Filomat 35:11 (2021), 3733–3743
https://doi.org/10.2298/FIL2111733A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filter Bornological Convergence in Topological Vector Spaces

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Abstract. The concept of ε-enlargement defined on metric spaces is generalized to the concept of U-enlargement by using neighborhoods U of the zero of the space on topological vector spaces. By using U-enlargement, we define the bornological convergence for nets of sets in topological vector spaces and we examine some of their properties. By using filters defined on natural numbers, we define the concept of filter bornological convergence on sequences of sets, which is a more general concept than the bornological convergence defined on topological vector spaces. We give similar results for the filter bornological convergence.

1. Introduction

Let X be a vector space on the real numbers field $\mathbb{R}$ and let τ be a topology on X. The topology τ is said to be a linear topology on the vector space X if the operations addition and scalar multiplication are τ-continuous. Then (X, τ) is called a topological vector space (see [15, 16, 23]). In this study, we briefly denote the topological vector space as TVS and indicate the zero of the space by θ. Each linear topologies on a vector space X has a base $\mathcal{N}$ of neighbourhoods of zero, providing the following properties:

(a) Each $V \in \mathcal{N}$ is a *balanced set* (i.e., $\lambda x \in V$ for each $x \in V$ and each $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$).

(b) Each $V \in \mathcal{N}$ is an *absorbing set* (i.e., for each $x \in X$ there is a $\lambda > 0$ such that $\lambda x \in V$).

(c) For each $V \in \mathcal{N}$ there is a set $W \in \mathcal{N}$ such that $W + W \subseteq V$. Here, the operation $W + W$ is defined as $W + W := \{x + y : x, y \in W\}$ (see [15, 16, 23]).

As an alternative to the ordinary convergence of sequences, a number of convergence methods, primarily statistical convergence, have been defined since 1951 and still continue to defined new convergence methods. The most common ones of these types of convergence are the ideal convergence and the filter convergence which are dual to each other.

Definition 1.1. Let $\mathcal{F}$ be a family of subsets of $\mathbb{N}$ and let $\mathcal{F} \neq \emptyset$. The family $\mathcal{F}$ is said to be a filter on $\mathbb{N}$, if it provides the following conditions (see [10, 25]):

i) $\emptyset \notin \mathcal{F}$,

ii) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$,

iii) If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

2020 *Mathematics Subject Classification*. Primary 40A35; Secondary 46A17, 54A20, 57N17
Keywords. Topological vector spaces, filter convergence, bornological convergence, filter bornological convergence
Received: 15 August 2020; Revised: 11 December 2020; Accepted: 15 December 2020
Communicated by Ljubiša D.R. Kočinac
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Definition 1.2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a topological space X, let $x_0 \in X$ and let $\mathcal{F}$ be a filter on $\mathbb{N}$. The sequence (x_n) is said to be filter convergent (or $\mathcal{F}$-convergent) to the point x_0, if for every neighborhood U of x_0 we have

$$\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F} \tag{1}$$

([1, 2, 14]). In this case, we write $\mathcal{F} - \lim x_n = x_0$ or briefly $x_n \xrightarrow{\mathcal{F}} x_0$.

In the following, we give some examples of filters and filter convergence. $|A|$ denotes the cardinality of the set A.

1. Fréchet Filter: The family $\mathcal{F}_r = \{A \subseteq \mathbb{N} : |\mathbb{N} \setminus A| < \infty\}$ is called the *Fréchet filter*. $\mathcal{F}_r$-convergence coincides with the ordinary convergence.

2. Statistical Convergence Filter: Let $A \subseteq \mathbb{N}$. Let $A(n) = |\{1, ..., n\} \cap A|$ indicate the number of elements in the set A from 1 to n. The functions

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{A(n)}{n} \text{ and } \overline{\delta}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$$

are called the *lower asymptotic density* and *upper asymptotic density* of the set A, respectively. If $\underline{\delta}(A) = \overline{\delta}(A)$, that is, the limit

$$\lim_{n\to\infty} \frac{A(n)}{n}$$

exists, then the value of this limit is called the *asymptotic density* of the set A, it is denoted by $\delta(A)$ ([5, 12, 18]). The family $\mathcal{F}_{st} = \{A \subseteq \mathbb{N} : \delta(A) = 1\}$ is called the *statistical convergence filter*. $\mathcal{F}_{st}$-convergence coincides with the statistical convergence ([9, 11, 22]).

In 1964, Wijsman ([24]) defined a new convergence on sequences of sets and later this convergence was called Wijsman convergence. Nuray and Rhoades ([19]) introduced Kuratowski, Wijsman and Hausdorff statistical convergences of sequences of sets. Sağıroğlu and Ünver ([20]) gave some results about the statistical convergence of sequences of sets in Wijsman topology. Savaş ([21]) gave some results about $\mathcal{I}$-lacunary statistical convergence of order α for sequences of sets.

In recent years, there has been an increasing interest on bornological spaces, bornological convergence and different types of convergence on bornologies. In [4], Beer and Levi introduced the concept of strong uniform convergence on a bornology (see also [3, 6, 7]). In [3], Beer defined the bornological Alexandroff property for function nets and gave the relationship between the bornological Alexandroff property and strong uniform convergence on bornology. In [6], Caserta et al. gave the relationship between Alexandroff convergence and strong uniform convergence on a bornology.

Now, we recall the concept of bornological convergence on metric spaces.

Definition 1.3. Let $X = (X, d)$ be a metric space. Let $d(x, A)$ denote the distance from the set A to a point $x \in X$, where A is a non-empty subset of X. Also, let $B(x, \varepsilon)$ indicate the open ball centered at x of radius ε. The set

$$A^{\varepsilon} = \{x : d(x, A) < \varepsilon\} = \bigcup_{x\in A} B(x, \varepsilon)$$

is called the ε-enlargement of the set A (see [8, 17]).

