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Quantale-valued Convergence Tower Spaces: Diagonal Axioms and Continuous Extension

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Abstract. We generalize a result on continuous extension of a mapping on a dense subspace from the category of convergence spaces to the category of quantale-valued convergence tower spaces. To this end, we introduce and study diagonal axioms which characterize topologicalness and regularity for quantale-valued convergence tower spaces.

1. Introduction

A quantale-valued convergence tower space [18] is a set X with a family of convergence structures indexed by a quantale. The quantale is considered in this way as a variable of the theory and - depending on the value of this variable, i.e. on the choice of the quantale - we obtain generalizations of different kinds of convergence spaces. E.g. the choice $L = \{0, 1\}$ leads to classical convergence spaces [6, 10, 20, 23], for the so-called Lawvere quantale we obtain limit tower spaces or convergence approach spaces [2], the unit interval with a left-continuous t-norm as the quantale leads to probabilistic convergence spaces [25] and the quantale of distance distribution functions leads to probabilistic convergence spaces as studied in [16]. It was shown in [18] that also L-metric spaces can be characterized internally as quantale-valued convergence tower spaces.

It is the aim of this paper to obtain an extension theorem for a continuous mapping from a dense subspace to a continuous mapping defined on the whole space. Such a result was shown by Cook [4] in the category CONV of convergence spaces. Suitable diagonal axioms characterizing topologicalness and regularity of spaces are crucial in the proof. Generalizations of such diagonal axioms were studied in many of the examples for quantale-valued convergence tower spaces. E.g. for convergence spaces such diagonal axioms characterize topological spaces [19], for convergence approach spaces or limit tower spaces, approach spaces [22] are characterized and in the probabilistic case, so-called probabilistic topological spaces are obtained. Furthermore, regularity axioms can be defined using "dual" diagonal axioms.

Aiming to generalize Cook's extension theorem to the category of quantale-valued convergence tower spaces, in a first step, we have a careful look at such diagonal axioms and regularity axioms in the general framework of quantale-valued convergence tower spaces. Most of the results are extensions of known results and we recapture the special instances if we choose a suitable quantale. Furthermore,

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we formulate the diagonal axioms depending on a mapping $\gamma : L \times L \longrightarrow L$. In this way, we can e.g. simultaneously accommodate "level-wise spaces" (for $\gamma(\alpha, \beta) = \alpha \wedge \beta$) and probabilistic convergence spaces (with $\gamma(\alpha, \beta) = \alpha * \beta$ for a t-norm $*$ on $[0, 1]$.) The importance of this further variable of our theory, however, becomes only clear if we consider also quantale-valued generalization of uniform convergence spaces or Cauchy spaces. We have to postpone this, however, to future work.

In this paper, we shall discuss the following.

- Diagonal axioms, generalizing Kowalsky's axiom, Fischer's axiom and Gähler's neighbourhood condition and which give rise to what we shall call *topological* L-convergence tower spaces. In particular we show that quantale-valued metric spaces satisfy these diagonal axioms.
- Regularity axioms and its characterizations via the convergence of closures of filters. Again, we show that quantale-valued metric spaces are regular.
- A generalization of an extension theorem in CONV, for continuous mappings from a dense subset. This result ties the diagonal and dual diagonal axioms nicely together.

2. Preliminaries

Let L be a complete lattice with distinct top and bottom elements $\top \neq \bot$. In any complete lattice L we can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and $\alpha \lhd \bigvee_{j \in J} \beta_j$ iff $\alpha \lhd \beta_i$ for some $i \in J$. A complete lattice is completely distributive if and only if we have $\alpha = \bigvee \{\beta : \beta \lhd \alpha\}$ for any $\alpha \in L$, [24]. Similarly, we can define the *way-below relation* $\alpha \prec \beta$ if for all *directed* subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. This relation has similar properties as the well-below relation if arbitrary subsets are replaced by directed subsets. However, we have that $\alpha, \beta \prec \gamma$ implies $\alpha \vee \beta \prec \gamma$. For more details and results on lattices we refer to [11].

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice with order relation $\leq$, is called a *commutative and integral quantale* if $(L, *)$ is a commutative semigroup for which the top element of L acts as the unit, i.e. $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e. $(\bigvee_{i \in J} \alpha_i) * \beta = \bigvee_{i \in J} (\alpha_i * \beta)$, see e.g. [13].

We consider in this paper only commutative and integral quantales $\mathsf{L} = (L, \leq, *)$ with completely distributive lattices $(L, \leq)$. Typical examples of such quantales are e.g. the unit interval $[0, 1]$ with a left-continuous *t-norm* [26]. Another important example is given by *Lawvere's quantale*, the interval $[0, \infty]$ with the opposite order and addition $\alpha * \beta = \alpha + \beta$, extended by $\alpha + \infty = \infty + a = \infty$, see e.g. [7]. A further important example is the quantale of distance distribution functions. A *distance distribution function* $\varphi : [0, \infty] \longrightarrow [0, 1]$, satisfies $\varphi(x) = \sup\{\varphi(y) : y < x\}$ for all $x \in [0, \infty]$. The set of all distance distribution functions is denoted by Δ^+ and with the pointwise order Δ^+ becomes a completely distributive lattice [7]. A quantale operation $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ is also called a *sup-continuous triangle function* [26].

For a set X, we denote its power set by $\mathsf{P}(X)$ and the set of all filters $\mathbb{F}, \mathbb{G}, ...$ on X by $\mathsf{F}(X)$. The set $\mathsf{F}(X)$ is ordered by set inclusion and maximal elements of $\mathsf{F}(X)$ in this order are called *ultrafilters*. In particular, for each $x \in X$, the point filter $[x] = \{A \subseteq X : x \in A\}$ is an ultrafilter. If $\mathbb{F} \in \mathsf{F}(X)$ and $f : X \longrightarrow Y$ is a mapping, then we define $f(\mathbb{F}) \in \mathsf{F}(Y)$ by $f(\mathbb{F}) = \{G \subseteq Y : f(F) \subseteq G$ for some $F \in \mathbb{F}\}$. In particular, we have $f([x]) = [f(x)]$ for any $x \in X$. For a filter $\mathbb{G} \in \mathsf{F}(Y)$ the set $\{f^{-1}(G) : G \in \mathbb{G}\}$ is a filter basis whenever none of the $f^{-1}(G)$ is empty. In this case we denote by $f^{-1}(\mathbb{G})$ the filter on X generated by this filter basis and say that $f^{-1}(\mathbb{G})$ exists. We then have $f^{-1}(f(\mathbb{F})) \leq \mathbb{F}$ and $\mathbb{G} \leq f(f^{-1}(\mathbb{G}))$ in case $f^{-1}(\mathbb{G})$ exists. For a family of filters $(\mathbb{F}_i)_{i \in I}$ we define their join, $\bigvee_{i \in I} \mathbb{F}_i$, as the filter generated by the filter basis of finite intersections $F_{i_1} \cap ... \cap F_{i_n}$ with $F_{i_k} \in \mathbb{F}_{i_k}$ for $k = 1, ..., n$, whenever all these intersections are non-empty. For $\mathbb{G} \in \mathsf{F}(J)$ and $\mathbb{F}_j \in \mathsf{F}(X)$ for each $j \in J$, we denote $\kappa(\mathbb{G}, (\mathbb{F}_j)_{j \in J}) = \bigvee_{G \in \mathbb{G}} \bigwedge_{j \in G} \mathbb{F}_j \in \mathsf{F}(X)$ the *diagonal filter* [20].

For notions from category theory we refer to the textbooks [1] and [23]. A *construct* is a category C with a faithful functor $U : C \longrightarrow SET$, from C to the category of sets. We always consider a construct as a category whose objects are structured sets (S, ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e. if for every source $(f_i : S \longrightarrow (S_i, \xi_i))_{i \in I}$

there is a unique structure ξ on S, such that a mapping $g : (T, \eta) \longrightarrow (S, \xi)$ is a morphism if and only if for each $i \in I$ the composition $f_i \circ g : (T, \eta) \longrightarrow (S_i, \xi_i)$ is a morphism.

