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Topologically Stable Equicontinuous Non-Autonomous Systems

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Abstract. We obtain sufficient conditions for commutative non-autonomous systems on certain metric spaces (not necessarily compact) to be topologically stable. In particular, we prove that: (i) Every mean equicontinuous, mean expansive system with strong average shadowing property is topologically stable. (ii) Every equicontinuous, recurrently expansive system with eventual shadowing property is topologically stable. (iii) Every equicontinuous, expansive system with shadowing property is topologically stable.

1. Introduction

In experiments, it is seldom possible to measure a physical quantity without any error. Therefore only those properties that are unchanged under small perturbations are physically relevant. In topological dynamics, a meaningful way to perturb a system is through a continuous map.

A homeomorphism (resp. continuous map) f on a metric space X is said to be topologically stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if h is another homeomorphism (resp. continuous map) on X satisfying $d(f(x), h(x)) < \delta$, for each $x \in X$, then there exists a continuous map $k : X \to X$ satisfying $f \circ k = k \circ h$ and $d(k(x), x) < \epsilon$, for each $x \in X$.

This particular concept of stability is popularly known as topological stability which was originally introduced for a diffeomorphism on compact smooth manifold [15]. By looking at the significance of the concept, it is worth to identify those dynamical properties which imply topological stability. One of such result in topological dynamics is "Walters stability theorem" which states that expansive homeomorphisms with shadowing property on compact metric spaces are topologically stable [16].

The notion of expansivity expresses the worse case unpredictability of a system. Although such unpredictable behaviour of symbolic flows was recognized earlier, the concept of expansivity for homeomorphisms on general metric spaces was introduced in the middle of the twentieth century [14]. The expansive behaviour of continuous maps is popularly known as positive expansivity [4].

For a continuous map f on a metric space X and fixed $x_0 \in X$, identifying those $x \in X$ whose orbit follow that of x_0 for a long time and hence understanding the asymptotic behaviour of $f^n(x)$ relative to $f^n(x_0)$ can provide deep insight of the system. Anosov closing lemma provides us with such information for a differentiable map on compact smooth manifold [1]. The notion of shadowing property originated

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from this lemma in differentiable dynamics has played a significant role in topological dynamics. The idea behind the notion of shadowing is to guarantee the existence of an actual orbit with a particular behaviour by giving evidence of the existence of a pseudo orbit with the same behaviour. For general qualitative study on shadowing property, one may refer to [2].

In recent time, the study of non-autonomous systems has gained significant attention with the publication of [7]. Such systems occur as mathematical models of real-life problems affected by two or more (possibly distinct) external forces in different time span. These systems appear in various branches like informatics, quantum mechanics, biology etcetera. The earliest known example with connection to biology arise while solving the famous "mathematical rabbit problem" which was appeared in the book "Liber Abaci" written by Fibonacci in the year 1202. This problem can be written in the form of second order difference equation [5] which is also known as Fibonacci sequence and in this representation every state depends explicitly on the current time and therefore it can be seen from the context of non-autonomous systems. Because of such occurrences of non-autonomous systems in practical problems, it is important to know about stability of such systems.

In [13], authors have studied expansivity and shadowing property in the context of non-autonomous systems and proved a version of "Walters stability theorem" in this setting. In this paper, we prove the following results which provide sufficient conditions for non-autonomous systems to be topologically stable.

Let F be a commutative non-autonomous system on a Mandelkern locally compact metric space. Then the following statements hold:

(i) If F is mean equicontinuous, mean expansive and has strong average shadowing property, then F is topologically stable.

(ii) If F is equicontinuous, recurrently expansive and has eventual shadowing property, then F is topologically stable.

(iii) If F is equicontinuous, expansive and has shadowing property, then F is topologically stable.

The first part of the above result shows that the average shadowing property is not only useful in the investigation of chaos [3] but also in the investigation of stability. Although several notions that describes the chaotic behaviour of commutative non-autonomous systems have been studied in [10–12] recently, the usefulness of average shadowing property in the investigation of chaos in such systems is still to be understood. The second part shows the connection of topological stability with a weaker form of shadowing property called eventual shadowing property which is useful when one is not interested in the initial behaviour of the system. The final part improves the conclusion of [13, Theorem 4.1], but under a stronger hypothesis.