Definition 1.4. A family $\mathcal{B}$ of subsets of a set X is said to be a bornology, if it provides the following conditions ([4, 13]):

i) $\mathcal{B}$ is a cover of X, i.e. $X = \bigcup_{\check{B}\in\mathcal{B}} \check{B}$,

ii) $\mathcal{B}$ is closed under subsets, i.e. $\check{B} \in \mathcal{B}$ and $\check{A} \subseteq \check{B} \Longrightarrow \check{A} \in \mathcal{B}$,

iii) $\mathcal{B}$ is closed under finite unions, i.e. $\check{A}, \check{B} \in \mathcal{B} \Longrightarrow (\check{A} \cup \check{B}) \in \mathcal{B}$.

Definition 1.5. Let X be a real vector space and let $\mathcal{B}$ be a bornology on X. $\mathcal{B}$ is called a vector bornology, if it provides the following conditions ([13]):

i) If $\check{B}_1, \check{B}_2 \in \mathcal{B}$ then $\check{B}_1 + \check{B}_2 \in \mathcal{B}$ where $\check{B}_1 + \check{B}_2 = \left\{x_1 + x_2 : x_1 \in \check{B}_1, x_2 \in \check{B}_2\right\}$

ii) If $\check{B} \in \mathcal{B}$ then $\lambda\check{B} \in \mathcal{B}$ for every $\lambda \in \mathbb{R}$ and $\bigcup_{|\lambda|\leq 1} \lambda\check{B} \in \mathcal{B}$ where $\lambda\check{B} = \left\{\lambda x : x \in \check{B}\right\}$.

Example 1.6. The following families are bornologies on X.

1) The power set $\mathcal{P}(X)$.

2) $\mathcal{B}_f = \left\{\check{B} \subseteq X : \check{B} \text{ is finite}\right\}$.

3) $\mathcal{B}_b = \left\{\check{B} \subseteq X : \check{B} \text{ is bounded}\right\}$.

Definition 1.7. Let (X, d) be a metric space and let the family $\mathcal{B}$ be a bornology on X. Take a net $(A_\lambda)_{\lambda\in\Lambda}$ of non-empty subsets of X and a set $A \subseteq X$.

i) The net $(A_\lambda)_{\lambda\in\Lambda}$ is said to be lower bornological convergent to the set A, if for each $\varepsilon > 0$ and each $\check{B} \in \mathcal{B}$ there is a $\lambda_0 \in \Lambda$ such that we have

$$A \cap \check{B} \subseteq A_\lambda^\varepsilon \text{ for every } \lambda \geq \lambda_0.$$

Then we write $\mathcal{B}^- - \lim A_\lambda = A$.

ii) The net $(A_\lambda)_{\lambda\in\Lambda}$ is said to be upper bornological convergent to the set A, if for each $\varepsilon > 0$ and each $\check{B} \in \mathcal{B}$ there is a $\lambda_0 \in \Lambda$ such that we have

$$A_\lambda \cap \check{B} \subseteq A^\varepsilon \text{ for every } \lambda \geq \lambda_0.$$

Then we write $\mathcal{B}^+ - \lim A_\lambda = A$.

iii) If the net $(A_\lambda)_{\lambda\in\Lambda}$ is both lower bornological convergent and upper bornological convergent to the set A then the net $(A_\lambda)_{\lambda\in\Lambda}$ is called bornological convergent to the set A. In this case, it is denoted by $\mathcal{B} - \lim A_\lambda = A$ (see [8, 17]).

2. Bornological Convergence in Topological Vector Spaces

In this section, we extend the concept of ε-enlargement for sets in metric spaces to the U-enlargement concept in topological vector spaces and thus we define the concept of bornological convergence in topological vector spaces. We give some basic results about bornological convergence on topological vector spaces.

Definition 2.1. Let (X, τ) be a TVS and let U be a neighborhood of θ. For $A \subseteq X$, the set

$$A^U = \{x \in X : x - y \in U \text{ for some } y \in A\}$$

is called U-enlargement of the set A.

Some Properties of U-enlargement

Let (X, τ) be a TVS. Let U be a neighborhood of θ and let $A, B \subseteq X$.

1) If $A \subseteq B$ then $A^U \subseteq B^U$.

2) $(A \cup B)^U = A^U \cup B^U$.

3) $(A \cap B)^U \subseteq A^U \cap B^U$.

4) Let U_1 and U_2 be two neighborhoods of θ. The following implication is provided:

$$U_1 \subseteq U_2 \Longrightarrow A^{U_1} \subseteq A^{U_2}.$$

Definition 2.2. Let (X, τ) be a TVS and let $\mathcal{B}$ be a bornology on X. Take a net $(A_\lambda)_{\lambda\in\Lambda}$ of non-empty subsets of X and a set $A \subseteq X$.

i) The net $(A_\lambda)_{\lambda\in\Lambda}$ is said to be lower bornological convergent to the set A, if for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ there is a $\lambda_0 \in \Lambda$ such that we have

$$A \cap \check{B} \subseteq A_\lambda^U \text{ for every } \lambda \in \Lambda \text{ with } \lambda \geq \lambda_0$$

and then we write $\mathcal{B}^- - \lim A_\lambda = A$.

ii) The net $(A_\lambda)_{\lambda\in\Lambda}$ is said to be upper bornological convergent to the set A, if for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ there is a $\lambda_0 \in \Lambda$ such that we have

$$A_\lambda \cap \check{B} \subseteq A^U \text{ for every } \lambda \in \Lambda \text{ with } \lambda \geq \lambda_0$$

and then we write $\mathcal{B}^+ - \lim A_\lambda = A$.

iii) If the net $(A_\lambda)_{\lambda\in\Lambda}$ is both lower bornological convergent and upper bornological convergent to the set A then the net $(A_\lambda)_{\lambda\in\Lambda}$ is called bornological convergent to the set A. In this case, it is denoted by $\mathcal{B} - \lim A_\lambda = A$.