3. L-Convergence Tower Spaces

Definition 3.1. ([18]) Let $\mathsf{L} = (L, \leq, *)$ be a quantale. A pair $(X, \overline{q})$ of a set X and a family of mappings $\overline{q} = (q_\alpha \colon \mathsf{F}(X) \longrightarrow \mathsf{P}(X))_{\alpha \in L}$ is called an L-*convergence tower space* [18] if the following axioms are satisfied:

(LC1) $x \in q_\alpha([x])$, $\forall x \in X$ and $\forall \alpha \in L$;

(LC2) $\forall \mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$, with $\mathbb{F} \leq \mathbb{G}$, and $\alpha \in L$ implies $q_\alpha(\mathbb{F}) \subseteq q_\alpha(\mathbb{G})$;

(LC3) $\forall \alpha, \beta \in L$ with $\alpha \leq \beta$ implies $q_\beta(\mathbb{F}) \subseteq q_\alpha(\mathbb{F})$, $\forall \mathbb{F} \in \mathsf{F}(X)$;

(LC4) $x \in q_\perp(\mathbb{F})$, $\forall x \in X$, $\mathbb{F} \in \mathsf{F}(X)$.

If $(X, \overline{q})$ satisfies $x \in q_{\vee A}(\mathbb{F})$ whenever $x \in q_\alpha(\mathbb{F})\ \ \forall \alpha \in A$, it is called *left-continuous*. A mapping $f \colon (X, \overline{q}) \longrightarrow \left(X', \overline{q'}\right)$ between L-convergence tower spaces is called *continuous* if, for all $x \in X$, and for all $\mathbb{F} \in \mathsf{F}(X)$, $f(x) \in q'_\alpha(f(\mathbb{F}))$ whenever $x \in q_\alpha(\mathbb{F})$. The category of all L-convergence tower spaces and continuous mappings is denoted by L-CTS.

If $L = \{0, 1\}$, then L-convergence tower spaces can be identified with classical convergence spaces, [6, 23]. If $\mathsf{L} = ([0, \infty], \geq +)$ is Lawvere's quantale, then, demanding an additional axiom, an L-convergence tower space is a limit tower space [2] and a left-continuous L-convergence tower space is an approach limit spaces in the sense of Lowen, [22]. For $\mathsf{L} = ([0, 1], \leq, *)$, we obtain probabilistic convergence spaces in the sense of Richardson and Kent, [25] and if $\mathsf{L} = (\Delta^+ \leq, *)$, then an L-convergence tower space is a probabilistic convergence space in the definition of [16].

We call a space $(X, \overline{q}) \in |\text{L-CTS}|$ a *T1-space* if $x \in q_\top([y])$ implies $x = y$, and we call it a *T2-space* if $x, y \in q_\top(\mathbb{F})$ implies $x = y$. Obviously a T2-space is a T1-space.

The category L-CTS is topological and initial constructions are done as follows. For $(f_j : X \longrightarrow (X_j, \overline{q^j}))_{j \in J}$ we define for $\mathbb{F} \in \mathbb{F}(X)$, $x \in q_\alpha(\mathbb{F}) \iff f_j(x) \in q_\alpha(f_j(\mathbb{F}))$ for all $j \in J$. In particular, in L-CTS, we have *subspaces*, taking the source $\iota_A : A \longrightarrow X$, $x \longmapsto x$ for $x \in A \subseteq X$, and *product spaces*, taking the source of projection mappings $p_j : \prod_{i \in J} X_i \longrightarrow X_j$.

If all $(X_j, \overline{q^j}) \in |\text{L-CTS}|$ are T1-spaces, respectively T2-spaces for all $j \in J$ and if the family $(f_j : X \longrightarrow (X_j, \overline{q^j}))_{j \in J}$ is *point-separating*, i.e. if for $x \neq y$ there is $j \in J$ such that $f_j(x) \neq f_j(y)$, then the initial construction $(X, \overline{q})$ is also a T1-space, respectively a T2-space. This applies in particular for subspaces and product spaces.

Example 3.2 (L-metric spaces). For a quantale $\mathsf{L} = (L, \leq, *)$, an L-*metric space* is a pair (X, d) of a set X and an L-*metric* $d : X \times X \longrightarrow L$ such that

(LM1) $d(x, x) = \top$ for all $x \in X$ (*reflexivity*);

(LM2) $d(x, y) * d(y, z) \leq d(x, z)$ for all $x, y, z \in X$ (*transitivity*).

A mapping between two L-metric spaces, $f : (X, d) \longrightarrow (X', d')$ is called an L-*metric morphism* if $d(x_1, x_2) \leq d'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. We denote the category of L-metric spaces with L-metric morphisms by L-MET. If the L-metric satisfies $d(x, y) = d(y, x)$ for all $x, y \in X$, it is called *symmetric*. If $d(x, y) = \top$ implies $x = y$, it is called *separated*.

Other names for L-metric spaces are *continuity spaces* [7], L-*categories* [13, 21], or L-*preordered sets* [27]. In case $L = \{0, 1\}$, an L-metric space is a preordered set. If $\mathsf{L} = ([0, \infty], \geq, +)$, an L-metric space is a quasimetric space. If $\mathsf{L} = (\Delta^+, \leq, *)$, an L-metric space is a probabilistic quasimetric space, see [7].

Let $(X, d) \in |\text{L-MET}|$. We define $x \in q^d_\alpha(\mathbb{F}) \iff \bigvee_{F \in \mathbb{F}} \bigwedge_{xy \in F} d(x, y) \geq \alpha$. Then $(X, \overline{q^d}) \in |\text{L-CTS}|$ and an L-metric morphism $f : (X, d^X) \longrightarrow (Y, d^Y)$ becomes continuous as a morphism $f : (X, \overline{q^{d^X}}) \longrightarrow (Y, \overline{q^{d^Y}})$, [18]. It can be shown that if (X, d) is symmetric and separated, then $(X, \overline{q^d})$ is a T2-space.

Given $(X, \overline{q}) \in |\text{L-CTS}|$, we define $d^{\overline{q}}(x, y) = \bigvee\limits_{x \in q_\alpha([y])} \alpha$. Then $(X, d^{\overline{q}}) \in |\text{L-MET}|$ and a continuous mapping $f : (X, \overline{q^X}) \longrightarrow (Y, \overline{q^Y})$ becomes an L-metric morphism $f : (X, d^{\overline{q^X}}) \longrightarrow (Y, d^{\overline{q^Y}})$. It was further shown in

[18] that for $(X,d) \in |\mathsf{L\text{-}MET}|$ we have $d^{\overline{q^d}} = d$ and for $(X,\overline{q}) \in |\mathsf{L\text{-}CTS}|$ we have $q_\alpha^{(d^{\overline{q}})}(\mathbb{F}) \subseteq q_\alpha(\mathbb{F})$ for all $\alpha \in L, \mathbb{F} \in \mathsf{F}(X)$. Hence, the functor $F : \mathsf{L\text{-}MET} \longrightarrow \mathsf{L\text{-}CTS}$, $F((X,d)) = (X,\overline{q^d})$, $F(f) = f$, embeds $\mathsf{L\text{-}MET}$ into $\mathsf{L\text{-}CTS}$ as a coreflective subcategory. It is straightforward to show that if $(X,\overline{q})$ is a T2-space, then $(X,d^{\overline{q}})$ is separated. □

4. Diagonal Axioms for L-Convergence Tower Spaces: Topologicalness

We first characterize a "pretopological axiom".

Proposition 4.1. *Let* $(X,\overline{q}) \in |\mathsf{L\text{-}CTS}|$. *The following are equivalent:*
(LCP) $\bigcap_{j\in J} q_\alpha(\mathbb{F}_j) = q_\alpha(\bigwedge_{j\in J} \mathbb{F}_j)$;
(LCP') $x \in q_\alpha(\mathbb{U}_\alpha^x)$, *with the* α-neighbourhood filter at x, $\mathbb{U}_\alpha^x = \bigwedge_{x\in q_\alpha(\mathbb{F})} \mathbb{F}$.

