a}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^2\left[\frac{\left|f'(b)\right|^q + \left|f'(a+b-x)\right|^q}{s+1}\right]^{\frac{1}{q}} + (b-x)^2\left[\frac{\left|f'(a)\right|^q + \left|f'(a+b-x)\right|^q}{s+1}\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 2.4. *(i) If we take $h(t) = t$ in Theorem 2.1, then Theorem 2.1 reduces to [4, Theorem 3].*
(ii) If we take $h(t) = t$ in Corollary 2.3, then Corollary 2.3 reduces to [4, Corollary 1]

Theorem 2.5. *$f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $\left|f'\right|^q$ is h-convex on $[a, b]$ for some fixed $q \geq 1$, then for all $x \in [a, b]$ the following fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x)\right] - f(a+b-x)\right| \tag{12}$$

$$\leq \frac{1}{b-a}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{(x-a)^2}{(b-a)}\left(I_1\left|f'(b)\right|^q + I_2\left|f'(a+b-x)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left. + \frac{(b-x)^2}{(b-a)}\left(I_1\left|f'(a)\right|^q + I_2\left|f'(a+b-x)\right|^q\right)^{\frac{1}{q}}\right],$$

where $I_1 = \int_0^1 (1-t^\alpha)\, h(t)$ and $I_2 = \int_0^1 (1-t^\alpha)\, h(1-t)dt$.

Proof. By the Lemma 1.6 and the power mean inequality, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x)\right] - f(a+b-x)\right| \tag{13}$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 |1-t^\alpha| \left|f'(tb+(1-t)(a+b-x))\right| dt$$

$$+\frac{(b-x)^2}{b-a}\int_0^1 |t^\alpha - 1| \left|f'(ta+(1-t)(a+b-x))\right| dt$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 |1-t^\alpha|\, dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-t^\alpha| \left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 |t^\alpha - 1|\, dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t^\alpha - 1| \left|f'(ta+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}.$$

Using the h-convexity of $\left|f'\right|^q$, we obtain

$$\int_0^1 |1-t^\alpha| \left|f'(tb+(1-t)(a+b-x))\right|^q dt$$

$$\leq \int_0^1 (1-t^\alpha)\left[h(t)\left|f'(b)\right|^q + h(1-t)\left|f'(a+b-x)\right|^q\right] dt$$

$$= \left|f'(b)\right|^q \int_0^1 (1-t^\alpha)\, h(t)dt + \left|f'(a+b-x)\right|^q \int_0^1 (1-t^\alpha)\, h(1-t)dt,$$

and similarly, we have

$$\begin{aligned}
&\int_0^1 |t^\alpha - 1| \left|f'\left(ta+(1-t)(a+b-x)\right)\right|^q dt \\
\leq\ & \int_0^1 (1-t^\alpha)\left[h(t)\left|f'(a)\right|^q + h(1-t)\left|f'(a+b-x)\right|^q\right] dt \\
=\ & \left|f'(a)\right|^q \int_0^1 (1-t^\alpha) h(t) dt + \left|f'(a+b-x)\right|^q \int_0^1 (1-t^\alpha) h(1-t) dt,
\end{aligned}$$

which completes the proof. □

Corollary 2.6. *Under assumption of Theorem 2.5 with $x=\frac{a+b}{2}$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^\alpha f\left(\frac{a+b}{2}\right)+J_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\left(I_1+h\left(\frac{1}{2}\right)I_2\right)\left|f'(b)\right|^q + h\left(\frac{1}{2}\right)I_2\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left(I_1+h\left(\frac{1}{2}\right)I_2\right)\left|f'(a)\right|^q + h\left(\frac{1}{2}\right)I_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{14}$$

Corollary 2.7. *By taking $h(t)=t^s$ in (12), the following inequality holds for s-convexity:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha} J_{b-}^\alpha f(a+b-x)+(b-x)^{1-\alpha} J_{a+}^\alpha f(a+b-x)\right]-f(a+b-x)\right| \\
\leq\ & \frac{1}{b-a}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{(x-a)^2}{(b-a)}\left(|f'(a+b-x)|\left[\frac{1}{s+1}-B(\alpha+1,s+1)\right]+\frac{\alpha|f'(b)|}{(s+1)(s+\alpha+1)}\right)^{\frac{1}{q}}\right. \\
&\left.+\frac{(b-x)^2}{(b-a)}\left(|f'(a+b-x)|\left[\frac{1}{s+1}-B(\alpha+1,s+1)\right]+\frac{\alpha|f'(a)|}{(s+1)(s+\alpha+1)}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $B(x,y)$ is Euler's Beta function.

