



Approximation by Using the Meyer-König and Zeller Operators Based on (p, q) -Analogue

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Abstract. In this paper, a generalization of the q -Meyer-König and Zeller operators by means of the (p, q) -calculus is introduced. Some approximation results for (p, q) -analogue of Meyer-König and Zeller operators denoted by $M_{n,p,q}$ for $0 < q < p \leq 1$ are obtained. Also we investigate classical and statistical versions of Korovkin type approximation results based on proposed operator. Furthermore, some graphical examples for convergence of the operators are presented.

1. Introduction

In 1960, starting from the identity

$$(1-u)^{m+1} \sum_{\ell=0}^{\infty} \binom{m+\ell}{\ell} u^{\ell} = 1 \quad \text{for all } u \in [0, 1),$$

Meyer-König and Zeller [36] constructed a sequence of positive linear operators by using real continuous functions defined on $[0, 1)$. Further modified Meyer-König and Zeller (MKZ) operators [12] on $C[0, 1]$ are defined by

$$M_m(f; u) = \sum_{\ell=0}^{\infty} f\left(\frac{\ell}{m+\ell}\right) \binom{m+\ell}{\ell} u^{\ell} (1-u)^{m+1} \quad \text{if } u \in [0, 1), \quad (1)$$

$$M_m(f; 1) = f(1) \quad \text{if } u = 1, \quad \text{for every } m \in \mathbb{N}.$$

Many authors defined q -modifications of several operators [19, 22, 47, 48]. Very first q -analogue [7] was defined by Lupuş [35]:

$$L_{m,q}(f; u) = \sum_{\ell=0}^m \frac{f\left(\frac{[\ell]_q}{[m]_q}\right) [m]_q q^{\frac{\ell(\ell-1)}{2}} u^{\ell} (1-u)^{m-\ell}}{\prod_{j=1}^m \{(1-u) + q^{j-1}(u)\}}$$

2020 Mathematics Subject Classification. 41A25, 41A36, 41A10, 41A30, 47B39, 47A58

Keywords. (p, q) -integers; statistical convergence; Korovkin type approximation theorem; linear positive operators; rate of convergence.

Received: 19 August 2020; Accepted: 17 February 2021

Communicated by Naseer Shahzad

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which are known as Lupaş q -analogue of Bernstein operators.

Later on, Phillips [48] proposed another q -analogue of these Bernstein operators, as

$$B_{m,q}(f; u) = \sum_{\ell=0}^m \begin{bmatrix} m \\ \ell \end{bmatrix}_q u^\ell \prod_{s=0}^{m-\ell-1} (1 - q^s u) f\left(\frac{[\ell]_q}{[m]_q}\right), \quad u \in [0, 1], \quad n \in \mathbb{N}, \quad f \in C[0, 1].$$

In a recent paper [50], Trif constructed the q -MKZ operators by

$$\begin{aligned} M_{m,q}(f, u) &= \sum_{\ell=0}^{\infty} f\left(\frac{[\ell]_q}{[m + \ell]_q}\right) \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_q u^\ell (1 - u)_q^{m+1} \quad \text{if } u \in [0, 1), \\ M_{m,q}(f, 1) &= f(1), \quad \text{if } u = 1, \end{aligned} \tag{2}$$

for $m \in \mathbb{N}$ and for any $f \in C[0, 1]$. For $q = 1$, the operators $M_{m,q}$ reduced to the classical Meyer-König and Zeller operators given in (1). Furthermore, in the slight modification by Dogru and Duman in [14] the q -variant of Meyer-König and Zeller operators are defined on $C[0, a]$, $a \in (0, 1)$, by

$$M_{m,q}(f, u) = \prod_{s=0}^m (1 - uq^s) \sum_{\ell=0}^{\infty} f\left(\frac{q^m [\ell]_q}{[m + \ell]_q}\right) \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_q u^\ell, \quad (q \in (0, 1)), \quad m \in \mathbb{N}.$$

For some other works on Meyer-König and Zeller operator, one can see [1].

Recently, Mursaleen et al. [41] firstly used the concept of (p, q) -integers in approximation and gave the (p, q) -analogue of Bernstein polynomials. For $1 \geq p > q > 0$ and $m \in \mathbb{N}$,

$$B_{m,p,q}(f; u) = \frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{\ell=0}^m \begin{bmatrix} m \\ \ell \end{bmatrix}_{p,q} p^{\frac{\ell(\ell-1)}{2}} u^\ell \prod_{s=0}^{m-\ell-1} (p^s - q^s u) f\left(\frac{[\ell]_{p,q}}{p^{\ell-m} [m]_{p,q}}\right), \quad u \in [0, 1]. \tag{3}$$

For more study we can see [9, 30, 40, 42, 46].

If $p = 1$, the operators (3) reduce to Phillips q -Bernstein operators.