Example 2.3. Let $(\mathbb{R}, \tau)$ be a TVS endowed with Euclidean topology τ and let $\mathcal{B}_f$ be a bornology of all finite subsets of $\mathbb{R}$. The sequence $(A_n)_{n\in\mathbb{N}}$ with $A_n = \left[\frac{1}{n}, 1 + \frac{1}{n}\right]$ is bornological convergent to the set $A = [0, 1]$.

Indeed, let us take the neighborhoods $U = (-\varepsilon, \varepsilon) \in \mathcal{N}$ where $\varepsilon > 0$ and let $\check{B} \in \mathcal{B}_f$.

Firstly, we will show that $A \cap \check{B} \subseteq A_n^U$. The U-enlargement of A_n is $A_n^U = (\frac{1}{n} - \varepsilon, 1 + \frac{1}{n} + \varepsilon)$. If $x \in A \cap \check{B}$ then $0 \leq x \leq 1$. The right side of the inequality

$$\frac{1}{n} - \varepsilon < x < 1 + \frac{1}{n} + \varepsilon$$

is provided for all $n \in \mathbb{N}$. For the left side, it must be

$$x > \frac{1}{n} - \varepsilon \Longrightarrow n > \frac{1}{x + \varepsilon}.$$

Take $N(x, \varepsilon) = \left[\!\left[\frac{1}{x+\varepsilon} \right]\!\right] + 1 \in \mathbb{N}$. $A \cap \check{B}$ is finite set and so let us take

$$N = \max_{x\in A\cap\check{B}} N(x, \varepsilon).$$

In this case, we get

$$\frac{1}{n} - \varepsilon < x < 1 + \frac{1}{n} + \varepsilon$$

for every $n \geq N$ and so $x \in A_n^U$. Thus we have $\mathcal{B}_f^- - \lim A_n = A$.

Now, we will show that $A_n \cap \check{B} \subseteq A^U$. The U-enlargement of A is $A^U = (-\varepsilon, 1 + \varepsilon)$. Let $x \in A_n \cap \check{B}$. For the inequality

$$-\varepsilon < \frac{1}{n} \leq x \leq 1 + \frac{1}{n} < 1 + \varepsilon,$$

it must be provided

$$\frac{1}{n} < \varepsilon \Longrightarrow n > \frac{1}{\varepsilon}.$$

Hence, we get

$$A_n \cap \check{B} \subseteq A^U$$

for every $n \geq N(\varepsilon)$ where $N(\varepsilon) = \left[\!\left[\frac{1}{\varepsilon} \right]\!\right] + 1 \in \mathbb{N}$. Thus we have $\mathcal{B}_f^+ - \lim A_n = A$.

Consequently, we have $\mathcal{B}_f - \lim A_n = A$.

Example 2.4. Take $\left(\mathbb{R}^2, \tau\right)$, where τ is the Euclidean topology . Let the sequence $(A_n)_{n\in\mathbb{N}}$ be defined as $A_n = \left\{(x, y) : y = \dfrac{x}{n}\right\}$ for each $n \in \mathbb{N}$. We get

$$\mathcal{B}_f - \lim A_n = A \text{ and } \mathcal{B}_b - \lim A_n = A$$

where $A = \{(x, y) : y = 0 \text{ and } x \in \mathbb{R}\}$.

In the following, we give an example of bornological convergence on a non-metrizable topological vector space.

Example 2.5. Let X be the set of all functions defined from $\mathbb{R}$ to $\mathbb{R}$, that is $X = \mathbb{R}^{\mathbb{R}}$, and τ_{pw} be the topology of pointwise convergence on X. It is known that $\left(X, \tau_{pw}\right)$ is a non-metrizable TVS. The zero element of this space is the zero function defined as $\theta(x) = 0$ for every $x \in \mathbb{R}$. The family of sets of the form

$$W(\theta, \mathtt{F}, \varepsilon) = \left\{g \in X : \left|g(x)\right| < \varepsilon, \forall x \in \mathtt{F}\right\}$$

for $\varepsilon > 0$ and a finite subset $\mathtt{F}$ of $\mathbb{R}$, forms a base at θ. Let $\mathcal{B}$ be any bornology on X. Let us take the net $(A_\lambda)_{\lambda\in\mathbb{R}}$ defined as $A_\lambda = \{\chi_\lambda\}$ for each $\lambda \in \mathbb{R}$ where

$$\chi_\lambda(x) = \begin{cases} 1 & \text{if } x = \lambda \\ 0 & \text{if } x \neq \lambda \end{cases}$$

• Let $A = \{\theta\}$. For every $\check{B} \in \mathcal{B}$, we have either $A \cap \check{B} = \emptyset$ or $A \cap \check{B} = \{\theta\}$. Let U be an arbitrary neighborhood of zero. Then there is an $\varepsilon > 0$ and a finite set $\mathtt{F}$ such that $U \supseteq W(\theta, \mathtt{F}, \varepsilon) = \left\{g \in X : \left|g(x)\right| < \varepsilon, \forall x \in \mathtt{F}\right\}$. Let us take $\lambda_0 := \max \mathtt{F}$. For every $\lambda > \lambda_0$,

$$|\theta(x) - \chi_\lambda(x)| = |0 - 0| = 0 < \varepsilon \text{ for every } x \in \mathtt{F}$$

$$\theta - \chi_\lambda \in W(\theta, \mathtt{F}, \varepsilon) \subseteq U$$

$$\theta \in A_\lambda^U = \{f \in X : f - \chi_\lambda \in U\}$$

Thus, we get $A \cap \check{B} \subseteq A_\lambda^U$ for every $\lambda > \lambda_0$.

