



Homology of Contact CR-Submanifolds

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Abstract. In this paper, homology of a contact CR-submanifold of a real hypersurface, which has naturally almost contact metric structure induced from the complex Euclidean space \mathbb{C}^m , is examined. More precisely, nonexistence of stable integral currents on a compact contact CR-submanifold of real hypersurface of canonical complex space form \mathbb{C}^m is investigated and vanishing theorems concerning the homology groups are obtained.

1. Introduction

Recently, geometric and topological methods have important roles in learning neural networks, pattern recognition and signal processing [2], [24]. The main idea in these areas is to consider a flat or curved network parameter space endowed with a suitable geometric structure in the network learning algorithm. The non-Euclidean nature of data implies that manifold theory may be considered as new tool to use in these new research areas. One of the new methods in machine learning is depicted data as a submanifold (with curvature) of Euclidean space. Thus submanifold theory finds a place in machine learning and data mining, which are prominent research areas. Let \tilde{M} be an m -dimensional compact Riemannian manifold with Riemannian metric \tilde{g} and the Levi-Civita connection $\tilde{\nabla}$. Denote by (S, ρ) the oriented, p -rectifiable set in \tilde{M} . The set of rectifiable p -currents is

$$\mathcal{R}_p(\tilde{M}) = \left\{ S : \sum_{n=1}^{\infty} n S_n; S_n = (S_n, \rho_n), \tilde{M}(S) = \sum_{n=1}^{\infty} n \mathcal{H}^p(S_n) < \infty \right\}.$$

$S \in \mathcal{R}_p(\tilde{M})$ is called an integral p -current if S and ∂S are both rectifiable currents, for details see:[9]. After Federer and Fleming [10] showed that any non-trivial integral homology class in $H_p(M, \mathbb{Z})$ corresponds to a stable current, Lawson and Simons [12] obtained that there are no stable integral currents in the sphere S^n , and there is no integral current in a submanifold M^m of S^n when the second fundamental form of M^m satisfies a pinching condition. The result on submanifolds of S^n has been extended to Euclidean space [13], [27], to $S^{n_1} \times S^{n_2}$ [30] and to higher codimensional submanifolds in the Euclidean spaces [14]. In [8], the authors showed that when the second fundamental form of a compact submanifold N^n of a hypersurface M^m in \mathbb{R}^{m+1} or a submanifold M^m in \mathbb{R}^{m+2} satisfies some conditions, there is no stable integral p -current

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in N^n and $H_p(N, \mathbb{Z}) = H_{n-p}(N, \mathbb{Z}) = 0$, where $H_i(N, \mathbb{Z})$ is the i -th homology group of N with integer coefficients.

CR-submanifolds of a Kähler manifold were introduced by Bejancu [1]. The study of such submanifolds is still an active research area, see:[25], [26] for instances. On the other hand, Chen defined CR-warped product submanifolds of almost Hermitian manifolds in his papers [6] and [7]. Later such submanifolds have been studied by many authors, see book [5] and references therein. In almost contact manifolds, Hasegawa and Mihai [11] introduced and studied contact CR-warped submanifolds and obtained a sharp inequality for the squared norm of the second fundamental form (an extrinsic invariant) in terms of the warping function for contact CR-warped products isometrically immersed in Sasakian manifolds. Moreover Mihai [15] obtained a classification of contact CR-warped product submanifolds in spheres, which satisfy the equality case, see also [16–18].

In [22], the first author considered CR-warped product submanifolds in even-dimensional Euclidean spaces and obtained a sufficient condition for certain homology groups to be zero. In this article, a more general case is taken into account in calculating homology groups of CR-submanifolds immersed in an arbitrarily curved space. In this direction, first the non-existence of integral currents on a CR-warped submanifold of the real hypersurface of complex Euclidean space was found in terms of the warping function. Second, the non-existence of integral currents on a general contact CR-submanifold of the real hypersurface of complex Euclidean space was obtained by using mixed scalar curvature introduced in [20], [21] and related notions.