Proof. If (LCP) is true, then $x \in \bigcap_{x\in q_\alpha(\mathbb{F})} q_\alpha(\mathbb{F}) = q_\alpha(\bigwedge_{x\in q_\alpha(\mathbb{F})} \mathbb{F}) = q_\alpha(\mathbb{U}_\alpha^x)$ and we have (LCP'). Conversely, let $x \in \bigcap_{j\in J} q_\alpha(\mathbb{F}_j)$. Then for all $j \in J$ we have $\mathbb{U}_\alpha^x \leq \mathbb{F}_j$ and consequently also $\mathbb{U}_\alpha^x \leq \bigwedge_{j\in J} \mathbb{F}_j$. This implies $x \in q_\alpha(\mathbb{U}_\alpha^x) \subseteq q_\alpha(\bigwedge_{j\in J} \mathbb{F}_j)$ and (LCP) is satisfied. □

Proposition 4.2. ([18]) *Let* $(X,d) \in |\mathsf{L\text{-}MET}|$. *Then* $(X,\overline{q^d})$ *satisfies (LCP).*

Let $(X,\overline{q}) \in |\mathsf{L\text{-}CTS}|$ and $\gamma : L \times L \longrightarrow L$. We say that $(X,\overline{q})$ satisfies the axiom (LF-γ) if

$$\forall J, \psi : J \longrightarrow X, \mathbb{F}_j \in \mathsf{F}(X)\ (j \in J), \mathbb{G} \in \mathsf{F}(J) : x \in q_\alpha(\psi(\mathbb{G})), \psi(j) \in q_\beta(\mathbb{F}_j) \forall j \in J \Longrightarrow x \in q_{\gamma(\alpha,\beta)}(\kappa(\mathbb{G},(\mathbb{F}_j)_{j\in J})).$$

We call $(X,\overline{q}) \in |\mathsf{L\text{-}CTS}|$ *γ-topological* if the axiom (LF-γ) is satisfied. For $\gamma(\alpha,\beta) = \alpha * \beta$ we simply speak of a *topological* L*-convergence tower space*. If $\gamma(\alpha,\beta) = \alpha \wedge \beta$, then (LF-$\gamma$) is equivalent to all "level spaces" (X,q_α) satisfying the so-called Fischer diagonal axiom, [5]. If $\gamma' \leq \gamma$ pointwisely, then γ-topologicalness implies γ'-topologicalness.

Proposition 4.3. *Let* $(X_\lambda,\overline{q^\lambda}) \in |\mathsf{L\text{-}CTS}|$ *satisfy the axiom (LF-γ) for all* $\lambda \in \Lambda$ *and let* $(f_\lambda : X \longrightarrow X_\lambda)_{\lambda\in\Lambda}$ *be a source and let* $(X,\overline{q})$ *be the initial construction. Then* $(X,\overline{q})$ *satisfies (LF-γ).*

Proof. Let J be a set, $\psi : J \longrightarrow X$, $\mathbb{G} \in \mathsf{F}(J)$ and for all $j \in J$ let $\mathbb{F}_j \in \mathsf{F}(X)$. If $x \in q_\alpha(\psi(\mathbb{G}))$ and for all $j \in J$, $\psi(j) \in q_\beta(\mathbb{F}_j)$, then for all $\lambda \in \Lambda$ we have $f_\lambda(x) \in q_\alpha^\lambda(f_\lambda(\psi(\mathbb{G})))$ and $f_\lambda(\psi(j)) \in q_\beta^\lambda(f_\lambda(\mathbb{F}_j))$ for all $j \in J$. We denote $\psi_\lambda = f_\lambda \circ \psi : J \longrightarrow X_\lambda$ for all $\lambda \in \Lambda$. Then $f_\lambda(x) \in q_{\gamma(\alpha,\beta)}^\lambda(\kappa(\mathbb{G},(\psi_\lambda(\mathbb{F}_j))_{j\in J}))$ for all $\lambda \in \Lambda$. It is not difficult to show that $\kappa(\mathbb{G},(\psi_\lambda(\mathbb{F}_j))_{j\in J}) = f_\lambda(\kappa(\mathbb{G};(\mathbb{F}_j)_{j\in J}))$. Hence $f_\lambda(x) \in q_{\gamma(\alpha,\beta)}^\lambda(f_\lambda(\kappa(\mathbb{G};(\mathbb{F}_j)_{j\in J})))$ for all $\lambda \in \Lambda$, i.e. $x \in q_{\gamma(\alpha,\beta)}(\kappa(\mathbb{G};(\mathbb{F}_j)_{j\in J}))$. □

A special case of the axiom (LF-γ) arises if we restrict to $J = X$ and $\psi = id_X$ in (LF-γ). We say that $(X,\overline{q})$ satisfies the axiom (LK-γ) if

$$\forall \mathbb{G}, \mathbb{F}_y \in \mathsf{F}(X), y \in X : x \in q_\alpha(\mathbb{G}), y \in q_\beta(\mathbb{F}_y) \forall y \in X \Longrightarrow x \in q_{\gamma(\alpha,\beta)}(\kappa(\mathbb{G},(\mathbb{F}_y)_{y\in X})).$$

Proposition 4.4. *Let* $(X_\lambda,\overline{q^\lambda}) \in |\mathsf{L\text{-}CTS}|$ *satisfy the axiom (LK-γ) and let* $f_\lambda : X \longrightarrow X_\lambda$ *be injective for all* $\lambda \in \Lambda$. *Then the initial construction* $(X,\overline{q})$ *satisfies (LK-γ).*

Proof. This proof is essentially from [25]. Let $\mathbb{G}, \mathbb{F}_y \in \mathsf{F}(X)$ for all $y \in X$ and let $x \in q_\alpha(\mathbb{G})$ and $y \in q_\beta(\mathbb{F}_y)$ for all $y \in Y$. For $\lambda \in \Lambda$ and $x_\lambda \in X_\lambda$ we define $\mathbb{H}_{x_\lambda} = f_\lambda(\mathbb{F}_y)$ if $f_\lambda(y) = x_\lambda$ and $\mathbb{H}_{x_\lambda} = [x_\lambda]$ if $x_\lambda \notin f_\lambda(X)$. We note that y is uniqueliy determined by the requirement $f_\lambda(y) = x_\lambda$ as the mappings f_λ are injections. We then have $f_\lambda(x) \in q_\alpha^\lambda(f_\lambda(\mathbb{G}))$ and $x_\lambda \in q_\beta^\lambda(\mathbb{H}_{x_\lambda})$ for all $\lambda \in \Lambda$ and hence, by (LK-γ) for $(X_\lambda,\overline{q^\lambda})$ we conclude $f_\lambda(x) \in q_{\gamma(\alpha,\beta)}^\lambda(\kappa(f_\lambda(\mathbb{G}),(\mathbb{H}_{x_\lambda})_{x_\lambda\in X_\lambda}))$ for all $\lambda \in \Lambda$. It is not difficult to show that $\kappa(f_\lambda(\mathbb{G}),(\mathbb{H}_{x_\lambda})_{x_\lambda\in X_\lambda}) \leq f_\lambda(\kappa(\mathbb{G},(\mathbb{F}_y)_{y\in X}))$ and hence we have $f_\lambda(x) \in q_{\gamma(\alpha,\beta)}^\lambda(f_\lambda(\kappa(\mathbb{G},(\mathbb{F}_y)_{y\in X}))$ for all $\lambda \in \Lambda$ from which $x \in q_{\gamma(\alpha,\beta)}(\kappa(\mathbb{G},(\mathbb{F}_y)_{y\in X}))$ follows. □

In particular, the axiom (LK-γ) is preserved by the formation of subspaces.

Proposition 4.5. *Let $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$ satisfy the axiom (LCP). Then (LK-γ) is equivalent to $\mathbb{U}^x_{\gamma(\alpha,\beta)} \le \kappa(\mathbb{U}^x_\alpha, (\mathbb{U}^y_\beta)_{y\in X})$.*

Proof. Let (LK-γ) be satisfied. By (LCP) we have $x \in q_\alpha(\mathbb{U}^x_\alpha)$ and $y \in q_\beta(\mathbb{U}^y_\beta)$ for all $y \in X$. Then $x \in q_{\gamma(\alpha,\beta)}(\kappa(\mathbb{U}^x_\alpha, (\mathbb{U}^y_\beta)_{y\in X}))$ which shows $\mathbb{U}^x_{\gamma(\alpha,\beta)} \le \kappa(\mathbb{U}^x_\alpha, (\mathbb{U}^y_\beta)_{y\in X})$.