Similarly, we have either $A_\lambda \cap \check{B} = \emptyset$ or $A_\lambda \cap \check{B} = \{\chi_\lambda\}$ for every $\check{B} \in \mathcal{B}$ and every $\lambda \in \mathbb{R}$. Let U be an arbitrary neighborhood of zero. Then there is an $\varepsilon > 0$ and a finite set $\mathtt{F}$ such that $U \supseteq W(\theta, \mathtt{F}, \varepsilon) = \left\{g \in X : \left|g(x)\right| < \varepsilon, \forall x \in \mathtt{F}\right\}$. Take $\lambda_0 := \max \mathtt{F}$. For every $\lambda > \lambda_0$,

$$|\chi_\lambda(x) - \theta(x)| = |0 - 0| = 0 < \varepsilon \text{ for every } x \in \mathtt{F}$$

$$\chi_\lambda - \theta \in W(\theta, \mathtt{F}, \varepsilon) \subseteq U$$

$$\chi_\lambda \in A^U = \{f \in X : f - \theta \in U\}$$

Thus, we get $A_\lambda \cap \check{B} \subseteq A^U$ for every $\lambda > \lambda_0$. Consequently, we have $\mathcal{B} - \lim A_\lambda = A$.

• Now, let's examine the bornological convergence to the set $A_{\lambda_*} = \{\chi_{\lambda_*}\}$ for a $\lambda_* \in \mathbb{R}$. For every $\check{B} \in \mathcal{B}$, we have either $A_{\lambda_*} \cap \check{B} = \emptyset$ or $A_{\lambda_*} \cap \check{B} = \{\chi_{\lambda_*}\}$. Let's choose a set $\check{B}_0 \in \mathcal{B}$ and a neighborhood $U_0 = W(\theta, \mathtt{F}, \varepsilon_0) = \left\{g \in X : \left|g(x)\right| < \varepsilon_0, \forall x \in \mathtt{F}\right\}$ of zero such that $\chi_{\lambda_*} \in \check{B}_0$, $\lambda_* \in \mathtt{F}$ (finite) and $\varepsilon_0 = \dfrac{1}{2}$. For every $\lambda \in \mathbb{R}$ with $\lambda \neq \lambda_*$,

$$\left|\chi_{\lambda_*}(\lambda_*) - \chi_\lambda(\lambda_*)\right| = |1 - 0| = 1 > \varepsilon_0 \text{ for } \lambda_* \in \mathtt{F}$$

$$\chi_{\lambda_*} - \chi_\lambda \notin U_0$$

$$\chi_{\lambda_*} \notin A_\lambda^{U_0} = \{f \in X : f - \chi_\lambda \in U_0 \text{ for } \chi_\lambda \in A_\lambda\}$$

Hence, we get $A_{\lambda_*} \cap \check{B}_0 \nsubseteq A_\lambda^{U_0}$ for every $\lambda \in \mathbb{R}$ with $\lambda \neq \lambda_*$, and so $\mathcal{B}^- - \lim A_\lambda \neq A_{\lambda_*}$. From this, we conclude that $\mathcal{B}^- - \lim A_\lambda \neq A_{\lambda_*}$ for every $\lambda_* \in \mathbb{R}$.

- Let us consider the bornology $\mathcal{B}_f$ on X. Take the set $A_{\lambda_*} = \{\chi_{\lambda_*}\}$ for a $\lambda_* \in \mathbb{R}$ again. Let $\check{B} \in \mathcal{B}_f$ and U be an arbitrary neighborhood of zero. Since $\check{B}$ is finite, the number of A_λ's with $A_\lambda \cap \check{B} \neq \emptyset$ is zero or finite. If it is zero then $A_\lambda \cap \check{B} \subseteq A_{\lambda_*}^U$ is provided for all $\lambda \in \mathbb{R}$. In other case, let λ_0 be the largest of the λ's that satisfy $A_\lambda \cap \check{B} \neq \emptyset$. Hence, for every $\lambda > \lambda_0$ we get $A_\lambda \cap \check{B} = \emptyset$ and so $A_\lambda \cap \check{B} \subseteq A_{\lambda_*}^U$. Therefore, we get $\mathcal{B}_f^+ - \lim A_\lambda = A_{\lambda_*}$. Consequently, we have $\mathcal{B}_f^+ - \lim A_\lambda = A_{\lambda_*}$ for each $\lambda_* \in \mathbb{R}$.

Theorem 2.6. *Let (X, τ) be a TVS and let $\mathcal{B}$ be a bornology on X. Let $(A_\lambda)_{\lambda \in \Lambda}$ be a net on X and let $A, B \subseteq X$.*
i) If $\mathcal{B}^- - \lim A_\lambda = A$ and $B \subseteq A$ then $\mathcal{B}^- - \lim A_\lambda = B$.
ii) If $\mathcal{B}^+ - \lim A_\lambda = A$ and $A \subseteq B$ then $\mathcal{B}^+ - \lim A_\lambda = B$.