2. Preliminaries

In this section, we will review basic materials from [29] for later sections. An almost contact manifold M is a manifold endowed with an $(1, 1)$ tensor field ϕ , a 1- form η and a vector field ξ such that

$$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0 \tag{1}$$

for $X \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the module of vector fields on M . An almost contact metric structure (ϕ, ξ, η) is called Sasakian structure if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

for $Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M . A Kähler manifold \bar{M} is a differentiable Riemannian manifold endowed with an almost complex structure J , and a Riemannian metric g such that

$$J^2 X = -X, \nabla J = 0, g(JX, JY) = g(X, Y) \tag{2}$$

for $X, Y \in \Gamma(T\bar{M})$. A complex space form is a simply connected complete Kähler manifold of constant holomorphic sectional curvature.

Let M be an orientable real hypersurface of the complex space form \mathbb{C}^m , which is a complex space form with constant holomorphic sectional curvature 0, with the unit normal vector field ζ . We denote the induced metric on M by g again. Then there is an almost contact metric structure (ϕ, ξ, g) defined on M , whose ξ is the characteristic unit vector field defined by $\xi = -J\zeta$, η is a smooth form which is dual to ξ that is $\eta(X) = g(X, \xi)$, $X \in \Gamma(TM)$ and ϕ is a tensor field of type $(1, 1)$ satisfying (1) and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Let A_ζ be the shape operator of M with respect to ζ . Then we also have the following identities

$$R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY, \tag{3}$$

$$JX = \phi X + \eta(X)\xi \tag{4}$$

where we have put $A = A_\zeta$. Let N be an $n + 1$ - dimensional submanifold, tangent to the structure vector field ξ of a real hypersurface M^{2m-1} of \mathbb{C}^m . The Gauss equation for a submanifold N is given by

$$g(R(X, Y)Z, W) = g(R'(X, Y)Z, W) - g(h'(X, W), h'(Y, Z)) + g(h'(Y, W), h'(X, Z)) \tag{5}$$

for $X, Y, Z, W \in \Gamma(TN)$, here h' is the second fundamental form of N , R' is the curvature tensor field of N and R is the curvature tensor field of M . From (1), (2) and (4), it follows that a real hypersurface of a Kähler manifold is Sasakian manifold if and only if A satisfies

$$A = -I + \beta\eta \otimes \xi$$

where $\beta = \eta(A\xi) + 1$. For research papers on the real hypersurfaces of a Kaehler manifold being a Sasakian manifold, see: [19] and [23], see also [4] for generalized Sasakian space form.

We now suppose N is a contact CR-submanifold of M^{2m-1} , then there are two distributions D and D^\perp on N such that

$$TM = D \oplus D^\perp \oplus \xi, \phi(D) = D, \phi(D^\perp) \subset T^\perp M.$$

We note that totally real submanifolds and invariant submanifolds are contact CR-submanifolds with $D = \{0\}$ and $D^\perp = \{0\}$, respectively. For examples of such submanifolds, see: [28]. Unless otherwise stated, we will consider proper contact CR-submanifolds throughout this paper. This means that $\dim(D) = p \neq 0$ and $\dim(D^\perp) = q \neq 0$.

3. Homology of Contact CR-Warped Submanifolds of a Sasakian Hypersurface

CR-warped product submanifolds of a Kähler manifold were first introduced by Chen [5–7] and then such submanifolds were studied by Hasegawa and Mihai[11], Mihai [15] and Munteanu [18] in the Sasakian setting. We first recall basic notions for warped product manifolds from [3] and [5]. Let (B, g_1) and (F, g_2) be two Riemannian manifolds, $f : B \rightarrow (0, \infty)$ and $\pi : B \times F \rightarrow B, \eta : B \times F \rightarrow F$ the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in B \times F$. The warped product [3] $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_*X, \pi_*Y) + (f \circ \pi)^2 g_2(\eta_*X, \eta_*Y)$$

for every X and Y of M , where $*$ denotes the tangent map. The function f is called the warping function of the warped product manifold. In particular, if the warping function is constant, then the warped product manifold M is said to be trivial. Let X, Y be vector fields on B and V, W vector fields on F , then from Lemma 7.3 of [3], we have

$$\nabla_X V = \nabla_V X + \left(\frac{Xf}{f}\right)V$$

where ∇ is the Levi-Civita connection on M .