For the converse, let $x \in q_\alpha(\mathbb{G})$ and for all $y \in X$, let $y \in q_\beta(\mathbb{F}_y)$. Then $\mathbb{U}^x_\alpha \le \mathbb{G}$ and $\mathbb{U}^y_\beta \le \mathbb{F}_y$ for all $y \in X$. Hence, $\mathbb{U}^x_{\gamma(\alpha,\beta)} \le \kappa(\mathbb{U}^x_\alpha, (\mathbb{U}^y_\beta)_{y\in X}) \le \kappa(\mathbb{G}, (\mathbb{F}_y)_{y\in X})$ which yields with (LCP) that $x \in q_{\gamma(\alpha,\beta)}(\kappa(\mathbb{G}, (\mathbb{F}_y)_{y\in X}))$. $\square$

Proposition 4.6. *Let $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$ and let $\gamma(\top, \alpha) = \alpha$ for all $\alpha \in L$. Then (LF-γ) is equivalent to (LK-γ) and (LCP).*

Proof. Clearly (LF-γ) implies (LK-γ). We need to show that it also implies (LCP). Let J be a set and consider $\mathbb{G} = [J]$, $\mathbb{F}_j \in \mathsf{F}(X)$ for $j \in J$. We define $\psi(j) = x$ for all $j \in J$. Then $\psi(\mathbb{G}) = [x]$ and $\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}) = \bigwedge_{j\in J} \mathbb{F}_j$. If $x \in q_\alpha(\mathbb{F}_j)$ for all $j \in J$, then $\psi(j) \in q_\alpha(\mathbb{F}_j)$ for all $j \in J$. Also $x \in q_\top([x]) = q_\top(\psi(\mathbb{G}))$. The axiom (LF-$\gamma$) implies $x \in q_{\gamma(\top,\alpha)}(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})) = q_\alpha(\bigwedge_{j\in J} \mathbb{F}_j)$.

Conversely, let (LCP) and (LK-γ) be satisfied. Consider $\psi : J \longrightarrow X$, $\mathbb{G} \in \mathsf{F}(J)$ and $\mathbb{F}_j \in \mathsf{F}(X)$ for all $j \in J$. If $x \in q_\alpha(\psi(\mathbb{G}))$ and $\psi(j) \in q_\beta(\mathbb{F}_j)$ for all $j \in J$, then $\mathbb{U}^x_\alpha \le \psi(\mathbb{G})$ and $\mathbb{U}^{\psi(j)}_\beta \le \mathbb{F}_j$ for all $j \in J$. Hence, by Proposition 4.2, $\mathbb{U}^x_{\gamma(\alpha,\beta)} \le \kappa(\mathbb{U}^x_\alpha, (\mathbb{U}^{\psi(j)}_\beta)_{j\in J}) \le \kappa(\psi(\mathbb{G}), (\mathbb{F}_j)_{j\in J})$. This means, by (LC2) that $x \in q_{\gamma(\alpha,\beta)}(\kappa(\psi(\mathbb{G}), (\mathbb{F}_j)_{j\in J}))$ and (LF-γ) is true. $\square$

Theorem 4.7. *Let $(X, d) \in |\mathsf{L\text{-}MET}|$. Then $(X, \overline{q^d})$ satisfies the axiom (LF-γ) for $\gamma(\alpha, \beta) = \alpha * \beta$.*

Proof. We need only show the topological axiom (LK-γ). Let $\bigvee_{G\in\mathbb{G}} \bigwedge_{y\in G} d(x, y) \ge \beta$ and for each $y \in X$, let $\bigvee_{F^y\in\mathbb{F}_y} \bigwedge_{z\in F^y} d(y, z) \ge \alpha$. Consider $\epsilon \lhd \beta$ and $\delta \lhd \alpha$. Then there is $G \in \mathbb{G}$ such that for all $y \in G$ we have $d(x, y) \ge \epsilon$ and for each $y \in X$ there is $F^y \in \mathbb{F}_y$ such that for all $z \in F^y$ we have $d(y, z) \ge \delta$. The set $H^G = \bigcup_{y\in G} F^y \in \bigwedge_{y\in G} \mathbb{F}_y \le \kappa(\mathbb{G}, (\mathbb{F}_y)_{y\in Y})$ and for $z \in H^G$ then $z \in F^y$ for some $y \in G$. Hence $d(x, z) \ge d(x, y) * d(y, z) \ge \epsilon * \delta$. We conclude $\bigwedge_{z\in H^G} d(x, z) \ge \epsilon * \delta$ and from this we obtain $\bigvee_{K\in\kappa(\mathbb{G},(\mathbb{F}_y)_{y\in Y})} \bigwedge_{u\in K} d(x, u) \ge \bigwedge_{z\in H^G} d(x, z) \ge \epsilon * \delta$. This is true for all $\epsilon \lhd \beta$ and all $\delta \lhd \alpha$ and L being a quantale and $(L, \le)$ being completely distributive, the claim follows. $\square$

We can generalize Proposition 4.5 to a neighbourhood condition à la Gähler [10]. In the case of Lawvere's quantale and convergence approach spaces such a condition was established in [14]. We denote, for $\mathbb{F} \in \mathsf{F}(X)$, the *$\alpha$-neighbourhood filter of* $\mathbb{F}$ by $\mathbb{U}_\alpha(\mathbb{F}) = \kappa(\mathbb{F}, (\mathbb{U}^y_\alpha)_{y\in X})$. Then $\mathbb{U}_\alpha(\mathbb{F}) \le \mathbb{F}$ and $\mathbb{U}_\alpha([x]) = \mathbb{U}^x_\alpha$.

Proposition 4.8. *Let $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$. Then the following are equivalent:*
(i) $(X, \overline{q})$ satisfies (LF-γ);
(ii) $(X, \overline{q})$ satisfies (LG-γ): $q_\beta(\mathbb{F}) \subseteq q_{\gamma(\alpha,\beta)}(\mathbb{U}_\alpha(\mathbb{F}))$ for all $\alpha, \beta \in L$.

Proof. Let first (LF-γ) be satisfied and let $x \in q_\beta(\mathbb{F})$. We define $J = \{(x, \mathbb{G}) : x \in q_\alpha(\mathbb{F})\}$ and the mapping $\psi : J \longrightarrow X$ by $\psi((x, \mathbb{G})) = x$. Furthermore, for $(x, \mathbb{G}) \in J$, we define $\mathbb{F}_{(x,\mathbb{G})} = \mathbb{G}$. From (LC1) we see that $(x, [x]) \in J$ and hence the mapping ψ is a surjection. We define $\mathbb{K} = \psi^{-1}(\mathbb{F}) \in \mathsf{F}(J)$. Then $\psi(\mathbb{K}) = \mathbb{F}$ and $\psi((x, \mathbb{G})) = x \in q_\alpha(\mathbb{G}) = q_\alpha(\mathbb{F}_{(x,\mathbb{G})})$ for all $(x, \mathbb{G}) \in J$. From the axiom (LF-γ) we obtain $x \in q_{\gamma(\alpha,\beta)}(\kappa(\mathbb{K}, (\mathbb{F}_j)_{j\in J}))$. As $\kappa(\mathbb{K}, (\mathbb{F}_j)_{j\in J}) = \bigvee_{H\in\psi^{-1}(\mathbb{F})} \bigwedge_{(x,\mathbb{G})\in H} \mathbb{F}_{(x,\mathbb{G})} \le \bigvee_{F\in\mathbb{F}} \bigwedge_{(x,\mathbb{G})\in\psi^{-1}(F)} \mathbb{G} = \bigvee_{F\in\mathbb{F}} \bigwedge_{x\in F} \bigwedge_{\mathbb{G}:(x,\mathbb{G})\in J} \mathbb{G} = \bigvee_{F\in\mathbb{F}} \bigwedge_{x\in F} \mathbb{U}^x_\alpha = \mathbb{U}_\alpha(\mathbb{F})$ we see that $x \in q_{\gamma(\alpha,\beta)}(\mathbb{U}_\alpha(\mathbb{F}))$ and (LG-γ) is true.