Proof. i) Since $\mathcal{B}^- \lim A_\lambda = A$, for each $\check{B} \in \mathcal{B}$ and each $U \in \mathcal{N}$ there is a $\lambda_0 \in \Lambda$ such that

$$A \cap \check{B} \subseteq A_\lambda^U$$

for every $\lambda \geq \lambda_0$. From $B \subseteq A$, we get $B \cap \check{B} \subseteq A \cap \check{B}$ and so

$$B \cap \check{B} \subseteq A_\lambda^U.$$

Thus we have $\mathcal{B}^- \lim A_\lambda = B$.

ii) Since $\mathcal{B}^+ \lim A_\lambda = A$, for each $\check{B} \in \mathcal{B}$ and each $U \in \mathcal{N}$ there is a $\lambda_0 \in \Lambda$ such that

$$A_\lambda \cap \check{B} \subseteq A^U$$

for every $\lambda \geq \lambda_0$. From $A \subseteq B$, we get $A^U \subseteq B^U$ and so

$$A_\lambda \cap \check{B} \subseteq B^U.$$

Thus we have $\mathcal{B}^+ \lim A_\lambda = B$. □

Theorem 2.7. *Let (X, τ) be a TVS and let $\mathcal{B}$ be a bornology on X. Let $(A_\lambda)_{\lambda \in \Lambda}$ and $(B_\lambda)_{\lambda \in \Lambda}$ be two nets of sets on X and let $A, B \subseteq X$. If $\mathcal{B} - \lim A_\lambda = A$ and $\mathcal{B} - \lim B_\lambda = B$ then we have*

$$\mathcal{B} - \lim(A_\lambda \cup B_\lambda) = A \cup B.$$

Proof. Take $U \in \mathcal{N}$ and $\check{B} \in \mathcal{B}$. From $\mathcal{B} - \lim A_\lambda = A$, there is a $\lambda_1 \in \Lambda$ such that

$$A \cap \check{B} \subseteq A_\lambda^U \text{ and } A_\lambda \cap \check{B} \subseteq A^U \tag{2}$$

for every $\lambda \geq \lambda_1$. Similarly, from $\mathcal{B} - \lim A_\lambda = B$, there is a $\lambda_2 \in \Lambda$ such that

$$B \cap \check{B} \subseteq B_\lambda^U \text{ and } B_\lambda \cap \check{B} \subseteq B^U \tag{3}$$

for every $\lambda \geq \lambda_2$. Take $\lambda_0 = \sup\{\lambda_1, \lambda_2\}$. From (2) and (3), for every $\lambda \geq \lambda_0$ we get

$$\begin{aligned} (A \cap \check{B}) \cup (B \cap \check{B}) &\subseteq A_\lambda^U \cup B_\lambda^U \\ (A \cup B) \cap \check{B} &\subseteq (A_\lambda \cup B_\lambda)^U \end{aligned}$$

and

$$\begin{aligned} (A_\lambda \cap \check{B}) \cup (B_\lambda \cap \check{B}) &\subseteq A^U \cup B^U \\ (A_\lambda \cup B_\lambda) \cap \check{B} &\subseteq (A \cup B)^U \end{aligned}$$

From these two results, we have $\mathcal{B} - \lim(A_\lambda \cup B_\lambda) = A \cup B$. □

Theorem 2.8. *Let (X, τ) be a TVS and let $\mathcal{B}_1, \mathcal{B}_2$ be two bornologies on X where $\mathcal{B}_1 \subseteq \mathcal{B}_2$. If $\mathcal{B}_2 - \lim A_\lambda = A$ then $\mathcal{B}_1 - \lim A_\lambda = A$.*

Proof. Take $U \in \mathcal{N}$ and $\check{B} \in \mathcal{B}_1$. From $\mathcal{B}_1 \subseteq \mathcal{B}_2$, we have $\check{B} \in \mathcal{B}_2$. If $\mathcal{B}_2 - \lim A_\lambda = A$ then for $U \in \mathcal{N}$ and $\check{B} \in \mathcal{B}_2$ there is a $\lambda_0 \in \Lambda$ such that

$$A \cap \check{B} \subseteq A_\lambda^U \text{ and } A_\lambda \cap \check{B} \subseteq A^U$$

for every $\lambda \geq \lambda_0$. Since $U \in \mathcal{N}$ and $\check{B} \in \mathcal{B}_1$ are arbitrary sets, we also get $\mathcal{B}_1 - \lim A_\lambda = A$. □

Theorem 2.9. *Let (X, τ) be a TVS and let $\mathcal{B}$ be a bornology on X. Let $(A_\lambda)_{\lambda\in\Lambda}$, $(B_\lambda)_{\lambda\in\Lambda}$ and $(C_\lambda)_{\lambda\in\Lambda}$ be three nets of sets on X where*

$$A_\lambda \subseteq B_\lambda \subseteq C_\lambda$$

for every $\lambda \in \Lambda$. If

$$\mathcal{B} - \lim A_\lambda = \mathcal{B} - \lim C_\lambda = A$$

then we have

$$\mathcal{B} - \lim B_\lambda = A.$$

Proof. Take $U \in \mathcal{N}$ and $\check{B} \in \mathcal{B}$.
From $\mathcal{B}^- - \lim A_\lambda = A$, there is a $\lambda_1 \in \Lambda$ such that

$$A \cap \check{B} \subseteq A_\lambda^U \subseteq B_\lambda^U \tag{4}$$

for every $\lambda \geq \lambda_1$. From $\mathcal{B}^+ - \lim C_\lambda = A$, there is a $\lambda_2 \in \Lambda$ such that

$$C_\lambda \cap \check{B} \subseteq A^U$$

for every $\lambda \geq \lambda_2$, and from $B_\lambda \subseteq C_\lambda$ we get

$$B_\lambda \cap \check{B} \subseteq A^U \tag{5}$$

for every $\lambda \geq \lambda_2$. From (4) and (5), we have $\mathcal{B} - \lim B_\lambda = A$. □