Also,

$$\begin{aligned} (1 - u)_{p,q}^m &= \prod_{s=0}^{m-1} (p^s - q^s u) = (1 - u)(p - qu)(p^2 - q^2 u) \dots (p^{m-1} - q^{m-1} u) \\ &= \sum_{\ell=0}^m (-1)^\ell p^{\frac{(m-\ell)(m-\ell-1)}{2}} q^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} m \\ \ell \end{bmatrix}_{p,q} u^\ell \end{aligned}$$

Very recently, Khalid et al. [31, 32] introduced (p, q) -analogue of the operators defined in [32] and provided an application in CAGD, a generalization of q -Bézier curves and surfaces [8, 48], are defined as follows:

The positive linear operators $L_{p,q}^m : C[0, 1] \rightarrow C[0, 1]$

$$L_{p,q}^m(f; u) = \sum_{\ell=0}^m \frac{f\left(\frac{p^{m-\ell} [\ell]_{p,q}}{[m]_{p,q}}\right) \begin{bmatrix} m \\ \ell \end{bmatrix}_{p,q} p^{\frac{(m-\ell)(m-\ell-1)}{2}} q^{\frac{\ell(\ell-1)}{2}} u^\ell (1 - u)^{m-\ell}}{\prod_{i=1}^m \{p^{i-1}(1 - u) + q^{i-1}u\}},$$

for any $p, q > 0$ and $u \in [0, 1]$.

Again for $p = 1$, these operators reduce to the operators as given in [47].

Based on (p, q) -calculus and its applications, here in the present paper we construct (p, q) -analogue of MKZ operators which is an extension of the operators in [50] and study some convergence properties. Contents of this paper are available on arXiv [27].

For detailed study, we refer [2–4, 7, 11, 13, 20, 22, 23, 25, 26, 28, 29, 37, 49].

The (p, q) -integer $[m]_{p,q}$ is defined by

$$[m]_{p,q} = p^{m-1} + p^{m-2}q + p^{m-3}q^2 + \dots + pq^{m-2} + q^{m-1} = \begin{cases} \frac{p^m - q^m}{p - q}, & \text{for } p \neq q \neq 1 \\ m p^{m-1}, & \text{for } p = q \neq 1 \\ q, & \text{for } p = 1 \\ m, & \text{for } p = q = 1. \end{cases}$$

Also, the (p, q) -binomial expansions are

$$(au + by)_{p,q}^m := \sum_{\ell=0}^m p^{\frac{(m-\ell)(m-\ell-1)}{2}} q^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} m \\ \ell \end{bmatrix}_{p,q} a^{m-\ell} b^\ell u^{m-\ell} y^\ell,$$

$$(u + y)_{p,q}^m = (u + y)(pu + qy)(p^2u + q^2y) \cdots (p^{m-1}u + q^{m-1}y),$$

$$(1)_{p,q}^m = (1)(p)(p^2) \cdots (p^{m-1}) = p^{\frac{m(m-1)}{2}}.$$

Thus we obtain,

$$\sum_{\ell=0}^m \begin{bmatrix} m \\ \ell \end{bmatrix}_{p,q} p^{\frac{\ell(\ell-1)}{2}} u^\ell \prod_{s=0}^{m-\ell-1} (p^s - q^s u) = p^{\frac{m(m-1)}{2}}, \quad u \in [0, 1]$$

where

$$\begin{bmatrix} m \\ \ell \end{bmatrix}_{p,q} = \frac{[m]_{p,q}!}{[\ell]_{p,q}! [m - \ell]_{p,q}!} \quad (m > \ell > 0).$$

Also

$$q^\ell [m - \ell + 1]_{p,q} = [m + 1]_{p,q} - p^{m-\ell+1} [\ell]_{p,q}$$

and

$$[m + 1]_{p,q} = q^m + p[m]_{p,q} = p^m + q[m]_{p,q}.$$

For details on q -calculus and (p, q) -calculus, we can refer to [21, 41].

One can easily verify by induction that

$$(1 + u)(p + qu)(p^2 + q^2u) \cdots (p^{m-1} + q^{\ell-1}u) = \sum_{r=0}^{\ell} p^{\frac{(\ell-r)(\ell-r-1)}{2}} q^{\frac{r(r-1)}{2}} \begin{bmatrix} \ell \\ r \end{bmatrix}_{p,q} u^r$$

Again by induction and (p, q) -integers, we obtain (p, q) -analogue of Pascal’s relation [31, 32] defined by

$$\begin{bmatrix} m \\ \ell \end{bmatrix}_{p,q} = q^{m-\ell} \begin{bmatrix} m - 1 \\ \ell - 1 \end{bmatrix}_{p,q} + p^\ell \begin{bmatrix} m - 1 \\ \ell \end{bmatrix}_{p,q}$$

and

$$\begin{bmatrix} m \\ \ell \end{bmatrix}_{p,q} = p^{m-\ell} \begin{bmatrix} m - 1 \\ \ell - 1 \end{bmatrix}_{p,q} + q^\ell \begin{bmatrix} m - 1 \\ \ell \end{bmatrix}_{p,q}.$$

In Section 2, we construct (p, q) -analogue of Meyer-König and Zeller operators and present two auxiliary lemmas for the proposed operators. Section 3 is devoted to Korovkin type approximation theorem involving (p, q) -Meyer-König and Zeller operators. In section 4, we present some direct theorems based on Peetre's K -functional and in addition the rates of convergence are estimated. In the final section of this article, we display some graphical examples for the convergence and comparisons of the operators under different parameters.