Let now (LG-γ) be satisfied and let $\psi : J \longrightarrow X$ be a mapping, $\mathbb{G} \in \mathsf{F}(J)$ with $x \in q_\beta(\psi(\mathbb{G}))$ and $\psi(j) \in q_\alpha(\mathbb{F}_j)$ for all $j \in J$. Then $\mathbb{F}_j \ge \mathbb{U}^{\psi(j)}_\alpha$ for all $j \in J$ and from (LG-γ) we infere $x \in q_{\gamma(\alpha,\beta)}(\mathbb{U}_\alpha(\psi(\mathbb{G})))$. Now we note that $\mathbb{U}_\alpha(\psi(\mathbb{G})) = \bigvee_{H\in\psi(\mathbb{G})} \bigwedge_{z\in H} \mathbb{U}^z_\alpha = \bigvee_{G\in\mathbb{G}} \bigwedge_{z\in\psi(G)} \mathbb{U}^z_\alpha = \bigvee_{G\in\mathbb{G}} \bigwedge_{j\in G} \mathbb{U}^{\psi(j)}_\alpha \le \bigvee_{G\in\mathbb{G}} \bigwedge_{j\in G} \mathbb{F}_j = \kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})$. Hence, by (LC2), we obtain $x \in q_{\gamma(\alpha,\beta)}(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}))$ and (LF-γ) is true. $\square$

We are finally going to characterize the topological axiom (LK-γ) by interior and closure operators. We define for $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$, $\alpha \in L$, the *α-interior of $A \subseteq X$*, $\underline{A}_\alpha \subseteq X$, by

$$x \in \underline{A}_\alpha \iff A \in \mathbb{U}^x_\alpha.$$

For the principal filter of $\emptyset \neq A \subseteq X$, $[A] = \{F \subseteq X \ : \ A \subseteq F\}$, we have $B \in \mathbb{U}_\beta([A])$ if and only if $A \subseteq \underline{B}_\beta$. The *$\alpha$-closure of A* is defined as $\overline{A}^\alpha = (\underline{A^c}_\alpha)^c$, with the complement $A^c = \{x \in X \ : \ x \notin A\}$. The usual characterization of the α-closure can be shown, i.e. for $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$, $A \subseteq X$ and $\alpha \in L$ we have $x \in \overline{A}^\alpha$ if and only if there is a (ultra-)filter $\mathbb{F} \in \mathsf{F}(X)$ such that $x \in q_\alpha(\mathbb{F})$ and $A \in \mathbb{F}$.

Proposition 4.9. *Let $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$ satisfy the axiom (LCP). Then the following are equivalent:*
(i) $(X, \overline{q})$ satisfies (LK-γ);
(ii) For all $A \subseteq X$, $\underline{A}_{\gamma(\alpha,\beta)} \subseteq \underline{\underline{A}_\beta}_\alpha$;
(iii) For all $A \subseteq X$, $\overline{\overline{A}^\beta}^\alpha \subseteq \overline{A}^{\gamma(\alpha,\beta)}$.

Proof. We show that (i) $\iff$ (ii). The equivalence of (ii) and (iii) follows in the usual way.

Let first the axiom (LK-γ) be satisfied and let $x \in \underline{A}_{\gamma(\alpha,\beta)}$. Then $A \in \mathbb{U}^x_{\gamma(\alpha,\beta)} \leq \mathbb{U}_\beta(\mathbb{U}^x_\alpha) = \bigvee_{U \in \mathbb{U}^x_\alpha} \bigwedge_{y \in U} \mathbb{U}^y_\beta$. Hence there is $U \in \mathbb{U}^x_\alpha$ such that for all $y \in U$ we have $A \in \mathbb{U}^y_\beta$, i.e. $y \in \underline{A}_\beta$. This means $U \subseteq \underline{A}_\beta$ and therefore $\underline{A}_\beta \in \mathbb{U}^x_\alpha$, i.e. $x \in \underline{\underline{A}_\beta}_\alpha$.

Let now $x \in \underline{\mathbb{U}^x_{\gamma(\alpha,\beta)}}_\alpha$. Then $x \in \underline{A}_{\gamma(\alpha,\beta)} \subseteq \underline{\underline{A}_\beta}_\alpha$, i.e. $\underline{A}_\beta \in \mathbb{U}^x_\alpha$. Hence there is $U \in \mathbb{U}^x_\alpha$ such that $U \subseteq \underline{A}_\beta$ and we have for all $y \in U$ that $A \in \mathbb{U}^y_\beta$. We conclude $A \in \bigwedge_{y \in U} \mathbb{U}^y_\beta \leq \kappa(\mathbb{U}^x_\alpha, (\mathbb{U}^y_\beta)_{y \in X})$ and (LK-γ) is true. □

In a slightly different lattice context, an early example of a closure operator satisfying property (iii) of Proposition 4.9 is given in [9].

Remark 4.10 (α-interior and α-closure in $\mathsf{L\text{-}MET}$). For an L-metric space (X, d), we define for $\epsilon \in L$, the *ϵ-ball at $x \in X$* by $B^d(x, \epsilon) = \{y \in X \ : \ d(x, y) \succ \epsilon\}$. Because $\bot \prec \alpha$ and by the interpolation property $\bot \prec \epsilon \prec \alpha$ for some $\epsilon \in L$, the set $\mathbb{B} = \{B^d(x, \epsilon) \ : \ \epsilon \prec \alpha\}$ is not empty. As $\epsilon_1, \epsilon_2 \prec \alpha$ implies $\epsilon_1 \vee \epsilon_2 \prec \alpha$ and $B^d(x, \epsilon_1 \vee \epsilon_2) \subseteq B^d(x, \epsilon_1) \cap B^d(x, \epsilon_2)$, the set $\mathbb{B}$ is a filter basis. We denote the generated filter by $\mathbb{U}^{d,x}_\alpha$. It is shown in [17] that $\mathbb{U}^{d,x}_\alpha = \mathbb{U}^{\overline{q^d},x}_\alpha$ with the α-neighbourhood filter in $(X, \overline{q^d})$. For $A \subseteq X$ then we have $x \in \underline{A}_\alpha$ if and only if $A \in \mathbb{U}^{d,x}_\alpha$ if and only if there is $\epsilon \prec \alpha$ such that $B^d(x, \epsilon) \subseteq A$.

In an L-metric space (X, d), we define the *α-d-closure of $A \subseteq X$* by $x \in \overline{A}^{d,\alpha}$ iff $d(x, A) = \bigvee_{a \in A} d(x, a) \geq \alpha$.

We show that $\overline{A}^\alpha \subseteq \overline{A}^{d,\alpha}$ with the α-closure $\overline{A}^\alpha$ in $(X, \overline{q^d})$. Let $x \in \overline{A}^\alpha$. Then there is $\mathbb{F} \in \mathsf{F}(X)$ with $A \in \mathbb{F}$ and $\bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d(x, y) \geq \alpha$. For $\epsilon \lhd \alpha$, there is $F_\epsilon \in \mathbb{F}$ such that for all $y \in F_\epsilon$ we have $d(x, y) \geq \epsilon$. The set $A \cap F_\epsilon \in \mathbb{F}$ and we have $\bigvee_{a \in A} d(x, a) \geq \bigvee_{y \in F_\epsilon \cap A} d(x, y) \geq \epsilon$. As $\epsilon \lhd \alpha$ was arbitrary, we obtain $\bigvee_{a \in A} d(x, a) \geq \alpha$ and $x \in \overline{A}^{d,\alpha}$.

It was shown in [17] that for a value quantale L, i.e. for a quantale for which $\alpha, \beta \lhd \top$ implies $\alpha \vee \beta \lhd \top$, we have $\overline{A}^{d,\top} = \overline{A}^\top$. If in $(L, \leq)$ the way-below and the well-below relations coincide, then we even have $\overline{A}^{d,\alpha} = \overline{A}^\alpha$ for all $\alpha \in L$. To see this, let $x \in \overline{A}^{d,\alpha}$, i.e. $\bigvee_{a \in A} d(x, a) \geq \alpha$. For $\epsilon \lhd \alpha$ then there is $y \in A$ such that $d(x, y) \rhd \epsilon$ and hence, for all $\epsilon \lhd \alpha$ there is $y \in B^d(x, \epsilon)$ with $y \notin A^c$, i.e. $x \notin \underline{A^c}_\alpha$ which means $x \in \overline{A}^\alpha$.