Theorem 2.10. *Let (X, τ_1) and (X, τ_2) be two TVS where $\tau_1 \subseteq \tau_2$ and let $\mathcal{B}$ be a bornology on X. Let $(A_\lambda)_{\lambda\in\Lambda}$ be a net on X and let $A \subseteq X$. If $\mathcal{B} - \lim A_\lambda = A$ in (X, τ_2) then $\mathcal{B} - \lim A_\lambda = A$ in (X, τ_1).*

Proof. Let θ_1 and θ_2 be the zero of (X, τ_1) and (X, τ_2), respectively. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be a base of neighborhoods of θ_1 and θ_2, respectively. Let $\mathcal{B} - \lim A_\lambda = A$ in (X, τ_2). Take $\check{B} \in \mathcal{B}$ and $U_1 \in \mathcal{N}_1$. From $\tau_1 \subseteq \tau_2$, there is an $U_2 \in \mathcal{N}_2$ such that $U_2 \subseteq U_1$. From $\mathcal{B} - \lim A_\lambda = A$ in (X, τ_2), there is a $\lambda_0 \in \Lambda$ such that

$$A \cap \check{B} \subseteq A_\lambda^{U_2} \text{ and } A_\lambda \cap \check{B} \subseteq A^{U_2}$$

for every $\lambda \geq \lambda_0$. From $U_2 \subseteq U_1$, we have $A_\lambda^{U_2} \subseteq A_\lambda^{U_1}$ and $A^{U_2} \subseteq A^{U_1}$. Then we get

$$A \cap \check{B} \subseteq A_\lambda^{U_1} \text{ and } A_\lambda \cap \check{B} \subseteq A^{U_1}$$

for every $\lambda \geq \lambda_0$. Consequently, we have $\mathcal{B} - \lim A_\lambda = A$ in (X, τ_1). □

Theorem 2.11. *Let (X, τ) be a TVS and let $\mathcal{B}$ be a bornology on X. Let $(\Lambda, \preceq)$ and $(M, \preccurlyeq)$ be two directed sets. Let $(A_\lambda)_{\lambda\in\Lambda}$ be a net on X, let $\left(A_{\lambda_\mu}\right)_{\mu\in M}$ be a subnet of (A_λ) and let $A \subseteq X$. If $\mathcal{B} - \lim A_\lambda = A$ then $\mathcal{B} - \lim A_{\lambda_\mu} = A$.*

Proof. Take $\check{B} \in \mathcal{B}$ and $U \in \mathcal{N}$. If $\mathcal{B} - \lim A_\lambda = A$ then there is a $\lambda_0 \in \Lambda$ such that

$$A \cap \check{B} \subseteq A_\lambda^U \text{ and } A_\lambda \cap \check{B} \subseteq A^U$$

for every $\lambda \geq \lambda_0$. There is a $\lambda_{\mu_0} \in \Lambda$ such that $\lambda_0 \preceq \lambda_{\mu_0}$ and $\mu_0 \in M$. Then for every $\mu \succcurlyeq \mu_0$ we have $\lambda_\mu \geq \lambda_{\mu_0} \geq \lambda_0$ and so

$$A \cap \check{B} \subseteq A_{\lambda_\mu}^U \text{ and } A_{\lambda_\mu} \cap \check{B} \subseteq A^U.$$

Consequently, we have $\mathcal{B} - \lim A_{\lambda_\mu} = A$. □

3. Filter Bornological Convergence

The concept of bornological convergence in the previous section is generalized to the concept of filter bornological convergence by means of filters defined on natural numbers in this section. Some results are given on the filter bornological convergence.

Definition 3.1. Let (X, τ) be a TVS, let $\mathcal{B}$ be a bornology on X and let $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of non-empty subsets of X and let $A \subseteq X$.

i) The sequence $(A_n)_{n\in\mathbb{N}}$ is said to be filter lower bornological convergent to the set A, if for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ we have

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U\right\} \in \mathcal{F}$$

and then we write $\mathcal{F}\mathcal{B}^{-}\text{-}\lim A_n = A$.

ii) The sequence $(A_n)_{n\in\mathbb{N}}$ is said to be filter upper bornological convergent to the set A, if for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ we have

$$\left\{n \in \mathbb{N} : A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}$$

and then we write $\mathcal{F}\mathcal{B}^{+}\text{-}\lim A_n = A$.

iii) If the sequence $(A_n)_{n\in\mathbb{N}}$ is both filter lower bornological convergent and filter upper bornological convergent to the set A, that is, for each neighborhood U of θ and each $\check{B} \in \mathcal{B}$ we have

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}$$

then the sequence $(A_n)_{n\in\mathbb{N}}$ is called filter bornological convergent to the set A. In this case, it is denoted by $\mathcal{F}\mathcal{B} - \lim A_n = A$.

When the Fréchet filter $\mathcal{F}_r$ is considered, $\mathcal{F}_r\mathcal{B}$−convergence is equivalent to bornological convergence on the sequence of sets.

Example 3.2. Take a sequence $(A_n)_{n\in\mathbb{N}}$ defined as

$$A_n = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : |x| + |y| = 1\right\} & , n \in \mathbb{P} \\ \left\{(x, y) \in \mathbb{R}^2 : |x| + \left|ny - n^2\right| = n^2\right\} & , n \notin \mathbb{P} \end{cases}$$

and take $A = \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\}$.

