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Almost Riemann Solitons and Gradient Almost Riemann Solitons on *LP*-Sasakian Manifolds

Krishnendu De[a]

[a] *Assistant Professor of Mathematics, Kabi Sukanta Mahavidyalaya, Bhadreswar,P.O.- Angus, Hooghly, Pin.712221, West Bengal, India.*

Abstract. The upcoming article aims to investigate almost Riemann solitons and gradient almost Riemann solitons in a *LP*-Sasakian manifold M^3. At first, it is proved that if (g, Z, λ) be an almost Riemann soliton on a *LP*-Sasakian manifold M^3, then it reduces to a Riemann soliton, provided the soliton vector Z has constant divergence. Also, we show that if Z is pointwise collinear with the characteristic vector field ξ, then Z is a constant multiple of ξ, and the ARS reduces to a Riemann soliton. Furthermore, it is proved that if a *LP*-Sasakian manifold M^3 admits gradient almost Riemann soliton, then the manifold is a space form. Also, we consider a non-trivial example and validate a result of our paper.

1. Introduction

The idea of Ricci flow was introduced by Hamilton [5] and defined by $\frac{\partial}{\partial_t} g(t) = -2S(t)$, where S denotes the Ricci tensor.

As a natural generalization, the concept of Riemann flow ([14],[15]) is defined by $\frac{\partial}{\partial_t} G(t) = -2Rg(t)$, $G = \frac{1}{2} g \otimes g$, where R is the Riemann curvature tensor and $\otimes$ is *Kulkarni-Nomizu* product (executed as (see Besse [2], p. 47),

$$(P \otimes Q)(X, Y, Z, W) = P(X, W)Q(Y, U) + P(Y, U)Q(X, W)$$
$$-P(X, U)Q(Y, W) - P(Y, W)Q(X, U)).$$

Similar to Ricci soliton, the interesting idea of Riemann soliton was introduced by Hirica and Udriste [6]. Analogous to Hirica and Udriste [6], a Lorentzian metric g on a Lorentzian manifold M is called a *Riemann solitons* if there exists a C^∞ vector field Z and a real scalar λ such that

$$2R + \lambda g \otimes g + g \otimes \pounds_Z g = 0. \tag{1}$$

On this occasion, we should mention that the space of constant sectional curvature is generalized by the Riemann soliton. If the vector field Z is the gradient of the potential function γ, then the manifold is called *gradient Riemann soliton*. Then the foregoing equation can be written as

$$2R + \lambda g \otimes g + g \otimes \nabla^2 \gamma = 0, \tag{2}$$

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Email address: krishnendu.de@outlook.in (Krishnendu De)

where $\nabla^2 f$ denotes the Hessian of γ. If we modified the equation (1) and (2) by fixing the condition on the parameter λ to be a variable function, then it reduces to *ARS* and *gradient ARS* respectively. Here the terminology "almost Riemann solitons" is written as *ARS* which will be applied throughout the article.

A general idea of Lorentzian para-Sasakian (briefly *LP*-Sasakian) manifold has been introduced by K. Matsumoto [7], in 1989 and several geometers in different context ([1], [8], [9], [10]) have studied *LP*-Sasakian manifolds. Riemann solitons and gradient Riemann solitons on Sasakian manifolds have been discussed in detail by Hirica and Udriste (see, [6]). Moreover, Riemann's soliton concerning infinitesimal harmonic transformation was investigated in [13]. Here it is appropriate to notice that Sharma in [11] investigated almost Ricci soliton in *K*-contact geometry and in [12], with divergence-free soliton vector field. Very recently in [4], the authors studied Riemann soliton within the framework of a contact manifold and proved various fascinating results.

The above studies motivate us to investigate an *ARS* and the *gradient ARS* in a 3-dimensional *LP*-Sasakian manifold.

The upcoming article is structured as follows: In section 2, we recall some fundamental facts and formulas of *LP*-Sasakian manifolds, which will be needed in later sections. Beginning from Section 3, after providing the proof, we will write our prime theorems. This article terminates with a concise bibliography which has been used during the formulation of the upcoming article.