We note that in general the way-below and the well-below relations are different, see e.g. [17]. In the general case we have $\overline{A}^{d,\alpha} = \bigcap_{\epsilon \lhd \alpha} \overline{A}^\epsilon$. To see this, let $\bigvee_{a \in A} d(x, a) \geq \alpha \rhd \epsilon$. Then there is $a_\epsilon \in A$ such that $d(x, a_\epsilon) \rhd \epsilon$ and hence also $d(x, a_\epsilon) \succ \epsilon$, i.e. $a \in B^d(x, \epsilon)$. Let $U \in \mathbb{U}^x_\epsilon$. Then there is $\epsilon' \prec \epsilon$ such that $B^d(x, \epsilon') \subseteq U$. Also, $d(x, a) \succ \epsilon'$, i.e. $a \in B^d(x, \epsilon')$ and hence $A \cap U \neq \emptyset$. This there is $\mathbb{F}_\epsilon \geq \mathbb{U}^x_\epsilon$ with $A \in \mathbb{F}_\epsilon$ which implies $x \in \overline{A}^\epsilon$. Hence $\overline{A}^{d,\alpha} \subseteq \bigcap_{\epsilon \lhd \alpha} \overline{A}^\epsilon$. The other inclusion is clear. □

5. Dual Diagonal Axioms for L-Convergence Tower Spaces: Regularity

Let $\gamma : L \times L \longrightarrow L$ be a mapping. We say that a space $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$ is *γ-regular* if it satisfies the following "dual diagonal axiom" (LDF-γ)

$$\forall J, \psi : J \longrightarrow X, \mathbb{F}_j \in \mathsf{F}(X)\ (j \in J), \mathbb{G} \in \mathsf{F}(J) : x \in q_\alpha(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})), \psi(j) \in q_\beta(\mathbb{F}_j) \forall j \in J \Longrightarrow x \in q_{\gamma(\alpha,\beta)}(\psi(\mathbb{G})).$$

For $\gamma(\alpha,\beta) = \alpha * \beta$, we call a γ-regular space *regular*. For $\gamma(\alpha,\beta) = \alpha \wedge \beta$, γ-regularity is equivalent to all "level spaces" (X, q_α) being regular convergence spaces [5]. If $\gamma' \le \gamma$ pointwisely, then γ-regularity implies γ'-regularity.

Proposition 5.1. *Let $(X_\lambda, \overline{q^\lambda}) \in |\mathsf{L\text{-}CTS}|$ be γ-regular for all $\lambda \in \Lambda$ and let $(f_\lambda : X \longrightarrow X_\lambda)_{\lambda\in\Lambda}$ be a source and let $(X, \overline{q})$ be the initial construction. Then $(X, \overline{q})$ is γ-regular.*

Proof. Let J be a set and let $\psi : J \longrightarrow X$ be a mapping, $\mathbb{G} \in \mathsf{F}(J)$ and for all $j \in J$ let $\mathbb{F}_j \in \mathsf{F}(X)$ such that $\psi(j) \in q_\beta(\mathbb{F}_j)$ and $x \in q_\alpha(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}))$. We define $\psi_\lambda = f_\lambda \circ \psi$ and $\mathbb{F}^\lambda_j = f_\lambda(\mathbb{F}_j)$. Then, noting $f_\lambda(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})) = \kappa(\mathbb{G}, (\mathbb{F}^\lambda_j)_{j\in J})$ we have for all $j \in J$ that $\psi_\lambda(j) \in q^\lambda_\beta(\mathbb{F}^\lambda_j)$ and $f_\lambda(x) \in q_\alpha(\kappa(\mathbb{G}, (\mathbb{F}^\lambda_j)_{j\in J}))$. The axiom (LDF-$\gamma$) for $(X_\lambda, \overline{q^\lambda})$ thus implies $f_\lambda(x) \in q^\lambda_{\gamma(\alpha,\beta)}(\psi_\lambda(\mathbb{G})) = q^\lambda_{\gamma(\alpha,\beta)}(f_\lambda(\psi(\mathbb{G})))$ for all $\lambda \in \Lambda$ and we get $x \in q_{\gamma(\alpha,\beta)}(\psi(\mathbb{G}))$. □

Proposition 5.2. *Let $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$ be a T1-space and let $\gamma : L \times L \longrightarrow L$ satisfy $\gamma(\top, \top) = \top$. If $(X, \overline{q})$ is γ-regular, then it is a T2-space.*

Proof. Let $x, y \in q_\top(\mathbb{F})$. We define $J = \{\mathbb{G} \in \mathsf{F}(X) : y \in q_\top(\mathbb{G})\}$ and for $\mathbb{G} \in J$ we define $\mathbb{F}_\mathbb{G} = \mathbb{G}$. Furthermore, we define the mapping $\psi : J \longrightarrow X$ by $\psi(\mathbb{G}) = y$. Then $[\mathbb{F}] = \{K \subseteq J : \mathbb{F} \in K\} \in \mathsf{F}(J)$ and we have $\psi([\mathbb{F}]) = [y]$ and $\kappa([\mathbb{F}], (\mathbb{F}_\mathbb{G})_{\mathbb{G}\in J}) = \mathbb{F}$. Hence we have $x \in q_\top(\kappa([\mathbb{F}], (\mathbb{F}_\mathbb{G})_{\mathbb{G}\in J}))$ and $\psi(\mathbb{G}) \in q_\top(\mathbb{F}_\mathbb{G})$ for all $\mathbb{G} \in J$. The axiom (LDF-γ) thus implies $x \in q_{\gamma(\top,\top)}(\psi([\mathbb{F}])) = q_\top([y])$ and $(X, \overline{q})$ being a T1-space, this implies $x = y$. □

Theorem 5.3. *Let $(X, d) \in |\mathsf{L\text{-}MET}|$. Then $(X, \overline{q^d})$ satisfies the axiom (LDF-γ) for $\gamma(\alpha,\beta) = \alpha * \beta$.*

Proof. Let $\psi : J \longrightarrow X$, $\psi(j) \in q^d_\beta(\mathbb{F}_j)$ for all $j \in J$ and $x \in q^d_\alpha(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}))$. Further, let $\alpha' \lhd \alpha$ and $\beta' \lhd \beta$. Then for all $j \in J$ there is $F_j \in \mathbb{F}_j$ such that for all $z \in F_j$ we have $d(\psi(j), z) \ge \beta'$ and there is $H \in \kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})$ such that for all $y \in H$ we have $d(x, y) \ge \alpha'$. Hence there is $G \in \mathbb{G}$ such that for all $j \in G$ we have $H \in \mathbb{F}_j$ and for all $y \in H$ we have $d(x, y) \ge \alpha'$. For all $j \in J$ the set $H_j = H \cap F_j \in \mathbb{F}_j$ and for all $u \in H_j$ we have $d(\psi(j), u) \ge \beta'$ and $d(x, u) \ge \alpha'$. From the symmetry of (X, d) and (LM2) we conclude for all $j \in g$, $d(x, \psi(j)) \ge d(x, u) * d(u, \psi(j)) \ge \alpha' * \beta'$ and hence $\bigwedge_{j\in G} d(x, \psi(j)) \ge \alpha' * \beta'$ from which we conclude $\bigvee_{H\in\psi(\mathbb{G})} \bigwedge_{y\in H} d(x, y) \ge \alpha' * \beta'$. Using the complete distributivity we obtain $x \in q^d_{\alpha*\beta}(\psi(\mathbb{G}))$. □

We can characterize γ-regularity by closures of filters. We define, for $\mathbb{F} \in \mathsf{F}(X)$ and $\alpha \in L$, the *α-closure* $\overline{\mathbb{F}}^\alpha$ as the filter on X generated by the filter basis $\{\overline{F}^\alpha : F \in \mathbb{F}\}$.

Proposition 5.4. *Let $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$. The following are equivalent:*
(i) $(X, \overline{q})$ satisfies (LDF-γ);
(ii) $(X, \overline{q})$ satisfies (LR-γ): $q_\alpha(\mathbb{F}) \subseteq q_{\gamma(\alpha,\beta)}(\overline{\mathbb{F}}^\beta)$ for all $\alpha, \beta \in L$, $\mathbb{F} \in \mathsf{F}(X)$.