In [11], Hasegawa and Mihai considered warped products $M = M_1 \times_f M_2$ which are contact CR-submanifolds of a Sasakian manifold \bar{M} . They showed that there do not exist warped product submanifolds $M = M_1 \times_f M_2$ of a Sasakian manifold \bar{M} such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 an invariant submanifold of \bar{M} . Therefore they considered contact CR-warped product submanifold of the form $M_1 \times_f M_2$ of \bar{M} such that M_1 is an invariant submanifold to \bar{M} and M_2 an anti-invariant submanifold of \bar{M} .

One of the generalized version of the Lawson-Simons’s result is obtained as follows.

Theorem 3.1. (Cheng-Shiohama)[8] *Let N be an n -dimensional compact submanifold of M^m which is an m -dimensional hypersurface of an $(m + 1)$ -dimensional Euclidean space \mathbb{R}^{m+1} . If*

$$\sum_{i,\alpha} [2\|h'(e_i, e_\alpha)\|^2 - \langle h'(e_i, e_i), h(e_\alpha, e_\alpha) \rangle] < 2p(p - n)\lambda^2$$

is satisfied for every $x \in N$ and any orthonormal basis $\{e_i, e_\alpha\}$ of $T_x N, i = 1, \dots, p, \alpha = p + 1, \dots, n$. Here h' is the second fundamental form of N in M^m , and λ^2 is the maximum at x of the square of principal curvatures of M^m . Then there is no integral p -current in N and $H_p(N, \mathbb{Z}) = H_{n-p}(N, \mathbb{Z}) = 0$, where $H_p(N, \mathbb{Z})$ is the p -th singular homology group with integer coefficients.

In this section, we are going to find suitable conditions for a contact CR-warped product such that $(2p + 1)$ th homology group of N is zero and as a result of this, we obtain a sphere theorem for such submanifolds. We note that the Laplacian of a function f is denoted by Δf .

Theorem 3.2. *Let N^m be a compact contact CR-warped product submanifold $M = M_1 \times_f M_2$ of a Sasakian hypersurface of \mathbb{C}^m such that M_1 is a $(2p + 1)$ -dimensional invariant submanifold tangent to ξ and M_2 a q -dimensional C-totally real submanifold of M . If*

$$\Delta f < \frac{f}{q} \{2(2p + 1)(2p + 1 - n)\lambda(x)^2 - q(2p - \eta(A\xi)) + \|\nabla \ln f\|^2 + 1\}$$

is satisfied for every $x \in N$, where $\lambda(x)^2$ is the maximum at x of the square of principal curvature of M^{2m-1} , then there is no stable integral $(2p + 1)$ -current in N and $H_{2p+1}(N, \mathbb{Z}) = H_{n-(2p+1)}(N, \mathbb{Z}) = 0$, where $H_{2p+1}(N, \mathbb{Z})$ is the $(2p + 1)$ th homology group of N , where $2p + 1 + q = \dim N = n$ and $2m - 1 - \dim N = q$.

Proof. From (3) and (5), we have

$$g(h'(X, W), h'(Y, Z)) = g(AX, Z)g(AY, W) - g(AY, Z)g(AX, W) + g(R'(X, Y)Z, W) + g(h'(Y, W), h'(X, Z)). \tag{6}$$

for any vector fields X, Y, Z and W on N . We take $X = Z = e_i, Y = W = e_\alpha$ in (6). Thus we obtain

$$\sum_{i=1}^{2p+1} \sum_{\alpha=1}^q g(h'(e_i, e_\alpha), h'(e_\alpha, e_i)) = \sum_{i=1}^{2p+1} \sum_{\alpha=1}^q (g(Ae_i, e_i)g(Ae_\alpha, e_\alpha) - g(Ae_\alpha, e_i)g(Ae_i, e_\alpha) + g(R'(e_i, e_\alpha)e_i, e_\alpha) + g(h'(e_\alpha, e_\alpha), h'(e_i, e_i))).$$