2. *LP*-Sasakian manifolds

Let η, ξ, ϕ are tensor fields on a smooth manifold M^n of types (0,1), (1,0) and (1,1) respectively, such that

$$\eta(\xi) = -1, \quad \phi^2 E = E + \eta(E)\xi. \tag{3}$$

The foregoing equations imply that

$$\phi\xi = 0, \quad \eta \circ \phi = 0. \tag{4}$$

Then M^n admits a Lorentzian metric g of type (0,2) such that

$$g(E, \xi) = \eta(E), \quad g(\phi E, \phi F) = g(E, F) + \eta(E)\eta(F) \tag{5}$$

for any vector fields E, F. Then the structure (η, ξ, ϕ, g) is called Lorentzian almost para-contact structure. The manifold M^n equipped with a Lorentzian almost para-contact structure (η, ξ, ϕ, g) is called a Lorentzian almost para-contact manifold(briefly LAP-manifold).

If we denote $\Phi(E, F) = g(E, \phi F)$, then we obtain [7]

$$\Phi(E, F) = g(E, \phi F) = g(\phi E, F) = \Phi(F, E), \tag{6}$$

where E, F are any vector fields.
An LAP-manifold M^n equipped with the structure (η, ξ, ϕ, g) is said to be a Lorentzian para-contact manifold(briefly *LP*-manifold) if

$$\Phi(E, F) = \frac{1}{2}\{(\nabla_E \eta)F + (\nabla_F \eta)E\}, \tag{7}$$

where Φ is defined by (6) and ∇ indicates the covariant differentiation operator with respect to the Lorentzian metric g. A Lorentzian almost para-contact manifold M^n is said to be a *LP*-Sasakian manifold if it satisfies

$$(\nabla_E \phi)F = \eta(F)E + g(E, F)\xi + 2\eta(E)\eta(F)\xi. \tag{8}$$

Also since the vector field, η is closed in an *LP*-Sasakian manifold we have

$$(\nabla_E \eta)F = \Phi(E, F) = g(E, \phi F), \quad \Phi(E, \xi) = 0, \quad \nabla_E \xi = \phi E. \tag{9}$$

Furthermore, we find that the eigen values of ϕ are -1, 0 and 1. Here the multiplicity of 0 is one. Let us assume that the multiplicities of -1 and 1 are k and l respectively. Then we get, $trace(\phi) = l - k$. Hence, if

$(trace(\phi))^2 = (n-1)$, then either $l=0$ or $k=0$. Then the structure is called a trivial LP-Sasakian structure. Throughout this article we presume that $trace(\phi) \neq 0$, i.e., ξ is not harmonic.

Let us presume that $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the well-known Ricci tensor S and the scalar curvature r are defined by

$$S(E,F) = \sum_{i=1}^{n} \epsilon_i g(R(e_i,E)F, e_i)$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1, \epsilon_2 = \cdots = \epsilon_n = 1$.

Also in an LP-Sasakian manifold M^n, the subsequent relations hold ([1], [7], [10]):

$$\eta(R(E,F)Z) = g(F,Z)\eta(E) - g(E,Z)\eta(F), \tag{10}$$

$$R(E,F)\xi = \eta(F)E - \eta(E)F, \tag{11}$$

$$R(\xi,E)F = g(E,F)\xi - \eta(F)E, \tag{12}$$

$$S(E,\xi) = (n-1)\eta(E), \tag{13}$$

$$\nabla_\xi \eta = 0, \tag{14}$$

for any vector fields E, F, Z where R is the Riemannian curvature tensor, S is the Ricci tensor and ∇ is the Levi-Civita connection associated to the metric g.