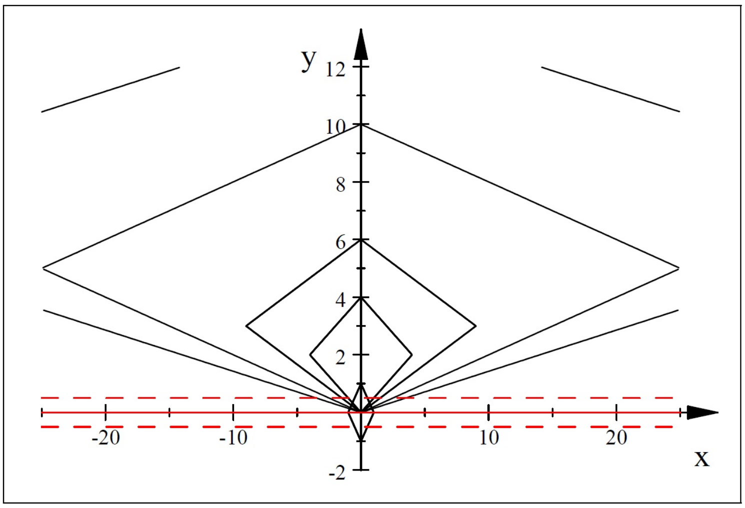

Figure 1:

The sequence $(A_n)_{n\in\mathbb{N}}$ is not $\mathcal{B}_f$-convergent and $\mathcal{B}_b$-convergent to A, but we have $\mathcal{F}_{st}\mathcal{B}_f - \lim A_n = A$ and $\mathcal{F}_{st}\mathcal{B}_b - \lim A_n = A$.

Theorem 3.3. *Let (X,τ) be a TVS, let $\mathcal{B}$ be a bornology on X and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filter on $\mathbb{N}$, where $\mathcal{F}_1 \subseteq \mathcal{F}_2$. If $\mathcal{F}_1\mathcal{B} - \lim A_n = A$ then $\mathcal{F}_2\mathcal{B} - \lim A_n = A$.*

Proof. Take $\check{B} \in \mathcal{B}$ and $U \in \mathcal{N}$. From $\mathcal{F}_1\mathcal{B} - \lim A_n = A$, we have

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}_1.$$

From $\mathcal{F}_1 \subseteq \mathcal{F}_2$, we get

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}_2.$$

Thus we get $\mathcal{F}_2\mathcal{B} - \lim A_n = A$. □

Theorem 3.4. *Let (X,τ) be a TVS, let $\mathcal{B}$ be a bornology on X and let $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $A \subseteq B$.*
i) If $\mathcal{F}\mathcal{B}^- - \lim A_n = B$ then $\mathcal{F}\mathcal{B}^- - \lim A_n = A$.
ii) If $\mathcal{F}\mathcal{B}^+ - \lim A_n = A$ then $\mathcal{F}\mathcal{B}^+ - \lim A_n = B$.

Proof. i) Let us assume that $\mathcal{F}\mathcal{B}^- - \lim A_n = B$. Take $\check{B} \in \mathcal{B}$ and $U \in \mathcal{N}$. Then we have

$$\left\{n \in \mathbb{N} : B \cap \check{B} \subseteq A_n^U\right\} \in \mathcal{F}.$$

From $A \subseteq B$, we get

$$\left\{n \in \mathbb{N} : B \cap \check{B} \subseteq A_n^U\right\} \subseteq \left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U\right\}$$

and so

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U\right\} \in \mathcal{F}.$$

Thus we get $\mathcal{F}\mathcal{B}^- - \lim A_n = A$.

ii) Let us assume that $\mathcal{F}\mathcal{B}^+ - \lim A_n = A$. Take $\check{B} \in \mathcal{B}$ and $U \in \mathcal{N}$. Then we have

$$\left\{n \in \mathbb{N} : A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}.$$

From $A \subseteq B$, we get

$$\left\{n \in \mathbb{N} : A_n \cap \check{B} \subseteq A^U\right\} \subseteq \left\{n \in \mathbb{N} : A_n \cap \check{B} \subseteq B^U\right\}$$

and so

$$\left\{n \in \mathbb{N} : A_n \cap \check{B} \subseteq B^U\right\} \in \mathcal{F}$$

Thus we get $\mathcal{F}\mathcal{B}^+ - \lim A_n = B$. □

Theorem 3.5. *Let (X,τ) be a TVS, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bornologies on X where $\mathcal{B}_1 \subseteq \mathcal{B}_2$, and let $\mathcal{F}$ be a filter on $\mathbb{N}$. If $\mathcal{F}\mathcal{B}_2 - \lim A_n = A$ then $\mathcal{F}\mathcal{B}_1 - \lim A_n = A$.*

Proof. Take $\check{B} \in \mathcal{B}_1$ and $U \in \mathcal{N}$. If $\mathcal{B}_1 \subseteq \mathcal{B}_2$ then $\check{B} \in \mathcal{B}_2$. From $\mathcal{F}\mathcal{B}_2 - \lim A_n = A$, we have

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U\right\} \in \mathcal{F}.$$

Then for each $\check{B} \in \mathcal{B}_1$ and each $U \in \mathcal{N}$, the above set is an element of the filter $\mathcal{F}$. We get $\mathcal{F}\mathcal{B}_1 - \lim A_n = A$. □

Theorem 3.6. *Let (X, τ_1) and (X, τ_2) be two TVS where $\tau_1 \subseteq \tau_2$, let $\mathcal{B}$ be a bornology on X and let $\mathcal{F}$ be a filter on $\mathbb{N}$. If $\mathcal{F}\mathcal{B} - \lim A_n = A$ in (X, τ_2) then $\mathcal{F}\mathcal{B} - \lim A_n = A$ in (X, τ_1).*