Remark 2.8. *(i) If we take $h(t)=t$ in Theorem 2.5, then Theorem 2.5 reduces to [4, Theorem 4].*
(ii) If we take $h(t)=t$ in Corollary 2.6, then Corollary 2.6 reduces to [4, Corollary 2].

Theorem 2.9. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$ and $f'\in L^1[a,b]$. If $|f'|$ is h-convex on $[a,b]$, then for all $x\in[a,b]$ the following fractional integrals inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\left((b-x)^{1-\alpha} J_{(a+b-x)-}^\alpha f(a)+(x-a)^{1-\alpha} J_{(a+b-x)^+}^\alpha f(b)\right)-f(a+b-x)\right| \\
=\ & \frac{(x-a)^2}{b-a}\left[I_3\left|f'(a+b-x)\right|+I_4\left|f'(b)\right|\right]+\frac{(b-x)^2}{b-a}\left[I_3\left|f'(a+b-x)\right|+I_4\left|f'(a)\right|\right],
\end{aligned} \tag{15}$$

where $I_3=\int_0^1 t^\alpha h(t)dt$ and $I_4=\int_0^1 t^\alpha h(1-t)dt$.

Proof. Taking the modulus in Lemma 1.7, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\left((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)^{+}}f(b)\right)-f(a+b-x)\right| \\
\leq\ & \frac{(x-a)^2}{b-a}\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)b)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)a)\right|dt.
\end{aligned}$$

Using h-convexity of $\left|f'\right|$, we get

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\left((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)^{+}}f(b)\right)-f(a+b-x)\right| \\
\leq\ & \frac{(x-a)^2}{b-a}\int_0^1\left[t^{\alpha}h(t)\left|f'(a+b-x)\right|+t^{\alpha}h(1-t)\left|f'(b)\right|\right]dt \\
&+\frac{(b-x)^2}{b-a}\int_0^1\left[t^{\alpha}h(t)\left|f'(a+b-x)\right|+t^{\alpha}h(1-t)\left|f'(a)\right|\right]dt \\
=\ & \frac{(x-a)^2}{b-a}\left[\left|f'(a+b-x)\right|\int_0^1 t^{\alpha}h(t)dt+\left|f'(b)\right|\int_0^1 t^{\alpha}h(1-t)dt\right] \\
&+\frac{(b-x)^2}{b-a}\left[\left|f'(a+b-x)\right|\int_0^1 t^{\alpha}h(t)dt+\left|f'(a)\right|\int_0^1 t^{\alpha}h(1-t)dt\right],
\end{aligned}$$

which completes the proof. □

Corollary 2.10. *Under assumptions of Theorem 2.9 with $x=\frac{a+b}{2}$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+b}{2})-}f(a)+J^{\alpha}_{(\frac{a+b}{2})+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right| \qquad (16)\\
\leq\ & \frac{(b-a)}{4}\left[2I_3\left|f'\left(\frac{a+b}{2}\right)\right|+I_4\left|f'(b)\right|+I_4\left|f'(a)\right|\right] \\
\leq\ & \frac{(b-a)}{4}\left[\left(2h\left(\frac{1}{2}\right)I_3+I_4\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\right].
\end{aligned}$$

Corollary 2.11. *By taking $h(t)=t^s$ in (15), the following inequality holds for s-convexity:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\left((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)^{+}}f(b)\right)-f(a+b-x)\right| \\
=\ & \frac{(x-a)^2}{b-a}\left[\frac{\left|f'(a+b-x)\right|}{\alpha+s+1}+\left|f'(b)\right|B(\alpha+1,s+1)\right] \\
+\ & \frac{(b-x)^2}{b-a}\left[\frac{\left|f'(a+b-x)\right|}{\alpha+s+1}+\left|f'(a)\right|B(\alpha+1,s+1)\right],
\end{aligned}$$

where $B(x,y)$ is Euler's Beta function.