It is well-known that a 3-dimensional Riemannian manifold M assumes the following curvature form

$$\begin{aligned} R(E,F)Z &= g(F,Z)QE - g(E,Z)QF + S(F,Z)E - S(E,Z)F \\ &\quad -\frac{r}{2}[g(F,Z)E - g(E,Z)F], \end{aligned} \tag{15}$$

for any vector fields E, F, Z where Q is the Ricci operator, i.e., $g(QE,F) = S(E,F)$ and r is the scalar curvature of the manifold. Replacing F=Z=ξ in the previous equation and utilizing (11) and (13) we get (see [10])

$$QE = \frac{1}{2}[(r-2)E + (r-6)\eta(E)\xi]. \tag{16}$$

In view of (16) the Ricci tensor is written as

$$S(E,F) = \frac{1}{2}[(r-2)g(E,F) + (r-6)\eta(E)\eta(F)]. \tag{17}$$

Using (17) and (16) in (15), we deduce

$$\begin{aligned} R(E,F)Z &= \frac{(r-4)}{2}\{g(F,Z)E - g(E,Z)F\} \\ &\quad +\frac{(r-6)}{2}\{g(F,Z)\eta(E)\xi - g(E,Z)\eta(F)\xi \\ &\quad +\eta(F)\eta(Z)E - \eta(E)\eta(Z)F\}. \end{aligned} \tag{18}$$

We first prove the following Lemma:

Lemma 2.1. *Let M^3 be a LP-Sasakian manifold. Then we have*

$$\xi r = -2(r-6)trace(\phi). \tag{19}$$

Proof. The equation (16) can be rewritten as:

$$QF = \frac{1}{2}[(r-2)F + (r-6)\eta(F)\xi]. \tag{20}$$

Taking covariant derivative along E and recalling (9) we write

$$\begin{aligned}(\nabla_E Q)F &= \frac{(Er)}{2}F + \frac{(Er)}{2}\eta(F)\xi + \frac{(r-6)}{2}g(E,\phi F)\xi \\ &+ \frac{(r-6)}{2}\eta(F)\phi E.\end{aligned} \tag{21}$$

Taking inner product operation with respect to Z in the foregoing equation, we obtain

$$\begin{aligned}g((\nabla_E Q)F, Z) &= \frac{(Er)}{2}g(F,Z) + \frac{(Er)}{2}\eta(F)\eta(Z) + \frac{(r-6)}{2}g(E,\phi F)\eta(Z) \\ & \frac{(r-6)}{2}\eta(F)g(\phi E, Z).\end{aligned} \tag{22}$$

Putting $E = Z = e_i$ (where $\{e_i\}$ is an orthonormal basis for the tangent space of M^3 and taking $\sum i$, $1 \leq i \leq 3$) in the above equation and utilizing the formula of Riemannian manifolds $divQ = \frac{1}{2}grad\ r$, we obtain

$$(\xi r)\eta(F) = -2(r-6)\eta(F)trace(\phi). \tag{23}$$

Substituting $F = \xi$ in the above equation we get the desired result. This finishes the proof. □

If an LP-Sasakian manifold M^3 is a space of constant curvature, then the manifold is said to be a space form.

Lemma 2.2. *(Lemma. 1.1 of [10]) A 3-dimensional LP-Sasakian manifold is a space form if and only if the scalar curvature $r = 6$.*

Lemma 2.3. *(Lemma. 3.8 of [4]) For any vector fields E, F on M^3, for a gradient ARS $(M, g, \gamma, m, \lambda)$, we have*

$$\begin{aligned}R(E,F)D\gamma &= (\nabla_F Q)E - (\nabla_E Q)F \\ &+ \{F(2\lambda + \triangle\gamma)E - E(2\lambda + \triangle\gamma)F\},\end{aligned} \tag{24}$$

where $\triangle\gamma = div\ D\gamma$, $\triangle$ is the Laplacian operator.

3. *ARS* on 3-dimensional *LP*-Sasakian manifolds

We consider a 3-dimensional para-Sasakian manifold M admitting an ARS defined by(1). Using *Kulkarni-Nomizu* product in (1) we write

$$\begin{aligned}& 2R(E,F,W,X) + 2\lambda\{g(E,X)g(F,W) - g(E,W)g(F,X)\} \\ + & \{g(E,X)(\pounds_Z g)(F,W) + g(F,W)(\pounds_Z g)(E,X) \\ - & g(E,W)(\pounds_Z g)(F,X) - g(F,X)(\pounds_Z g)(E,W)\} = 0.\end{aligned} \tag{25}$$