Proof. Let θ_1 and θ_2 be the zero of (X, τ_1) and (X, τ_2), respectively. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be a base of neighborhoods of θ_1 and θ_2, respectively. Let $\mathcal{F}\mathcal{B} - \lim A_n = A$ in (X, τ_2). Take $\check{B} \in \mathcal{B}$ and $U_1 \in \mathcal{N}_1$. From $\tau_1 \subseteq \tau_2$, there is an $U_2 \in \mathcal{N}_2$ such that $U_2 \subseteq U_1$. From $\mathcal{F}\mathcal{B} - \lim A_n = A$ in (X, τ_2), we have

$$F_2 = \left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^{U_2} \text{ and } A_n \cap \check{B} \subseteq A^{U_2}\right\} \in \mathcal{F}.$$

From $U_2 \subseteq U_1$, we have $A_n^{U_2} \subseteq A_n^{U_1}$ for every $n \in \mathbb{N}$ and $A^{U_2} \subseteq A^{U_1}$. We get $F_2 \subseteq F_1$ where

$$F_1 = \left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^{U_1} \text{ and } A_n \cap \check{B} \subseteq A^{U_1}\right\}.$$

From $F_2 \in \mathcal{F}$, it must be $F_1 \in \mathcal{F}$. Consequently, we have $\mathcal{F}\mathcal{B} - \lim A_n = A$ in (X, τ_1). □

Theorem 3.7. *Let (X, τ) be a TVS, let $\mathcal{B}$ be a bornology on X and let $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$,$(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ be three sequences of sets on X where*

$$A_n \subseteq B_n \subseteq C_n$$

for every $n \in \mathbb{N}$. If

$$\mathcal{F}\mathcal{B} - \lim A_n = \mathcal{F}\mathcal{B} - \lim C_n = A$$

then we have

$$\mathcal{F}\mathcal{B} - \lim B_n = A.$$

Proof. Take $U \in \mathcal{N}$ and $\check{B} \in \mathcal{B}$. From $\mathcal{F}\mathcal{B} - \lim A_n = A$, we have

$$F_1 = \left\{n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^{U} \text{ and } A_n \cap \check{B} \subseteq A^{U}\right\} \in \mathcal{F}.$$

Similarly, from $\mathcal{F}\mathcal{B} - \lim C_n = A$, we have

$$F_2 = \left\{n \in \mathbb{N} : A \cap \check{B} \subseteq C_n^{U} \text{ and } C_n \cap \check{B} \subseteq A^{U}\right\} \in \mathcal{F}.$$

Let $F_3 = F_1 \cap F_2 \in \mathcal{F}$. We get

$$A \cap \check{B} \subseteq A_n^U \subseteq B_n^U \text{ and } B_n \cap \check{B} \subseteq C_n \cap \check{B} \subseteq A^U$$

for each $n \in \mathcal{F}_3$. Then we get

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq B_n^{U} \text{ and } B_n \cap \check{B} \subseteq A^{U}\right\} \supseteq F_3 \in \mathcal{F}$$

and so

$$\left\{n \in \mathbb{N} : A \cap \check{B} \subseteq B_n^{U} \text{ and } B_n \cap \check{B} \subseteq A^{U}\right\} \in \mathcal{F}.$$

Consequently, we have $\mathcal{F}\mathcal{B} - \lim B_n = A$. □

Theorem 3.8. *Let (X, τ) be a TVS, let $\mathcal{B}$ be a bornology on X and let $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be two sequences of non-empty subsets of X and let $A, B \subseteq X$. If $\mathcal{F}\mathcal{B} - \lim A_n = A$ and $\mathcal{F}\mathcal{B} - \lim B_n = B$ then we have*

$$\mathcal{F}\mathcal{B} - \lim(A_n \cup B_n) = A \cup B.$$

Proof. Take $U \in \mathcal{N}$ and $\check{B} \in \mathcal{B}$. From $\mathcal{FB} - \lim A_n = A$, we have

$$F_1 = \left\{ n \in \mathbb{N} : A \cap \check{B} \subseteq A_n^U \text{ and } A_n \cap \check{B} \subseteq A^U \right\} \in \mathcal{F}.$$

Similarly, from $\mathcal{FB} - \lim B_n = B$, we have

$$F_2 = \left\{ n \in \mathbb{N} : B \cap \check{B} \subseteq B_n^U \text{ ve } B_n \cap \check{B} \subseteq B^U \right\} \in \mathcal{F}.$$

Let $F = F_1 \cap F_2$. For each $n \in F$ we get

$$\begin{aligned} (A \cap \check{B}) \cup (B \cap \check{B}) &\subseteq A_n^U \cup B_n^U \\ (A \cup B) \cap \check{B} &\subseteq (A_n \cup B_n)^U \end{aligned}$$

and

$$\begin{aligned} (A_n \cap \check{B}) \cup (B_n \cap \check{B}) &\subseteq A^U \cup B^U \\ (A_n \cup B_n) \cap \check{B} &\subseteq (A \cup B)^U. \end{aligned}$$

Then we get

$$\left\{ n \in \mathbb{N} : (A \cup B) \cap \check{B} \subseteq (A_n \cup B_n)^U \text{ and } (A_n \cup B_n) \cap \check{B} \subseteq (A \cup B)^U \right\} \supseteq F \in \mathcal{F}$$

and so

$$\left\{ n \in \mathbb{N} : (A \cup B) \cap \check{B} \subseteq (A_n \cup B_n)^U \text{ and } (A_n \cup B_n) \cap \check{B} \subseteq (A \cup B)^U \right\} \in \mathcal{F}.$$

Thus we have $\mathcal{FB} - \lim(A_n \cup B_n) = A \cup B$. □

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