Remark 2.12. *(i) If we take $h(t)=t$ in Theorem 2.9, then Theorem 2.9 reduces to [6, Theorem 2.2].*
(ii) If we take $h(t)=t$ in Corollary 2.10, then Corollary 2.10 reduces to [27, Theorem 5].
(iii) If we take $h(t)=t$ and $\alpha=1$ Theorem 2.9, then Theorem 2.9 reduces to [6, Corollary 2.4].

Theorem 2.13. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\le a<b$ and $f'\in L^1[a,b]$. If $\left|f'\right|^q$, $q>1$, is h-convex on $[a,b]$, then for all $x\in[a,b]$ the following fractional integral inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\right]-f(a+b-x)\right| \quad (17)\\ \le\ &\frac{1}{b-a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^2\left[\left|f'(a+b-x)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\ +\ &\left.(b-x)^2\left[\left|f'(a+b-x)\right|^q+\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right]\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By the Lemma 1.7, we have

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\right]-f(a+b-x)\right| \quad (18)\\ \le\ &\frac{(x-a)^2}{b-a}\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)b)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)a)\right|dt.\end{aligned}$$

Using the Hölder's inequality and h-convexity of $\left|f'\right|^q$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)b)\right|dt \quad (19)\\ \le\ &\left(\int_0^1 \left|t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(t(a+b-x)+(1-t)b)\right|^q\right)^{\frac{1}{q}}\\ \le\ &\left(\int_0^1 (t^{\alpha p})\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[h(t)\left|f'(a+b-x)\right|^q+h(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\\ =\ &\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left|f'(a+b-x)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)a)\right|dt \quad (20)\\ \le\ &\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left|f'(a+b-x)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}.\end{aligned}$$

Substituting the inequalities (19) and (20) in (18), the required result is obtained. □

Corollary 2.14. *Under assumption of Theorem 2.13 with $x = \frac{a+b}{2}$, the following inequality holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)-f\left(\frac{a+b}{2}\right)\right| \tag{21}$$

$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\left|f'(b)\right|^{q}\left(1+h\left(\frac{1}{2}\right)\right)+\left|f'(a)\right|^{q}h\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|f'(a)\right|^{q}\left(1+h\left(\frac{1}{2}\right)\right)+\left|f'(b)\right|^{q}h\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}\right]\left(\int_{0}^{1}h(t)dt\right)^{\frac{1}{q}}.$$

Corollary 2.15. *By taking $h(t) = t^s$ in (17), the following result holds for s-convexity:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\right]-f(a+b-x)\right|$$

$$\leq \frac{1}{b-a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^{2}\left[\frac{\left|f'(a+b-x)\right|^{q}+\left|f'(b)\right|^{q}}{s+1}\right]^{\frac{1}{q}}+(b-x)^{2}\left[\frac{\left|f'(a+b-x)\right|^{q}+\left|f'(a)\right|^{q}}{s+1}\right]^{\frac{1}{q}}\right],$$

Remark 2.16. *(i) If we take $h(t) = t$ in Theorem 2.13, then Theorem 2.13 reduces to [6, Theorem 2.5].*
(ii) If we take $h(t) = t$ in Corollary 2.14, then Corollary 2.14 reduces to [6, Corollary 2.6].
(iii) If we take $h(t) = t$ and $\alpha = 1$ Theorem 2.13, then Theorem 2.13 reduces to [25, Theorem 4].

Theorem 2.17. *Let $f : [a, b] \to \mathbb{R}$ be differentiable mapping on (a, b) with $0 \leq a < b$ and $f' \in L^1[a, b]$. If $\left|f'\right|^q$, $q \geq 1$, is h-convex on $[a, b]$, then for all $x \in [a, b]$ the following fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\right]-f(a+b-x)\right| \tag{22}$$

$$\leq \frac{1}{b-a}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[(x-a)^{2}\left(I_{3}\left|f'(a+b-x)\right|^{q}+I_{4}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+(b-x)^{2}\left(I_{3}\left|f'(a+b-x)\right|^{q}+I_{4}\left|f'(a)\right|^{q}\right)^{1-\frac{1}{q}}\right].$$