Contracting (25) over E and X, we get

$$(\pounds_Z g)(F,W) + 2S(F,W) + (4\lambda + 2divZ)g(F,W) = 0. \tag{26}$$

Utilizing (17) in the above equation we obtain

$$\begin{aligned}(\pounds_Z g)(F,W) &= -(r - 2 + 4\lambda + 2divZ)g(F,W) \\ &- (r-6)\eta(F)\eta(W) = 0.\end{aligned} \tag{27}$$

Applying Z has constant divergence and executing covariant derivative along E, we lead

$$\begin{aligned}(\nabla_E \pounds_Z g)(F,W) &= -[(Er)+4(E\lambda)]g(F,W)\\ &- (Er)\eta(F)\eta(W)\\ &- (r-6)[g(\phi E,F)\eta(W)+g(\phi E,W)\eta(F)]=0.\end{aligned} \tag{28}$$

Now we recall the formula by Yano (see, [16]):

$$(\pounds_Z \nabla_E g - \nabla_E \pounds_Z g - \nabla_{[Z,E]} g)(F,W) = -g((\pounds_Z \nabla)(E,F),W) - g((\pounds_Z \nabla)(E,W),F).$$

Hence by a straightforward calculation, we infer

$$(\nabla_E \pounds_Z g)(F,W) = g((\pounds_Z \nabla)(E,F),W) + g((\pounds_Z \nabla)(E,W),F). \tag{29}$$

Using symmetric property of $\pounds_F \nabla$, it reveals from (29) that

$$\begin{aligned}&g((\pounds_Z \nabla)(E,F),W)\\ &= \frac{1}{2}(\nabla_E \pounds_Z g)(F,W) + \frac{1}{2}(\nabla_F \pounds_Z g)(E,W) - \frac{1}{2}(\nabla_W \pounds_Z g)(E,F).\end{aligned} \tag{30}$$

Utilizing (28) in (30) we obtain

$$\begin{aligned}2g((\pounds_Z \nabla)(E,F),W) &= -[(Er)+4(E\lambda)]g(F,W)-(Er)\eta(F)\eta(W)\\ &- (r-6)[g(\phi E,F)\eta(W)+g(\phi E,W)\eta(F)]\\ &- [(Fr)+4(F\lambda)]g(E,W)-(Fr)\eta(E)\eta(W)\\ &- (r-6)[g(\phi F,E)\eta(W)+g(\phi F,W)\eta(E)]\\ &+ [(Wr)+4(W\lambda)]g(E,F)+(Wr)\eta(E)\eta(F)\\ &- (r-6)[g(\phi W,E)\eta(F)+g(\phi W,F)\eta(E)].\end{aligned} \tag{31}$$

After substituting $E = F = e_i$ in the foregoing equation and removing Z from both sides, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking $\sum i$, $1 \le i \le 3$, we have

$$(\pounds_Z \nabla)(e_i,e_i) = 2D\lambda - (\xi r)\xi - 2(r-6)trace(\phi)\xi, \tag{32}$$

where $E\alpha = g(D\alpha, E)$, D denotes the gradient operator with respect to g.
Now differentiating(1) and utilizing it in (29) we can easily determine

$$g((\pounds_Z \nabla)(E,F),W) = (\nabla_W S)(E,F) - (\nabla_E S)(F,W) - (\nabla_F S)(E,W). \tag{33}$$

Taking $E = F = e_i$ (where $\{e_i\}$ is an orthonormal frame) in (33) and summing over i we obtain

$$(\pounds_Z \nabla)(e_i,e_i) = 0, \tag{34}$$

for all vector fields Z. Combining (32) and (34) gives

$$-2D\lambda + (\xi r)\xi + 2(r-6)trace(\phi)\xi = 0. \tag{35}$$

Utilizing (19) in the previous equation, we get

$$D\lambda = 0. \tag{36}$$

This implies that λ is constant. This leads to the following theorem:

Theorem 3.1. *If the soliton vector Z has constant divergence in a LP-Sasakian manifold M^3, then an ARS reduces to a Riemann soliton.*

Now let the potential vector field Z be point-wise collinear with the characteristic vector field ξ (i.e., $Z = b\xi$, where b is a function on M^3)and has constant divergence. Therefore from (26) we lead

$$g(\nabla_E b\xi, F) + g(\nabla_F b\xi, E) + 2S(E, F) + 4\lambda g(E, F) = 0. \tag{37}$$