2. Construction of operators

Let $f \in C[0, 1]$ and $0 < q < p \leq 1$. We define (p, q) -analogue of the Meyer-König and Zeller operators as follows:

$$\begin{aligned}
 M_{m,p,q}(f; u) &= \frac{1}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell} p^{-\ell m} \prod_{s=0}^m (p^s - q^s u) f\left(\frac{[\ell]_{p,q} p^m}{[m + \ell]_{p,q}}\right) \text{ if } u \in [0, 1), \\
 M_{m,p,q}(f, 1) &= f(1), \text{ if } u = 1.
 \end{aligned}
 \tag{4}$$

Also if $p = 1$ in the equation (4), then $M_{m,p,q}$ turn out to be the q -MKZ operators $M_{m,q}$ defined by (2).

Note that, with the help of mathematical induction on m , it can be easily verified that

$$\frac{1}{\prod_{s=0}^m (p^s - q^s u)} = \frac{1}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} p^{-\ell m} u^{\ell}.$$

Lemma 2.1. For all $u \in [0, 1]$ and $1 \geq p > q > 0$, we have

1. $M_{m,p,q}(1; u) = 1;$
2. $M_{m,p,q}(\xi; u) = u;$
3. $u^2 \leq M_{m,p,q}(\xi^2; u) \leq \frac{p^m}{[m+1]_{p,q}} u + pu^2.$

Proof. Let $u \in [0, 1]$ and $1 \geq p > q > 0$. Then

(i)

$$M_{m,p,q}(1; u) = \frac{1}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell} p^{-\ell m} \prod_{s=0}^m (p^s - q^s u) = 1.
 \tag{5}$$

(ii) Using the fact that $\begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} \frac{[\ell]_{p,q}}{[m + \ell]_{p,q}} = \begin{bmatrix} m + \ell - 1 \\ \ell - 1 \end{bmatrix}_{p,q}$ we get

$$\begin{aligned}
 M_{m,p,q}(\xi; u) &= \frac{1}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell} p^{-\ell m} \prod_{s=0}^m (p^s - q^s u) \frac{p^m [\ell]_{p,q}}{[m + \ell]_{p,q}} \\
 &= \frac{p^m}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=1}^{\infty} \begin{bmatrix} m + \ell - 1 \\ \ell - 1 \end{bmatrix}_{p,q} u^{\ell} p^{-\ell m} \prod_{s=0}^m (p^s - q^s u) \\
 &= \frac{p^m}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell+1} p^{-m(\ell+1)} \prod_{s=0}^m (p^s - q^s u) \\
 &= \frac{u}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell} p^{-\ell m} \prod_{s=0}^m (p^s - q^s u) = u.
 \end{aligned}$$

(iii) By using $[\ell + 1]_{p,q} = p[\ell]_{p,q} + q^\ell$, we obtain

$$\begin{aligned} M_{m,p,q}(\xi^2; u) &= \frac{1}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix} u^\ell p^{-\ell m} \prod_{s=0}^m (p^s - q^s u) \left(\frac{p^{2m} [\ell]_{p,q}^2}{[m + \ell]_{p,q}^2} \right) \\ &= \frac{p^{2m}}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=1}^{\infty} u^\ell p^{-\ell m} \prod_{s=0}^m (p^s - q^s u) \left(\frac{[\ell]_{p,q}}{[m + \ell]_{p,q}} \right) \\ &= \frac{p^{2m}}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell+1} p^{-m(\ell+1)} \prod_{s=0}^m (p^s - q^s u) \left(\frac{[\ell + 1]_{p,q}}{[m + \ell + 1]_{p,q}} \right) \\ &= \frac{p^m}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell+1} p^{-m\ell} \prod_{s=0}^m (p^s - q^s u) \left(\frac{p[\ell]_{p,q} + q^\ell}{[m + \ell + 1]_{p,q}} \right). \end{aligned}$$