Proof. By the Lemma 1.7 and the power mean inequality, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\right]-f(a+b-x)\right| \tag{23}$$

$$\leq \frac{(x-a)^{2}}{b-a}\int_{0}^{1}t^{\alpha}\left|f'(t(a+b-x)+(1-t)b)\right|dt$$

$$+\frac{(b-x)^{2}}{b-a}\int_{0}^{1}t^{\alpha}\left|f'(t(a+b-x)+(1-t)a)\right|dt$$

$$\leq \frac{(x-a)^{2}}{b-a}\left(\int_{0}^{1}t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}t^{\alpha}\left|f'(t(a+b-x)+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{2}}{b-a}\left(\int_{0}^{1}t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}t^{\alpha}\left|f'(t(a+b-x)+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}.$$

Using the h-convexity of $\left|f'\right|^q$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha}\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)\right|^q dt\\
\leq\ &\int_0^1 t^{\alpha}\left[h(t)\left|f'\left(a+b-x\right)\right|^q+h\left(1-t\right)\left|f'\left(b\right)\right|^q\right]dt\\
=\ &\left|f'\left(a+b-x\right)\right|^q\int_0^1 t^{\alpha}h(t)dt+\left|f'(b)\right|^q\int_0^1 t^{\alpha}h(1-t)dt.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 t^{\alpha}\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)a\right)\right|^q dt\\
\leq\ &\left|f'\left(a+b-x\right)\right|^q\int_0^1 t^{\alpha}h(t)dt+\left|f'(a)\right|^q\int_0^1 t^{\alpha}h(1-t)dt.
\end{aligned}$$

This completes the proof. □

Corollary 2.18. *Under assumption of Theorem 2.17 with $x=\frac{a+b}{2}$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f\left(a\right)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f\left(b\right)\right]-f\left(\frac{a+b}{2}\right)\right| \qquad (24)\\
\leq\ &\frac{b-a}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[I_3\left|f'\left(\frac{a+b}{2}\right)\right|^q+I_4\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}\\
+\ &\frac{b-a}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[I_3\left|f'\left(\frac{a+b}{2}\right)\right|^q+I_4\left|f'\left(a\right)\right|^q\right]^{\frac{1}{q}}\\
\leq\ &\frac{b-a}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[h\left(\frac{1}{2}\right)I_3\left|f'\left(a\right)\right|^q+\left(h\left(\frac{1}{2}\right)I_3+I_4\right)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}\\
+\ &\frac{b-a}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(h\left(\frac{1}{2}\right)I_3+I_4\right)\left|f'\left(a\right)\right|^q+h\left(\frac{1}{2}\right)I_3\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.19. *By putting $h(t)=t^s$ (22), the following inequality holds for s-convexity:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f\left(a\right)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f\left(b\right)\right]-f\left(\frac{a+b}{2}\right)\right|\\
\leq\ &\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{\left(x-a\right)^2}{b-a}\left[\frac{\left|f'\left(a+b-x\right)\right|^q}{\alpha+s+1}+B\left(\alpha+1,s+1\right)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}\right.\\
+\ &\left.\frac{\left(b-x\right)^2}{b-a}\left[\frac{\left|f'\left(a+b-x\right)\right|^q}{\alpha+s+1}+B\left(\alpha+1,s+1\right)\left|f'\left(a\right)\right|^q\right]^{\frac{1}{q}}\right],
\end{aligned}$$

where $B(x,y)$ is Euler's Beta function.

Remark 2.20. *(i) If we take $h(t) = t$ in Theorem 2.17, then Theorem 2.17 reduces to [6, Theorem 2.7].*
(ii) If we take $h(t) = t$ in Corollary 2.18, then Corollary 2.18 reduces to [6, Corollary 2.8].
(iii) If we take $h(t) = t$ and $\alpha = 1$ Theorem 2.17, then Theorem 2.17 reduces to [25, Theorem 5].

3. Conclusions

The generalized midpoint inequalities and some related results have been obtained for h-convex functions. The obtained inequalities have direct consequences in midpoint type inequalities for Riemann-Liouville fractional integral operators via convex and s-convex functions.

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