First of all for a contact CR-warped product submanifold, we have the following

$$g(h(\phi X, Z), \phi W) = X(\ln f)g(Z, W), \tag{7}$$

for $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$, See:[11]. Now using (7), we also have

$$g(h'(X, Z), \phi W) = -\phi X(\ln f)g(Z, W). \tag{8}$$

Hence for $\dim(D^\perp) = \dim(TM^\perp)$, we derive

$$\sum_{i=1}^{2p+1} \sum_{\alpha=1}^q g(h'(e_i, e_\alpha), h'(e_i, e_\alpha)) = \sum_{i=1}^{2p} \sum_{\alpha=1}^q g(h'(e_i, e_\alpha), h'(e_i, e_\alpha)) + g(h'(e_\alpha, \xi), h'(e_\alpha, \xi)).$$

Since $h'(Y, \xi) = -\phi Y$ (see:[1, Page 102, Lemma 1.2]) for $Y \in \Gamma(D^\perp)$, we obtain

$$\sum_{i=1}^{2p+1} \sum_{\alpha=1}^q g(h'(e_i, e_\alpha), h'(e_i, e_\alpha)) = \sum_{i=1}^{2p} \sum_{\alpha=1}^q g(h'(e_i, e_\alpha), h'(e_i, e_\alpha)) + q. \tag{9}$$

Using (7) and (8), we derive

$$\begin{aligned} \sum_{i=1}^{2p} \sum_{\alpha=1}^q g(h'(e_i, e_\alpha), h'(e_i, e_\alpha)) &= \sum_{i=1}^p \sum_{\alpha, \gamma=1}^q (-\phi e_i(\ln f))^2 g(e_\alpha, e_\gamma)^2 \\ &\quad + e_i(\ln f)^2 g(e_\alpha, e_\gamma)^2 \\ &= \|\nabla \ln f\|^2 q. \end{aligned} \tag{10}$$

Since

$$\sum_{i=1}^{2p+1} \sum_{\alpha=1}^q g(Ae_i, e_i)g(Ae_\alpha, e_\alpha) = (2p + 1 - \beta)q,$$

and

$$\sum_{i=1}^{2p+1} \sum_{\alpha=1}^q g(Ae_{\alpha}, e_i)g(Ae_{\alpha}, e_i) = 0,$$

using (9) and (10), we have

$$\begin{aligned} & \sum_{i=1}^{2p+1} \sum_{\alpha=1}^q 2 \| h'(e_i, e_{\alpha}), h'(e_i, e_{\alpha}) \|^2 - g(h'(e_{\alpha}, e_{\alpha}), h'(e_i, e_i)) \\ &= (2p + 1 - \beta)q + \sum_{i=1}^{2p+1} \sum_{\alpha=1}^q g(R'(e_i, e_{\alpha})e_i, e_{\alpha}) + \|\nabla \ln f\|^2 q + q. \end{aligned}$$

On the other hand since N is a contact CR-submanifold of a Sasakian manifold and it is a warped product submanifold, we have (see [5])

$$R(X, V)W = -\frac{g(V, W)}{f} \nabla_X \nabla f.$$

Hence we get

$$\sum_{i=1}^{2p+1} \sum_{\alpha=1}^q g(R'(e_i, e_{\alpha})e_i, e_{\alpha}) = \frac{q}{f} \Delta f.$$

Thus we arrive at

$$\begin{aligned} & \sum_{i=1}^{2p+1} \sum_{\alpha=1}^q (2\|h'(e_i, e_{\alpha})\|^2 - g(h'(e_{\alpha}, e_{\alpha}), h'(e_i, e_i))) = (2p + 1 - \beta)q \\ & \quad + \|\nabla \ln f\|^2 q + \frac{q}{f} \Delta f + q. \end{aligned}$$

Therefore, it follows that

$$\sum_{i=1}^{2p+1} \sum_{\alpha=1}^q (2\|h'(e_i, e_{\alpha})\|^2 - g(h'(e_{\alpha}, e_{\alpha}), h'(e_i, e_i))) < 2(2p + 1)(2p + 1 - n)\lambda(x)^2$$

for $x \in N$ if and only if the following inequality is satisfied

$$\Delta f < -\frac{f}{q} \{(2p + 1 - \beta)q + q\|\nabla \ln f\|^2 + q - 2(2p + 1)(2p + 1 - n)\lambda(x)^2\}$$

which completes proof due to $\beta = \eta(A\xi) + 1$. \square

Thus we also have the following result.