Using the facts that $[m + 1]_{p,q} < [m + \ell + 1]_{p,q}$, $q^\ell < 1$ and

$$\begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} \frac{[\ell]_{p,q}}{[m + \ell + 1]_{p,q}} = \frac{[m + \ell]_{p,q}}{[m + \ell + 1]_{p,q}} \begin{bmatrix} m + \ell - 1 \\ \ell - 1 \end{bmatrix}_{p,q} \leq \begin{bmatrix} m + \ell - 1 \\ \ell - 1 \end{bmatrix}_{p,q},$$

we obtain

$$\begin{aligned} M_{m,p,q}(\xi^2; u) &= \frac{p^m}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell+1} p^{-m\ell} \prod_{s=0}^m (p^s - q^s u) \left(\frac{[\ell]_{p,q} p}{[m + \ell + 1]_{p,q}} \right) \\ &+ \frac{p^m}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell+1} p^{-m\ell} \prod_{s=0}^m (p^s - q^s u) \left(\frac{q^\ell}{[m + \ell + 1]_{p,q}} \right) \\ &\leq \frac{p^m}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=1}^{\infty} \begin{bmatrix} m + \ell - 1 \\ \ell - 1 \end{bmatrix}_{p,q} u^{\ell+1} p^{1-m\ell} \prod_{s=0}^m (p^s - q^s u) \\ &+ \frac{p^m}{[m + 1]_{p,q} p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell+1} p^{-m\ell} \prod_{s=0}^m (p^s - q^s u) \\ &= \frac{p^m}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=1}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell+2} p^{1-m(\ell+1)} \prod_{s=0}^m (p^s - q^s u) \\ &+ \frac{p^m}{[m + 1]_{p,q} p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^{\ell+1} p^{-m\ell} \prod_{s=0}^m (p^s - q^s u) \\ &= \frac{pu^2}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^\ell p^{-m\ell} \prod_{s=0}^m (p^s - q^s u) \\ &+ \frac{p^m u}{[m + 1]_{p,q} p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}_{p,q} u^\ell p^{-m\ell} \prod_{s=0}^m (p^s - q^s u) \\ &= \frac{p^m}{[m + 1]_{p,q}} u + pu^2. \end{aligned}$$

□

Lemma 2.2. For all $u \in [0, 1]$ and $1 \geq p > q > 0$, we have

- (i) $M_{m,p,q}(\xi - u; u) = 0$;
- (ii) $M_{m,p,q}(\xi - u)^2; u \leq \frac{p^m}{[m+1]_{p,q}} u + (p - 1)u^2$.

Proof.

The proof of (i) is obviously.

(ii) Let $0 \leq u \leq 1$ and $1 \geq p > q > 0$. Taking into account the linearity of the operators we get

$$\begin{aligned} M_{m,p,q}((\xi - u)^2; u) &= M_{m,p,q}(t^2; u) - 2uM_{m,p,q}(t; u) + u^2M_{m,p,q}(1; u) \\ &\leq \frac{up^m}{[m + 1]_{p,q}} + pu^2 - u^2 \\ &= \frac{p^m}{[m + 1]_{p,q}}u + (p - 1)u^2. \end{aligned}$$

□

3. Korovkin type approximation theorems

In this section, we first recall some well known results based on classical Korovkin type approximation result. We also study classical and statistical versions of Korovkin’s results with respect to the (p, q) -Meyer-König and Zeller operators.

Let C_I be the space of all real valued continuous functions with the following norm

$$\|f\|_{C_I} := \sup_{u \in I} |f(u)|, \quad f \in C_I.$$

where $I = [a, b]$

The Bohman-Korovkin [34] type theorem can be stated as follows:

Let (T_m) be a sequence of positive linear operators from C_I to itself. Then

$$\lim_{m \rightarrow \infty} \|T_m(f; u) - f(u)\|_{\infty} = 0, \quad \text{for all } f \in C_I$$

if and only if

$$\lim_{m \rightarrow \infty} \|T_m(f_j; u) - f_j(u)\|_{\infty} = 0,$$

where $f_j(u) = u^j$, for each $j = 0, 1, 2$.

Theorem 3.1. *Let $0 < q_m < p_m \leq 1$ such that $\lim_{m \rightarrow \infty} p_m = 1$, $\lim_{m \rightarrow \infty} q_m = 1$ and $\lim_{m \rightarrow \infty} p_m^m = 1$, $\lim_{m \rightarrow \infty} q_m^m = 1$. Then for every $f \in C_I$, $M_{m,p_m,q_m}(f; u)$ converges uniformly to f on I .*

Proof. Using Bohman-Korovkin Theorem, we need to prove that

$$\lim_{m \rightarrow \infty} \|M_{m,p_m,q_m}(\xi^i; u) - u^i\|_{C_I} = 0, \quad (i = 0, 1, 2.)$$