Proof. Let $(X, \overline{q})$ satisfy (LDF-γ) and let $x \in q_\alpha(\mathbb{F})$. For $\beta \in L$ we define $J = \{(y, \mathbb{G}) : y \in q_\beta(\mathbb{G})\}$, the mapping $\psi : J \longrightarrow X$ by $\psi((y, \mathbb{G})) = y$ and $\mathbb{F}_{(y,\mathbb{G})} = \mathbb{G}$. We further denote by $\mathbb{S}$ the filter on J generated by the filter basis $\{S_F : F \in \mathbb{F}\}$ with $S_F = \{(y, \mathbb{G}) \in J : F \in \mathbb{G}\}$. Then for $F \in \mathbb{F}$ we have $\psi(S_F) = \{\psi((y, \mathbb{G})) : F \in G, (y, \mathbb{G}) \in J\} = \{y \in X : y \in q_\beta(\mathbb{G}), F \in \mathbb{G}\} = \overline{F}^\beta$ and hence $\psi(\mathbb{S}) = \overline{\mathbb{F}}^\beta$. Furthermore, we have for $F \in \mathbb{F}$, $\bigwedge_{F\in\mathbb{G}} \mathbb{G} = \bigwedge_{(y,\mathbb{G})\in S_F} \mathbb{F}_{(y,\mathbb{G})} \le \kappa(\mathbb{S}, (\mathbb{F}_{(y,\mathbb{G})})_{(y,\mathbb{G})\in J})$ and hence $x \in q_\alpha(\mathbb{F})$ implies $x \in q_\alpha(\kappa(\mathbb{S}, (\mathbb{F}_{(y,\mathbb{G})})_{(y,\mathbb{G})\in J}))$. From $\psi((y, \mathbb{G})) = y \in q_\beta(\mathbb{G}) = q_\beta(\mathbb{F}_{(y,\mathbb{G})})$ for all $(y, \mathbb{G}) \in J$ we conclude with (LDF-γ) that $x \in q_{\gamma(\alpha,\beta)}(\psi(\mathbb{S})) = q_{\gamma(\alpha,\beta)}(\overline{\mathbb{F}}^\beta)$ and the condition (ii) is true.

Let now the condition (ii) be satisfied and let $x \in q_\alpha(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}))$ and $\psi(j) \in q_\beta(\mathbb{F}_j)$ for all $j \in J$. We first show that $\overline{\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})}^\beta \le \psi(\mathbb{G})$. Let $H \in \kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})$. Then there is $K \in \kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})$ such that $\overline{K}^\beta \subseteq H$ and hence there is $G \in \mathbb{G}$ such that for all $j \in G$ we have $K \in \mathbb{F}_j$. As $\psi(j) \in q_\beta(\mathbb{F}_j)$ we conclude $\psi(j) \in \overline{K}^\beta$ for all $j \in G$, i.e. $\psi(\mathbb{G}) \subseteq \overline{K}^\beta$. Therefore, $\overline{K}^\beta \in \psi(\mathbb{G})$ and hence $H \in \psi(\mathbb{G})$. The condition (ii) thus implies $x \in q_{\gamma(\alpha,\beta)}(\overline{\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})}^\beta) \subseteq q_{\gamma(\alpha,\beta)}(\psi(\mathbb{G}))$ and (LDF-γ) is satisfied. □

Proposition 5.5. *Let $(X, \overline{q}) \in |\mathsf{L\text{-}CTS}|$ satisfy the axiom (LCP). Then (LDF-γ) is equivalent to $\mathbb{U}^x_{\gamma(\alpha,\beta)} \le \overline{\mathbb{U}^x_\alpha}^\beta$.*

Proof. Let the axiom (LDF-γ) be satisfied. From (LCP) we know that $x \in q_\alpha(\mathbb{U}^x_\alpha)$ and hence $x \in q_{\gamma(\alpha,\beta)}(\overline{\mathbb{U}^x_\alpha}^\beta)$, which implies $\mathbb{U}^x_{\gamma(\alpha,\beta)} \le \overline{\mathbb{U}^x_\alpha}^\beta$. For the converse implication, let $x \in q_\alpha(\mathbb{F})$. Then $\mathbb{F} \ge \mathbb{U}^x_\alpha$ and hence $\overline{\mathbb{F}}^\beta \ge \overline{\mathbb{U}^x_\alpha}^\beta \ge \mathbb{U}^x_{\gamma(\alpha,\beta)}$. By (LCP) then $x \in q_{\gamma(\alpha,\beta)}(\overline{\mathbb{F}}^\beta)$. □

Remark 5.6 (Regularity in L-MET). From Proposition 5.5 we immediately conclude that for $(X, d) \in |\mathsf{L\text{-}MET}|$ the space $(X, \overline{q^d})$ is γ-regular if and only if for all $\epsilon \succ \gamma(\alpha, \beta)$ there is $\delta \succ \alpha$ such that $\overline{B^d(x, \delta)}^\beta \subseteq B^d(x, \epsilon)$. We note here that the occuring β-closure is taken in $(X, \overline{q^d})$. □

6. An Extension Theorem

Let $(X, \overline{q^X})$ and $(Y, \overline{q^Y})$ be L-convergence tower spaces and let $A \subseteq X$. The subspace $(A, \overline{q^X|_A})$ is defined as initial construction for the embedding $\iota_A : A \longrightarrow X$, $\iota(a) = a$ for $a \in A$, i.e. we have $x \in (q^X|A)_\alpha(\mathbb{F})$ iff $x \in q^X_\alpha(\iota_A(\mathbb{F}))$ for $\mathbb{F} \in \mathsf{F}(A)$. For simplicity, we write $[\mathbb{F}] = \iota_A(\mathbb{F})$ for the filter on X with filterbasis $\mathbb{F}$. We consider the following *extension problem*: if $f : ((A, \overline{q^X|_A}) \longrightarrow (Y, \overline{q^Y})$ is continuous, find conditions such that f can be extended to a continuous mapping $g : (X, \overline{q^X}) \longrightarrow (Y, \overline{q^Y})$ such that $g \circ \iota_A = f$. This problem was treated for the case of convergence approach spaces, i.e. left-continuous spaces for Lawvere's quantale, in [15], where a classical result of Cook [4] for convergence spaces was generalized. A notable improvement of the results in [15] was obtained in [3], where the extension problem was related to function spaces. We adapt the theory developped in [15], as it applies our diagonal axioms.

We first introduce the following notation. For $x \in X$, $A \subseteq X$ and $\alpha \in L$ we denote

$$H^\alpha_A(x) = \{\mathbb{F} \in \mathsf{F}(X) \ : \ \mathbb{F}_A \in \mathsf{F}(A), x \in q^X_\alpha(\mathbb{F})\}$$

$$F^\alpha_A(x) = \begin{cases} \{y \in Y \ : \ y \in q^Y_\alpha(f(\mathbb{F}_A)) \forall \mathbb{F} \in H^\alpha_A(x)\} & \text{if} \quad H^\alpha_A(x) \neq \emptyset \\ Y & \text{if} \quad H^\alpha_A(x) = \emptyset \end{cases}$$

We note that $H^\beta_A(x) \subseteq H^\alpha_A(x)$ whenever $\alpha \le \beta$ amd that $x \in \overline{A}^\alpha$ if and only if $H^\alpha_A(x) \neq \emptyset$. If we call $A \subseteq X$ *dense in* $(X, \overline{q^X})$ if $\overline{A}^\top = X$, then for a dense subset $A \subseteq X$ all $H^\alpha_A(x)$ are non-empty.

Lemma 6.1. *Let $(X, \overline{q^X}), (Y, \overline{q^Y}) \in |\mathsf{L\text{-}CTS}|$, $A \subseteq X$ and let $f : (A, \overline{q^X|_A}) \longrightarrow (Y, \overline{q^Y})$ be continuous. Then $A \subseteq \{x \in \overline{A}^\top \ : \ \bigcap_{\alpha\in L} F^\alpha_A(x) \neq \emptyset\}$.*

Proof. Let $x \in A$. From (LC1) we immediately conclude that $A \subseteq \overline{A}^\top$. Hence $H^\alpha_A(x) \neq \emptyset$ for all $\alpha \in L$. For $\alpha \in L$ and $\mathbb{F} \in H^\alpha_A(x)$ we have $[\mathbb{F}_A] \ge \mathbb{F}$ and hence $x \in (q^X|_A)_\alpha(\mathbb{F}_A)$. As f is continuous, we conclude $f(x) \in q^Y_\alpha(f(\mathbb{F}_A))$, i.e. we have $f(x) \in F^\alpha_A(x)$. This shows $\bigcap_{\alpha\in L} F^\alpha_A(x) \neq \emptyset$. □

In the sequel we will need to demand the following property from a mapping $\gamma : L \times L \longrightarrow L$:

$$\bigvee_{\beta \lhd \top} \gamma(\gamma(\alpha, \beta), \beta) \ge \alpha \quad \text{for all } \alpha, \beta \in L.$$

A simple example for such a mapping is $\gamma(\alpha,\beta) = \alpha \wedge \beta$. A further example is $\gamma(\alpha,\beta) = \alpha * \beta$ for a value quantale $\mathsf{L} = (L, \leq, *)$. Then we have $\bigvee_{\beta \lhd \top} \beta * \beta = \top$: For $\delta \lhd \top$ there is $\beta_\delta \lhd \top$ such that $\delta \lhd \beta_\delta * \beta_\delta$, see [7]. Hence $\top = \bigvee_{\delta \lhd \top} \delta \leq \bigvee_{\delta \lhd \top} \beta_\delta * \beta_\delta \leq \bigvee_{\beta \lhd \top} \beta * \beta$ and we conclude $\bigvee_{\beta \lhd \top} (\alpha * \beta) * \beta = \alpha * \bigvee_{\beta \lhd \top} \beta * \beta = \alpha * \top = \alpha$.