Corollary 3.3. *Let N^n be a compact contact CR-warped product submanifold $M = M_1 \times_f M_2$ of a Sasakian hypersurface of \mathbb{C}^m such that M_1 is a $(2p + 1)$ -dimensional invariant submanifold tangent to ξ and M_2 a q -dimensional C-totally real submanifold of M . If*

$$\Delta f < \frac{f}{q} \{2(2p + 1)(2p + 1 - n)\lambda(x)^2 - q(2p - \eta(A\xi)) + \|\nabla \ln f\|^2 + 1\}$$

is satisfied for every $x \in N$, where $\lambda(x)^2$ is the maximum at x of the square of principal curvature of M^{2m-1} , then N^n is homeomorphic to a sphere when $n \neq 3$ and N^n is homotopic to a sphere when $n = 3$.

4. Homology of Contact CR-submanifolds of a Sasakian hypersurface

In this section, we will consider an arbitrary contact CR-submanifold N of a Sasakian hypersurface M^{2m-1} immersed in \mathbb{C}^m . By definition of contact CR-submanifold, we use two differentiable distributions in our study. Therefore, we will recall some notions related to distributions which are introduced recently in [20] and [21]. Let D and D^\perp be two distributions on a submanifold M of a Riemannian manifold \bar{M} . Let X^T be the D -component of $X \in \chi(M)$ (resp., X^\perp the D^\perp -component of X). Then the second fundamental form of D^\perp and D , respectively, are defined as

$$h^\perp(u, v) = \frac{1}{2}(\nabla_u v + \nabla_v u)^T, h^T(X, Y) = \frac{1}{2}(\nabla_X Y + \nabla_Y X)^\perp$$

for $u, v \in \Gamma(D^\perp)$ and $X, Y \in \Gamma(D)$. The mean curvature of D^\perp and D are defined as

$$H^T = \text{trace}_g h^T, H^\perp = \text{trace}_g h^\perp.$$

We also define the following maps

$$T^\perp(u, v) = \frac{1}{2}[u, v]^T, T^T(X, Y) = \frac{1}{2}[X, Y]^\perp$$

where T^\perp and T^T denote the components of D^\perp and D , respectively. Finally mixed scalar curvature of submanifold of M is given

$$S_{mix} = \sum_{\alpha, i} g(R(E_\alpha, \varepsilon_i)E_\alpha, \varepsilon_i)$$

We note that this notion is not depend on the order of distribution on the choice of a local frame. By direct computation, we have [20]

$$S_{mix} = \|H^\perp\|^2 - \|h^\perp\|^2 + \|T^\perp\|^2 + \|H^T\|^2 - \|h^T\|^2 + \text{div}(H^T + H^\perp). \tag{11}$$

In this section, we are going to investigate certain conditions for a contact CR-submanifold of a Sasakian hypersurface such that its $2p$ -th homology group is zero and as a result of this we obtain a sphere theorem for contact CR-submanifold.

Theorem 4.1. *Let N be an n -dimensional compact contact CR-submanifold of a Sasakian hypersurface M^{2m-1} of \mathbb{C}^m . If $p = \dim(D)$ and $q = \dim(D^\perp)$, if*

$$S_{mix} < 2p(\eta(A\xi) - q) + \sum_{i=1}^p \sum_{s=1}^q -\|(\nabla'_{e_s} \phi)e_i\|^2 + 2g((\nabla'_{e_s} \phi)e_i, \phi h'(e_s^*, e_i)) - 2\|h'(e_i, e_s^*)\|^2 + 4p(2p - n)\lambda(x)^2$$

is satisfied for every $x \in N$ and orthonormal basis $\{e_i, e_s^*\}$ of $D \oplus D^\perp$, $i = 1, \dots, 2p$, $s = 1, \dots, q$, then there is no integral $2p$ -current in N and $H_{2p}(N, \mathbb{Z}) = H_{n-2p}(N, \mathbb{Z}) = 0$, where $H_p(N, \mathbb{Z})$ is the p -th singular homology group with integer coefficients and h' is the second fundamental form of N in M .