By using Lemma 2.1 (i)-(ii), we have

$$\lim_{m \rightarrow \infty} \|M_{m,p_m,q_m}(1; u) - 1\|_{C_I} = 0$$

and

$$\lim_{m \rightarrow \infty} \|M_{m,p_m,q_m}(\xi; u) - u\|_{C_I} = 0.$$

From Lemma 2.1 (iii), we get

$$\|M_{m,p_m,q_m}(\xi^2; u) - u^2\|_{C_I} \leq \left| \frac{p_m^m}{[m + 1]_{p_m,q_m}}u + (p_m - 1)u^2 \right|.$$

Hence

$$\|M_{m,p_m,q_m}(\xi^2; u) - u^2\|_{C_I} \leq \left| \frac{p_m^m}{[m+1]_{p_m,q_m}} + p_m - 1 \right|$$

which yields

$$\lim_{m \rightarrow \infty} \|M_{m,p_m,q_m}(\xi^2; u) - u^2\|_{C_I} = 0.$$

□

Remark 3.2. For $0 < q < 1$ and $q < p \leq 1$, it is clear that $\lim_{m \rightarrow \infty} [m]_{p,q} = 0$ (or $\frac{1}{p-q}$). We need to take two sequences $p = (p_m)$, $q = (q_m)$ such that $q_m \in (0, 1)$, $p_m \in (q_m, 1]$ and $\lim_{m \rightarrow \infty} p_m = 1$, $\lim_{m \rightarrow \infty} q_m = 1$, $\lim_{m \rightarrow \infty} p_m^m = 1$, $\lim_{m \rightarrow \infty} q_m^m = 1$. We obtain $\lim_{m \rightarrow \infty} [m]_{p_m,q_m} = \infty$.

3.1. Statistical approximation:

For the sequence of positive linear operators, Gadjiev and Orhan [18] were the first who gave statistical version of Korovkin theorem. For some recent work on statistical convergence, one can refer [5, 25, 38, 39, 43].

Here, we give a statistical approximation theorem of $M_{m,p,q}$.

A sequence $u = (u_j)$ is said to be statistically convergent to the number \mathcal{L} if for every $\varepsilon > 0$ (see [17]),

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{j \leq m : |u_j - \mathcal{L}| \geq \varepsilon\}| = 0,$$

where $|\cdot|$ denotes the number of elements in the set. Also, we write $st - \lim u = \mathcal{L}$.

Theorem 3.3. Let $0 < q_m < p_m \leq 1$ such that $st - \lim_{m \rightarrow \infty} p_m = 1$, $st - \lim_{m \rightarrow \infty} q_m = 1$ and $st - \lim_{m \rightarrow \infty} p_m^m = 1$, $st - \lim_{m \rightarrow \infty} q_m^m = 1$. Then for the operators M_{n,p_n,q_n} satisfying the condition

$$st - \lim_{m \rightarrow \infty} \|M_{m,p_m,q_m}(f_i(\xi); u) - f_i(u)\|_{C_I} = 0, \text{ for each } i = 0, 1, 2,$$

we have

$$st - \lim_{m \rightarrow \infty} \|M_{m,p_m,q_m}(f(\xi); u) - f(u)\|_{C_I} = 0, \text{ for all } f \in C_I.$$

Proof. Let $f \in C_I$ and $u \in I$ be fixed. Since f is bounded on the interval I there exists a number M such that $|f(u)| \leq M$ for all $-\infty < u < \infty$. Thus,

$$|f(\xi) - f(u)| \leq 2M, \quad -\infty < \xi, u < \infty.$$

From the continuity of the function f , there exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$|f(\xi) - f(u)| < \varepsilon \text{ whenever } |\xi - u| < \delta,$$

for $\varepsilon > 0$. Also, for all $\xi, u \in [0, 1]$ satisfying $|\xi - u| > \delta$ that

$$|f(\xi) - f(u)| \leq \varepsilon + \frac{2M}{\delta^2}(\xi - u)^2.$$

From the above relations, we get, for all $\xi, u \in [0, 1]$, that

$$|f(\xi) - f(u)| < \varepsilon + \frac{2M}{\delta^2}(t - u)^2.$$

By taking into account the properties of M_{m,p_m,q_m} , we obtain that

$$\begin{aligned} & |M_{m,p_m,q_m}(f(\xi); u) - f(u)| \\ & \leq M_{m,p_m,q_m}(|f(\xi) - f(u)|; u) + |f(u)| |M_{m,p_m,q_m}(1; u) - 1| \\ & \leq \left| M_{m,p_m,q_m} \left(\varepsilon + \frac{2M}{\delta^2} \varphi(u); u \right) \right| + M |M_{m,p_m,q_m}(1; u) - 1| \\ & \leq \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2} a^2 \right) |M_{m,p_m,q_m}(1; u) - 1| + \frac{4M}{\delta^2} a \left| M_{m,p_m,q_m}(\xi; u) - u \right| \\ & \quad + \frac{2M}{\delta^2} \left| M_{m,p_m,q_m}(\xi^2; u) - u^2 \right| \end{aligned}$$