Using (9) in (37), we get

$$(Eb)\eta(F) + (Fb)\eta(E) + 2S(E, F) + (4\lambda + 2divZ)g(E, F) = 0. \tag{38}$$

Putting $F = \xi$ in (38) yields

$$-(Eb) + (\xi b)\eta(E) + 4\eta(E) + (4\lambda + 2divZ)\eta(E) = 0. \tag{39}$$

Putting $E = \xi$ in (39) we have

$$(\xi b) = (2\lambda + divZ - 2). \tag{40}$$

Putting the value of ξb in (39) gives

$$db = -(6\lambda + 3divZ + 2)\eta. \tag{41}$$

Operating (41) by d and utilizing Poincare lemma $d^2 \equiv 0$, we infer

$$0 = d^2 b = -(6\lambda + 3divZ + 2)d\eta - 6d\lambda\eta. \tag{42}$$

Executing wedge product of (42) with η, we have

$$-(6\lambda + 3divZ + 2)\eta \wedge d\eta = 0. \tag{43}$$

Since $\eta \wedge d\eta \neq 0$ in a LP-Sasakian manifold M^3, therefore

$$\lambda = -(\frac{1}{2}divZ + \frac{1}{3}). \tag{44}$$

Using (44) in (41) gives $db = 0$ i.e., b =constant. Also from (32) we obtain

$$\lambda = -(\frac{1}{2}divZ + \frac{1}{3}) = constant. \tag{45}$$

Hence we write the following:

Theorem 3.2. *If the metric of a LP-Sasakian manifold M^3 is ARS and Z is pointwise collinear with ξ and has constant divergence, then Z is a constant multiple of ξ and the ARS reduces to a Riemann soliton.*

Corollary 3.3. *If a LP-Sasakian manifold M^3 admits an ARS of type (g, ξ), then the ARS reduces to a Riemann soliton.*

4. Gradient Almost Riemann soliton

This section is devoted to investigate a LP-Sasakian manifold M^3 admitting gradient ARS. Now before producing the detailed proof of our main theorems, we first write the following results without proof (Since the result can be obtained directly from (21)):

Lemma 4.1. *For a LP-Sasakian manifold M^3, we have*

$$(\nabla_E Q)\xi = -(\frac{r}{2} - 3)\phi E, (\nabla_\xi Q)E = -2(r - 6)trace\phi[E + \eta(E)\xi]. \tag{46}$$

Replacing F by ξ in (24) and utilizing the foregoing Lemma, we obtain

$$\begin{aligned} R(E,\xi)D\gamma &= (\frac{r}{2}-3)\phi E - 2(r-6)trace\phi[E+\eta(E)\xi] \\ &+\{\xi(2\lambda+\triangle\gamma)E - E(2\lambda+\triangle\gamma)\xi\}. \end{aligned} \tag{47}$$

Then using (8), we infer

$$\begin{aligned} g(E, D\gamma + D(2\lambda+\triangle\gamma))\xi &= (\frac{r}{2}-3)\phi E - 2(r-6)trace\phi[E+\eta(E)\xi] \\ &+\{(\xi\gamma)+\xi(2\lambda+\triangle\gamma)\}E. \end{aligned} \tag{48}$$

Executing the inner product of the previous equation with ξ gives

$$E(\gamma+(2\lambda+\triangle\gamma)) = \{(\xi\gamma)+\xi(2\lambda+\triangle\gamma)\}\eta(E), \tag{49}$$

from which easily we obtain

$$d(\gamma+(2\lambda+\triangle\gamma)) = \{(\xi\gamma)+\xi(2\lambda+\triangle\gamma)\}\eta, \tag{50}$$

where d indicates the exterior derivative. From the previous equation we see that $\gamma+(2\lambda+\triangle\gamma)$ is invariant along the distribution $\mathcal{D}$. In other terms, $E(\gamma+(2\lambda+\triangle\gamma)) = 0$ for any $E\in\mathcal{D}$. Using (49) in (48), we lead

$$\begin{aligned} &\{(\xi\gamma)+\xi(2\lambda+\triangle\gamma)\}[\eta(E)\xi - E] \\ &= (\frac{r}{2}-3)\phi E - 2(r-6)trace\phi[E+\eta(E)\xi]. \end{aligned} \tag{51}$$