Proof. Taking $X = Z = e_i$, $Y = W = e_\alpha$, $\alpha = 1, \dots, q + 1$, in (6), we have

$$\sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} g(h'(e_i, e_\alpha), h'(e_\alpha, e_i)) = \sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} (g(Ae_i, e_i)g(Ae_\alpha, e_\alpha) - g(Ae_\alpha, e_i)g(Ae_i, e_\alpha) + g(R'(e_i, e_\alpha)e_i, e_\alpha) + g(h'(e_\alpha, e_\alpha), h'(e_i, e_i))).$$

Since

$$\sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} g(Ae_i, e_i)g(Ae_\alpha, e_\alpha) = -2p((-q + 1) + \beta) = -2p(\beta - (q + 1)),$$

and

$$\sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} g(Ae_{\alpha}, e_i)g(Ae_{\alpha}, e_i) = 0,$$

we derive

$$\sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} g(h'(e_i, e_{\alpha}), h'(e_i, e_{\alpha})) = -2p(\beta - (q + 1)) + \sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} \{g(R'(e_i, e_{\alpha})e_i, e_{\alpha}) + g(h'(e_{\alpha}, e_{\alpha}), h'(e_i, e_i))\}.$$

Thus we get

$$\sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} \{2 \| h'(e_i, e_{\alpha}) \|^2 - g(h'(e_{\alpha}, e_{\alpha}), h'(e_i, e_i))\} = -2p(\beta - (q + 1)) + S_{mix} + \sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} g(h'(e_i, e_{\alpha}), h'(e_i, e_{\alpha})). \tag{12}$$

Since $h'(e_i, \xi) = 0$, we obtain

$$\sum_{i,\alpha} g(h'(e_i, e_{\alpha}), h'(e_i, e_{\alpha})) = \sum_{i,s} g(h'(e_i, e_s^*), h'(e_i, e_s^*)) \tag{13}$$

Putting (13) in (12), we get

$$\sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} \{2 \| h'(e_i, e_{\alpha}) \|^2 - g(h'(e_{\alpha}, e_{\alpha}), h'(e_i, e_i))\} = -2p(\beta - (q + 1)) + S_{mix} + \sum_{i=1}^{2p} \sum_{\alpha=1}^q g(h'(e_i, e_{\alpha}^*), h'(e_i, e_{\alpha}^*)).$$

On the other hand, by direct calculation we have

$$h'(\phi X, Z) = -(\nabla'_Z \phi)X + \phi h'(Z, X),$$

where ∇' is the induced Levi-Civita connection on M . Thus we have

$$g(h'(\phi X, Z), h'(\phi X, Z)) = \|(\nabla'_Z \phi)X\|^2 - 2g((\nabla'_Z \phi)X, h'(Z, X)) + g(\phi h'(Z, X), \phi h'(Z, X)).$$

for $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Since $\eta(h'(Z, X)) = 0$, we obtain

$$g(\phi h'(Z, X), \phi h'(Z, X)) = g(h'(Z, X), h'(Z, X)) = g(h'(Z, X), h'(Z, X)).$$

Putting above equality in (13), we have

$$\sum_{i=1}^{2p} \sum_{s=1}^q g(h'(e_i, e_s^*), h'(e_i, e_s^*)) = \sum_{i=1}^p \sum_{s=1}^q (\|(\nabla'_{e_s^*} \phi)e_i\|^2 - 2g((\nabla'_{e_s^*} \phi)e_i, \phi h'(e_s^*, e_i)) + 2g(h'(e_i, e_s^*), h'(e_i, e_s^*))).$$

Thus we arrive at

$$\begin{aligned} \sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} (2\|h'(e_i, e_\alpha)\|^2 - g(h'(e_\alpha e_\alpha), h'(e_i, e_i))) &= -2p(\beta - (q + 1)) + S_{mix} \\ &+ \sum_{i=1}^p \sum_{s=1}^q (\|(\nabla'_{e_s} \phi)e_i\|^2 - 2g((\nabla'_{e_s} \phi)e_i, \phi h'(e_s^*, e_i)) \\ &+ 2\|h'(e_i, e_s^*)\|^2). \end{aligned} \tag{14}$$

From above equality and Theorem 3.1, the proof is complete due to $\beta = \eta(A\xi) + 1$. \square

From Theorem 4.1, (14) and (11), we have the following result.