where $a = \max\{|u|\}$. Then taking supremum over $u \in [0, 1]$ in last inequality, we have

$$\|M_{m,p_m,q_m}(f(t); u) - f(u)\|_{C(I)} \leq \varepsilon + \eta \sum_{i=0}^2 \|M_{m,p_m,q_m}(f_i(t); u) - f_i(u)\|_{C(I)}$$

where $\eta = \max \left\{ \varepsilon + M + \frac{2M}{\delta^2} a^2, \frac{4M}{\delta^2} a, \frac{2M}{\delta^2} \right\}$. Now, for a given $\varepsilon' > 0$, choose a number $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$. Then, the following sets can be defined as

$$\begin{aligned} \mathcal{A} & := \left\{ m \in \mathbb{N} : \|M_{m,p_m,q_m}(f(\xi); u) - f(u)\|_{C(I)} \geq \varepsilon' \right\}, \\ \mathcal{A}_i & := \left\{ m \in \mathbb{N} : \|M_{m,p_m,q_m}(f_i(\xi); u) - f_i(u)\|_{C(I)} \geq \frac{\varepsilon' - \varepsilon}{3\eta} \right\}, \quad i = 0, 1, 2. \end{aligned}$$

Then it is easy to see that $\mathcal{A} \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ and hence $\delta(\mathcal{A}) \leq \delta(\mathcal{A}_0) + \delta(\mathcal{A}_1) + \delta(\mathcal{A}_2)$. Letting $m \rightarrow \infty$, we immediately get that

$$st - \lim_{m \rightarrow \infty} \|M_{m,p_m,q_m}(f(t); u) - f(u)\|_{C(I)} = 0, \quad \text{for each } f \in C(I).$$

Thus, the proof is completed. \square

4. Direct Theorems

Here, we present some direct theorems and obtain the rate of convergence. The Peetre's \mathcal{K} -functional of $f \in C_I$ is defined by

$$\mathcal{K}_2(f, \delta) = \inf_{g \in W^2} \{ \|f - g\|_{C(I)} + \delta \|g''\|_{C_I}, \quad \delta > 0 \}$$

where

$$W^2 = \{g \in C_I : g', g'' \in C_I\}.$$

By [15], there exists a positive constant $M > 0$ such that

$$\mathcal{K}_2(f, \delta) \leq M w_2(f, \delta^{\frac{1}{2}}) \quad (\delta > 0)$$

where w_2 represent the second order modulus of continuity given by

$$w_2(f, \delta^{\frac{1}{2}}) = \sup_{0 < h \leq \delta^{\frac{1}{2}}} \sup_{u, u+2h \in [0,1]} |f(u+2h) - 2f(u+h) + f(u)|.$$

Theorem 4.1. Let f be a real continuous function defined in I and $0 < q < p \leq 1$. Then $\forall n \in \mathbb{N}$, there is $M > 0$ such that

$$|M_{m,p,q}(f; u) - f(u)| \leq Mw_2(f, \delta_m(u))$$

where

$$\delta_m^2(u) = u^2(p - 1) + \frac{p^m}{[m + 1]_{p,q}}u.$$

Proof. Let $g \in W_2$. From Taylor’s expansion, we obtain

$$g(\xi) = g(u) + g'(u)(\xi - u) + \int_u^\xi (\xi - u) g''(u) du, \quad (\xi \in [0, 1])$$

By Lemma 2.2 (1), we have

$$M_{m,p,q}(g(\xi); u) = g(u) + M_{m,p,q}\left(\int_u^\xi (\xi - u) g''(u) du; u\right)$$

which yields that

$$\begin{aligned} |M_{m,p,q}(g(\xi); u) - g(u)| &\leq \left| M_{m,p,q}\left(\int_u^\xi (\xi - u) g''(u) du; u\right) \right| \\ &\leq M_{m,p,q}\left(\int_u^\xi |(\xi - u)| |g''(u)| du; u\right) \\ &\leq M_{m,p,q}((\xi - u)^2; u) \|g''\|. \end{aligned}$$

Using Lemma 2.2 (ii), we get

$$|M_{m,p,q}(g(\xi); u) - g(u)| \leq u^2(p - 1)\|g''\| + \frac{p^m}{[m + 1]_{p,q}}u \|g''\|.$$

From the definition of $M_{m,p,q}(f; u)$, we obtain

$$|M_{m,p,q}(f; u)| \leq \|f\|.$$

$$\begin{aligned} |M_{m,p,q}(f; u) - f(u)| &\leq |M_{m,p,q}((f - g); u) - (f - g)(u)| + |M_{m,p,q}(g; u) - g(u)| \\ &\leq \|f - g\| + u^2(p - 1)\|g''\| + u \frac{p^m}{[m + 1]_{p,q}}\|g''\|. \end{aligned}$$