Before proving these results we introduce the above mentioned notions and study their general properties.

2. Definitions and General Properties

Throughout this paper $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^+$ denote the set of all integers, non-negative integers and positive integers, respectively. A pair (X, d) denotes a metric space with the metric d and if no confusion arises of the concerned metric d, then we simply write X is a metric space. All maps between metric spaces are assumed to be uniformly continuous. A map f between metric spaces is said to be uniform equivalence if both f and f^{-1} are uniformly continuous. Let $\{f_i : X \to X\}_{i\in\mathbb{N}^+}$ be a sequence of maps on X. The family $F = \{f_i\}_{i\in\mathbb{N}^+}$ is said to be a non-autonomous system (NAS) on X generated by $\{f_i\}_{i\in\mathbb{N}^+}$. Moreover, F is said to be an autonomous system generated by a continuous map f on X if $f_i = f$, for each $i \in \mathbb{N}^+$ and in this case, we simply write $F = \langle f\rangle$. The NAS F is said to be periodic if there exists $m \in \mathbb{N}^+$ such that if $f_{mi+j} = f_j$, for each pair $i, j \in \mathbb{N}^+$ and the smallest such m is called the period of F. The NAS F is said to be commutative if $f_i \circ f_j = f_j \circ f_i$, for each pair $i, j \in \mathbb{N}^+$. The NAS F is called surjective if f_i is surjective, for each $i \in \mathbb{N}^+$.

The set of all NAS on X, NAS of period m on X and commutative NAS on X are denoted by $N(X)$, $N_m(X)$ and $NC(X)$, respectively.

For $F \in N(X)$ and $n \in \mathbb{N}$, we define $F_n = f_n \circ f_{n-1} \circ ... f_1 \circ f_0$, where f_0 denotes the identity map on X. For $0 \leq j \leq k$, we define $F_{[j,k]} = f_k \circ f_{k-1} \circ ... \circ f_{j+1} \circ f_j$. For $k \in \mathbb{N}^+$, the k^{th} iterate of F is given by $F^k = \{F_{[(i-1)k+1,ik]}\}_{i \in \mathbb{N}^+}$.

Let (X, d) and (Y, p) be metric spaces. The product of $F \in N(X)$ and $G \in N(Y)$ is defined as $F \times G = \{f_i \times g_i : X \times Y \to X \times Y\}_{i \in \mathbb{N}^+}$, where $X \times Y$ is equipped with the metric $q((x_1, y_1), (x_2, y_2)) = \max \{d(x_1, x_2), p(y_1, y_2)\}$.

The NAS $F \in N(X)$ is said to be equicontinuous if the family $\{f_i\}_{i \in \mathbb{N}^+}$ is equicontinuous. We now introduce mean equicontinuity for non-autonomous systems.

Definition 2.1. The NAS $F \in N(X)$ is said to be mean equicontinuous (ME) if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if a pair of sequences $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ of elements of X satisfies $\frac{1}{n}\sum_{i=0}^{n-1} d(x_i, y_i) < \delta$, then $\frac{1}{n}\sum_{i=0}^{n-1} d(f_j(x_i), f_j(y_i)) < \epsilon$, for each $j \in \mathbb{N}^+$. A continuous map f on X is said to be mean continuous (MC) if $F = \langle f \rangle$ is ME. A homeomorphism f on X is said to be mean equivalence (MEQ) if both f and f^{-1} are MC.

Example 2.2. (i) Every isometry is MEQ and every contraction map is MC.

(ii) The tent map $f : [0, 1] \to [0, 1]$ given by $f(x) = 2 \min\{x, 1 - x\}$ is MC.