Contracting the above equation yields

$$\{(\xi\gamma)+\xi(2\lambda+\triangle\gamma)\} = 0. \tag{52}$$

Utilizing (52) in (51), we get

$$(r-6)\{\phi E - 4trace\phi[E+\eta(E)\xi]\} = 0. \tag{53}$$

If $\{\phi E - 4trace\phi[E+\eta(E)\xi]\} = 0$, operating ϕ we can easily obtain $\phi^2 E = 4trace\phi\ (\phi E)$, which is obviously a contradiction. Thus we have $r = 6$. Hence by Lemma 2.2, the manifold is a space form.

Hence we write the following:

Theorem 4.2. *If a LP-Sasakian manifold M^3 admits a gradient ARS, then the manifold is a space form.*

5. Example

Here we consider a known example of our paper [3]. In this article, we considers a 3-dimensional manifold $M = \{(u,v,w)\in\mathbb{R}^3, w\neq 0\}$ and The vector fields

$$e_1 = e^w\frac{\partial}{\partial v},\quad e_2 = e^w(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}),\quad e_3 = \frac{\partial}{\partial w}$$

are linearly independent at each point of M and shows that the manifold is a LP-Sasakian manifold. Further, the well-known Koszul's formula gives

$$\begin{aligned} &\nabla_{\delta_1}\delta_1 = -\delta_3,\quad \nabla_{\delta_1}\delta_2 = 0,\quad \nabla_{\delta_1}\delta_3 = -\delta_1, \\ &\nabla_{\delta_2}\delta_1 = 0,\quad \nabla_{\delta_2}\delta_2 = -\delta_3,\quad \nabla_{\delta_2}\delta_3 = -\delta_2, \\ &\nabla_{\delta_3}\delta_1 = 0,\quad \nabla_{\delta_3}\delta_2 = 0,\quad \nabla_{\delta_3}\delta_3 = 0. \end{aligned} \tag{54}$$

Also, we have obtained the expressions of the curvature tensor and the Ricci tensor, respectively, as follows:

$$R(\delta_1,\delta_2)\delta_3 = 0, \quad R(\delta_2,\delta_3)\delta_3 = -\delta_2, \quad R(\delta_1,\delta_3)\delta_3 = -\delta_1,$$
$$R(\delta_1,\delta_2)\delta_2 = \delta_1, \quad R(\delta_2,\delta_3)\delta_2 = -\delta_3, \quad R(\delta_1,\delta_3)\delta_2 = 0,$$
$$R(\delta_1,\delta_2)\delta_1 = -\delta_2, \quad R(\delta_2,\delta_3)\delta_1 = 0, \quad R(\delta_1,\delta_3)\delta_1 = -\delta_3,$$

and

$$\begin{aligned} S(\delta_1,\delta_1) &= g(R(\delta_1,\delta_2)\delta_2,\delta_1) - g(R(\delta_1,\delta_3)\delta_3,\delta_1) \\ &= 2. \end{aligned}$$

Similarly we have

$$S(\delta_2,\delta_2) = 2, S(\delta_3,\delta_3) = -2$$

and

$$S(\delta_i,\delta_j) = 0 (i \neq j).$$

Therefore,

$$r = S(\delta_1,\delta_1) + S(\delta_2,\delta_2) - S(\delta_3,\delta_3) = 6.$$

From the expressions of the Ricci tensor, we find that M is an Einstein manifold.

Suppose $f : M^3 \to \mathbb{R}$ be a smooth function such that $f = w$. Then we can obtain

$$Df = \frac{\partial}{\partial w} = \delta_3.$$

Using (54) we get

$$Hess f(\delta_3,\delta_3) = 0.$$

Thus from (2) we can easily see that g is a gradient Riemann soliton with $f = w$ and $\lambda = -1$. Hence the **Theorem 4.2.** is verified.

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