Now taking infimum over all $g \in W^2$, we have

$$|M_{m,p,q}(f; u) - f(u)| \leq C\mathcal{K}_2(f, \delta_m^2(u))$$

Hence, we obtain

$$|M_{m,p,q}(f; u) - f(u)| \leq Cw_2(f, \delta_m(u)).$$

□

Theorem 4.2. If $f \in C_I$, then

$$|M_{m,p,q}(f; u) - f(u)| \leq 2w\left(f; \sqrt{\frac{p^m}{[m + 1]_{p,q}} + p - 1}\right).$$

Proof. Since $M_{m,p,q}(1, u) = 1$, we have

$$\begin{aligned}
 |M_{m,p,q}(f; u) - f(u)| &\leq \left| \frac{1}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}_{p,q} u^\ell p^{-\ell m} P_{m-1}(u) \left| f\left(\frac{[\ell]_{p,q} p^m}{[m+\ell]_{p,q}}\right) - f(u) \right| \right| \\
 &\leq \left\{ \frac{1}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}_{p,q} u^\ell p^{-\ell m} P_{m-1}(u) \left(\frac{|\frac{[\ell]_{p,q} p^m}{[m+\ell]_{p,q}} - u|^2}{\delta^2} + 1 \right) \right\} w_f(\delta) \\
 &= \left\{ \frac{1}{\delta^2} \frac{1}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}_{p,q} u^\ell p^{-\ell m} P_{m-1}(u) \left(\frac{[\ell]_{p,q} p^m}{[m+\ell]_{p,q}} - u \right)^2 \right. \\
 &\quad \left. + \frac{1}{p^{\frac{m(m+1)}{2}}} \sum_{\ell=0}^{\infty} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}_{p,q} u^\ell p^{-\ell m} P_{m-1}(u) \right\} w_f(\delta) \\
 &= \left\{ \frac{1}{\delta^2} (M_{m,p,q}(t^2; u) - 2uM_{m,p,q}(t; u) + u^2M_{m,p,q}(1; u)) + 1 \right\} w_f(\delta) \\
 &\leq \left\{ \frac{1}{\delta^2} \left(\frac{p^m}{[m+1]_{p,q}} + p - 1 \right) + 1 \right\} w_f(\delta),
 \end{aligned}$$

where $P_{m-1}(u) = \prod_{s=0}^m (p^s - q^s u)$. Choosing $\delta = \delta_m = \sqrt{\frac{p^m}{[m+1]_{p,q}} + p - 1}$, we get

$$|M_{m,p,q}(f; u) - f(u)| \leq 2w_f(\delta_m).$$

□

Further, we estimate the rate of convergence in terms of elements of the usual Lipschitz class $Lip_M(\alpha)$ of the operators $M_{m,p,q}$.

Let $f \in C_I$, $M > 0$ and $0 < \alpha \leq 1$. A function $f \in Lip_M(\alpha)$ if the following inequality

$$|f(\xi) - f(u)| \leq M|\xi - u|^\alpha, \quad (t, u \in [0, 1])$$

hold.

Theorem 4.3. *Let $1 \geq p > q > 0$. Then for every $f \in Lip_M(\alpha)$ we get*

$$|M_{m,p,q}(f; u) - f(u)| \leq M\delta_m(u)^\alpha$$

where

$$\delta_m^2(u) = u^2(p - 1) + \frac{p^m}{[m+1]_{p,q}} u.$$

Proof. Let us denote

$$P_{m,\ell}(u) = \frac{1}{p^{\frac{m(m+1)}{2}}} \begin{bmatrix} m+\ell \\ \ell \end{bmatrix}_{p,q} u^\ell p^{-\ell m} \prod_{s=0}^m (p^s - q^s u)$$

. From monotonicity of the operators $M_{m,p,q}$, we can write

$$\begin{aligned} |M_{m,p,q}(f; u) - f(u)| &\leq M_{m,p,q}(|f(\xi) - f(u)|; u) \\ &\leq \sum_{\ell=0}^{\infty} P_{m,\ell}(u) \left| f\left(\frac{p^m [\ell]_{p,q}}{[m + \ell]_{p,q}}\right) - f(u) \right| \\ &\leq \mathcal{M} \sum_{\ell=0}^{\infty} P_{m,\ell}(u) \left| \frac{p^m [\ell]_{p,q}}{[m + \ell]_{p,q}} - u \right|^\alpha \\ &= \mathcal{M} \sum_{\ell=0}^{\infty} \left(P_{m,\ell}(u) \left(\frac{p^m [\ell]_{p,q}}{[m + \ell]_{p,q}} - u \right)^2 \right)^{\alpha/2} P_{m,\ell}^{\frac{2-\alpha}{2}}(u). \end{aligned}$$