For $L = [0,1]$ with the usual order, also the arithmetic mean $\gamma(\alpha,\beta) = \frac{\alpha+\beta}{2}$ satisfies this property. In fact we have

$$\sup_{\beta<1} \frac{\frac{\alpha+\beta}{2} + \beta}{2} = \frac{\alpha}{4} + \frac{3}{4} \geq \frac{\alpha}{4} + \frac{3\alpha}{4} = \alpha.$$

Our final example uses the implication operation $\alpha \to \beta = \bigvee\{\gamma \in L \ : \ \alpha * \gamma \leq \beta\}$ which is available in a quantale. Then $\delta \leq \alpha \to \beta$ iff $\alpha * \delta \leq \beta$. From this it immediately follows that $\alpha \leq (\alpha \to \beta) \to \beta$ and hence the mapping $\gamma(\alpha,\beta) = \alpha \to \beta$ satisfies the desired property.

Theorem 6.2. *Let $\gamma : L \times L \longrightarrow L$ satisfy $\bigvee_{\beta \lhd \top} \gamma(\gamma(\alpha,\beta),\beta) \geq \alpha$ for all $\alpha,\beta \in L$..*

Let $(X, \overline{q^X}), (Y, \overline{q^Y}) \in |\mathsf{L\text{-}CTS}|$ and let $(X, \overline{q^X})$ satisfy (LK-γ) and $(Y, \overline{q^Y})$ be left-continuous and satisfy (LDF-γ). Let further $A \subseteq X$, $f : (A, \overline{q^X|_A}) \longrightarrow (Y, \overline{q^Y})$ be continuous and denote $X_0 = \{x \in \overline{A}^\top \ : \ \bigcap_{\alpha \in L} F_A^\alpha(x) \neq \emptyset\}$. Then there is a continuous mapping $g : (X_0, \overline{q^X|_{X_0}}) \longrightarrow (Y, \overline{q^Y})$ such that $g \circ \iota_A = f$.

Proof. For $x \in X_0 \setminus A$ we choose a fixed $y_x \in \bigcap_{\alpha \in L} F_A^\alpha(x)$ and we define $g(x) = f(x)$ if $x \in A$ and $g(x) = y_x$ for $x \in X_0 \setminus A$. We show that g is continuous. First, we note that the subspace $(X_0, \overline{q^X|_{X_0}})$ satisfies the axiom (LK-γ). Let $x_0 \in (q^X|_{X_0})_\alpha(\mathbb{G})$ and let $\beta \lhd \top$. For each $x \in X_0$ we choose a filter $\mathbb{F}_x \in H_A^\beta(x)$. Then the trace $(\mathbb{F}_x)_A \in \mathsf{F}(A)$ and as $A \subseteq X_0$, the trace $\mathbb{H}_x = (\mathbb{F}_x)_{X_0} \in \mathsf{F}(X_0)$ and we have $x \in q_\beta^X(\mathbb{F}_x) = (q^X|_{X_0})_\beta(\mathbb{H}_x)$ for all $x \in X_0$. The axiom (LK-γ) then implies $x_0 \in (q^X|_{X_0})_{\gamma(\alpha,\beta)}(\kappa(\mathbb{G}, (\mathbb{H}_x)_{x \in X}))$. As every filter $\mathbb{H}_x$ has a trace on A, it is not difficult to prove that also $\kappa(\mathbb{G}, (\mathbb{H}_x)_{x \in X})$ has a trace on A and we conclude $[(\kappa(\mathbb{G}, (\mathbb{H}_x)_{x \in X}))_A] \in H_A^{\gamma(\alpha,\beta)}(x_0)$. Therefore $g(x_0) \in q^Y_{\gamma(\alpha,\beta)}(f([(\kappa(\mathbb{G}, (\mathbb{H}_x)_{x \in X}))_A])) = q^Y_{\gamma(\alpha,\beta)}(f(\kappa(\mathbb{G}, (\mathbb{H}_x)_{x \in X})))$.

We define now for $x \in X_0$ the filter $\mathbb{K} = f((\mathbb{H}_x)_A) \in \mathsf{F}(Y)$. As $[\mathbb{H}_x] \in H_A^\beta(x)$ we conclude $g(x_0) \in q_\beta^Y(f([\mathbb{H}_x]_A)) = q_\beta^Y(\mathbb{K}_x)$. Now we note that $\kappa(\mathbb{G}, (\mathbb{K}_x)_{x \in X_0}) = f(\kappa(\mathbb{G}, (\mathbb{H}_x)_{x \in X_0}))$ and with $J = X_0$, $\psi = g$ and $(Y, \overline{q^Y})$ being γ-regular we conclude $g(x_0) \in q^Y_{\gamma(\gamma(\alpha,\beta),\beta)}(g(\mathbb{G}))$. This is true for all $\beta \lhd \top$ and noting that $\bigvee_{\beta \lhd \top} \gamma(\alpha, \gamma(\beta,\beta)) \geq \alpha$, the left-contiunuity then yields $g(x_0) \in q_\alpha^Y(g(\mathbb{G}))$ and g is continuous. □

It is clear that if $(Y, \overline{q^Y})$ is a T2-space, then $F_A^\top(x)$ contains at most one point and hence also $\bigcap_{\alpha \in L} F_A^\alpha(x)$ contains at most one point. The extension g from Theorem 6.2 will then be unique. This yields the main theorem of this section.

Theorem 6.3. *Let $\gamma : L \times L \longrightarrow L$ satisfy $\bigvee_{\beta \lhd \top} \gamma(\gamma(\alpha,\beta),\beta) \geq \alpha$ for all $\alpha,\beta \in L$.*

Let $(X, \overline{q^X}), (Y, \overline{q^Y}) \in |\mathsf{L\text{-}CTS}|$ and let $(X, \overline{q^X})$ satisfy (LK-γ) and $(Y, \overline{q^Y})$ be a left-continuous T2-space and satisfy (LDF-γ). Let further $A \subseteq X$ be dense in $(X, \overline{q^X})$ and let $f : (A, \overline{q^X|_A}) \longrightarrow (Y, \overline{q^Y})$ be continuous. The following are equivalent:

(i) There is a unique continuous mapping $g : (X, \overline{q^X}) \longrightarrow (Y, \overline{q^Y})$ such that $g \circ \iota_A = f$.

(ii) for each $x \in X$, $\bigcap_{\alpha \in L} F_A^\alpha(x) \neq \emptyset$.

Proof. If we have a continuous extension $g : (X, \overline{q^X}) \longrightarrow (Y, \overline{q^Y})$, then because $\overline{A}^\top = X$ we see that $H_A^\alpha(x) \neq \emptyset$ for all $x \in X$. Let now $\mathbb{F} \in H_A^\alpha(x)$. Then $\mathbb{F}_A \in \mathsf{F}(A)$ and $x \in q_\alpha^X(\mathbb{F})$. Noting that $g([\mathbb{F}_A]) = g \circ \iota_A(\mathbb{F}_A) = f(\mathbb{F}_A)$ we conclude $g(x) \in q_\alpha^Y(g(\mathbb{F})) \subseteq q_\alpha^Y(g([\mathbb{F}_A])) = q_\alpha^Y(f(\mathbb{F}_A))$ and we have $g(x) \in F_A^\alpha(x)$.

The converse follows with $X_0 = \overline{A}^\top = X$ from Theorem 6.2. □

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