Corollary 4.2. *Let N be a n -dimensional compact CR-submanifolds of M^{2m-1} which is a $(2m - 1)$ -dimensional Sasakian hypersurface of \mathbb{C}^m . If $p = \dim(D)$ and $q = \dim(D^\perp)$, then*

$$\begin{aligned} \|H^\perp\|^2 + \|T^\perp\|^2 + \|H^T\|^2 + \operatorname{div}(H^T + H^\perp) + \sum_{i=1}^p \sum_{s=1}^q (\|(\nabla'_{e_s} \phi)e_i\|^2 + 2\|h'(e_i, e_s^*)\|^2) \\ < 2p(\eta(A\xi) - q) + 4p(2p - n)\lambda(x)^2 + \sum_{i=1}^p \sum_{s=1}^q 2g((\nabla'_{e_s} \phi)e_i, \phi h'(e_s^*, e_i)). \end{aligned}$$

is satisfied for every $x \in N$ and orthonormal basis $\{e_i, \dots, e_s^*\}$ of $D \oplus D^\perp$, $i = 1, \dots, 2p$, $s = 1, \dots, q$. Then there is no integral $2p$ -current in N and $H_{2p}(N, \mathbb{Z}) = H_{n-2p}(N, \mathbb{Z}) = 0$, where $H_p(N, \mathbb{Z})$ is the p -th singular homology group with integer coefficients and h' is the second fundamental form of N in M .

Proof. By using (11), (14) becomes

$$\begin{aligned} \sum_{i=1}^{2p} \sum_{\alpha=1}^{q+1} (2\|h'(e_i, e_\alpha)\|^2 - g(h'(e_\alpha e_\alpha), h'(e_i, e_i))) &= -2p(\beta - (q + 1)) \\ &\|H^\perp\|^2 - \|h^\perp\|^2 + \|T^\perp\|^2 + \|H^T\|^2 - \|h^T\|^2 + \\ &+ \operatorname{div}(H^T + H^\perp) + \sum_{i=1}^p \sum_{s=1}^q (\|(\nabla'_{e_s} \phi)e_i\|^2 \\ &- 2g((\nabla'_{e_s} \phi)e_i, \phi h'(e_s^*, e_i)) + 2\|h'(e_i, e_s^*)\|^2). \end{aligned}$$

By using $\beta = \eta(A\xi) + 1$ we obtain the assertion. \square

We also have the following result.

Corollary 4.3. *Let N be an n -dimensional compact contact CR-submanifold of a Sasakian hypersurface M^{2m-1} of \mathbb{C}^m . If $p = \dim(D)$ and $q = \dim(D^\perp)$, then*

$$\begin{aligned} S_{mix} < 2p(\eta(A\xi) - q) + \sum_{i=1}^p \sum_{s=1}^q (-\|(\nabla'_{e_s} \phi)e_i\|^2 + 2g((\nabla'_{e_s} \phi)e_i, \phi h'(e_s^*, e_i)) \\ - 2\|h'(e_i, e_s^*)\|^2) + 4p(2p - n)\lambda(x)^2 \end{aligned}$$

is satisfied for every $x \in N$ and orthonormal basis $\{e_i, \dots, e_s^*\}$ of $D \oplus D^\perp$, $i = 1, \dots, 2p$, $s = 1, \dots, q$, then N^n is homeomorphic to a sphere when $n \neq 3$ and N^n is homotopic to a sphere when $n = 3$, where h' is the second fundamental form of N in M .

5. Concluding Remarks

In this article, we have taken a step in understanding the homology groups of the contact CR-submanifold of a Sasakian hypersurface as the first step in understanding the homology groups of a general contact CR-submanifold of an arbitrary almost contact manifold. Our study will stimulate to obtain more results for more general situations.

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