Now for the sum, applying the Hölder’s inequality with $\tilde{p} = \frac{2}{\alpha}$ and $\tilde{q} = \frac{2}{2-\alpha}$ and Lemma 2.1 (i) and Lemma 2.2 (ii), we get

$$\begin{aligned} |M_{m,p,q}(f; u) - f(u)| &\leq \mathcal{M} \left(\sum_{\ell=0}^{\infty} P_{m,\ell}(u) \left(\frac{p^m [\ell]_{p,q}}{[m + \ell]_{p,q}} - u \right)^2 \right)^{\alpha/2} \left(\sum_{\ell=0}^{\infty} P_{m,\ell}(u) \right)^{\frac{2-\alpha}{2}} \\ &= \mathcal{M} \left\{ M_{m,p,q}((t - u)^2; u) \right\}^{\frac{\alpha}{2}} \end{aligned}$$

Choosing $\delta = \delta_m(u) = \sqrt{M_{m,p,q}((\xi - u)^2; u)}$, we obtain

$$|M_{m,p,q}(f; u) - f(u)| \leq \mathcal{M} \delta_m(u)^\alpha.$$

□

5. Example

In this section, we show the comparisons of the operators (4) for the convergence to the function

$$f(u) = \left(u - \frac{1}{3}\right)\left(u - \frac{1}{2}\right)\left(u - \frac{3}{4}\right),$$

and some illustrative graphics [44] for different parameters. Because of a complicated infinite series, we investigate our series only for finite terms.

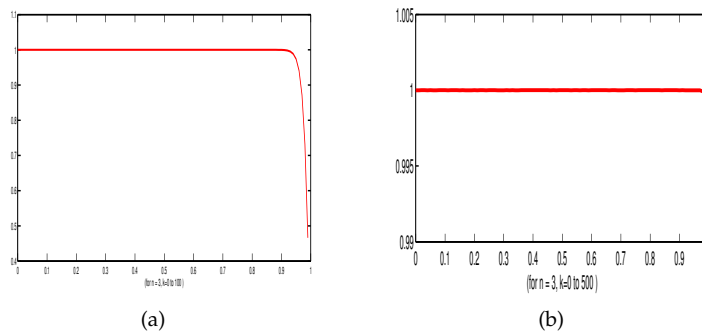
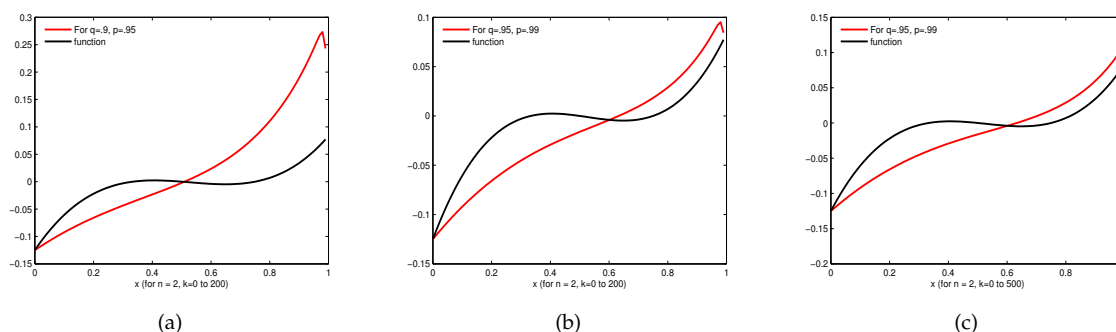
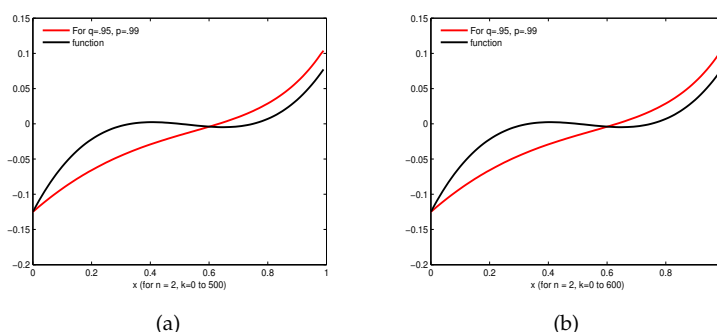


Figure 1: (p, q) -Meyer-König and Zeller operator

We can observe that in Figure (1), (p, q) -Meyer-König and Zeller operators (5) converges towards 1 as the value of ℓ increases. In Figure (1) (a), value of $\ell = 0$ to 100 whereas in Figure (1) (b), value of $\ell = 0$ to 500 for $n = 3$.

Figure 2: (p, q) -Meyer-König and Zeller operatorsFigure 3: (p, q) -Meyer-König and Zeller operator

Also, In Figure (2), we observed that the operators (5) converge towards the function as the value of p and q approaches to 1 provided $0 < q < p \leq 1$.

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