(iii) Let $X = \prod_{i \in \mathbb{Z}} X_i$ be equipped with the metric $d(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$, where $X_i = \{0, 1\}$, for each $i \in \mathbb{Z}$. Let $f : X \to X$ be given by $f(x) = y$, where $y_i = x_{i+1}$, for each $i \in \mathbb{Z}$. One can check that f is MEQ. Further, for each pair $x, y \in X$, we have $d(f(x), f(y)) \leq 2d(x, y)$ and $d(f^{-1}(x), f^{-1}(y)) \leq 2d(x, y)$. Therefore $F = \{f, f^{-1}, f, f^{-1}, \underbrace{f^{-1}, f^{-1}}_{\text{2-times}}, \underbrace{f, f}_{\text{2-times}}, f, f^{-1}, \underbrace{f^{-1}, f^{-1}}_{\text{2-times}}, \underbrace{f, f}_{\text{2-times}}, \underbrace{f, f, f}_{\text{3-times}}, \underbrace{f^{-1}, f^{-1}, f^{-1}}_{\text{3-times}}, \underbrace{f^{-1}, f^{-1}, f^{-1}, f^{-1}}_{\text{4-times}}, \underbrace{f, f, f, f}_{\text{4-times}} ...\}$ is ME.

Proposition 2.3. *Let $F \in N(X)$ and $G \in N(Y)$. Then the following statements hold:*

(i) F and G are ME if and only if $F \times G$ is ME.
(ii) If F is ME, then F^k is ME for each $k \in \mathbb{N}^+$.

Proof. For the proof of (i), use $p, q \leq max\{p, q\} \leq p + q$, for each pair $p, q \geq 0$. Proof of (ii) follows from the definition. □

The NAS $F \in N(X)$ is said to be expansive with expansivity constant $\mathfrak{c} \in (0, 1)$ if for each pair of distinct points $x, y \in X$, there is $n \in \mathbb{N}$ such that $d(F_n(x), F_n(y)) > \mathfrak{c}$ [13]. If $F = \langle f \rangle$ is expansive, then f is said to be positively expansive. Recall that if there exists a continuous injection f on a compact metric space X such that $F = \langle f \rangle$ is expansive, then X is finite [4]. The following examples justify that this fact need not be true for non-autonomous systems.

Example 2.4. Let $X = \{\frac{1}{m}, 1 - \frac{1}{m} : m \in \mathbb{N}^+\}$. Define $f : X \to X$ by $f(0) = 0$, $f(1) = 1$ and $f(x) = x^+$ otherwise, where x^+ is the immediate right element to x. Then $F = \{f, f^{-1}, f^{-2}, f^2, f^3, f^{-3}, f^{-4}, f^4, ...\}$ is expansive with expansivity constant $0 < \alpha < \frac{1}{6}$.

Example 2.5. If f is given as in Example 2.2 (iii), then $F = \{f, f^{-1}, f, f^{-1}, \underbrace{f^{-1}, f^{-1}}_{\text{2-times}}, \underbrace{f, f}_{\text{2-times}}, f, f^{-1}, \underbrace{f^{-1}, f^{-1}}_{\text{2-times}}, \underbrace{f, f}_{\text{2-times}}, \underbrace{f, f, f}_{\text{3-times}}, \underbrace{f^{-1}, f^{-1}, f^{-1}}_{\text{3-times}}, \underbrace{f^{-1}, f^{-1}, f^{-1}, f^{-1}}_{\text{4-times}}, ...\}$ and also, $G = \{f, f^{-1}, f^{-2}, f^2, f^3, f^{-3}, f^{-4}, f^4, ...\}$ are expansive with expansivity constant $0 < \alpha < \frac{1}{2}$.

We also study the following notions both of which are stronger than expansivity.

Definition 2.6. Let $F \in N(X)$. Then:

(i) F is said to be recurrently expansive if there exists a constant $\mathfrak{c} \in (0,1)$ such that each pair of distinct points $x, y \in X$ satisfies $\limsup_{n\to\infty} d(F_n(x), F_n(y)) > \mathfrak{c}$.

(ii) F is said to be mean expansive if there exists a constant $\mathfrak{c} \in (0,1)$ such that each pair of distinct points $x, y \in X$ satisfies $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(F_i(x), F_i(y)) > \mathfrak{c}$.

Remark 2.7. Let $F \in N(X)$. If F is mean expansive, then F is recurrently expansive.

Proposition 2.8. *The NAS $F = \langle f\rangle$ is recurrently expansive if and only if f is a positively expansive injective map.*

Proof. If F is recurrently expansive with expansivity constant $\mathfrak{c}$, then it is clear that f is positively expansive with expansivity constant $\mathfrak{c}$. Further, if $f(x) = f(y)$, then $f^n(x) = f^n(y)$, for each $n \in \mathbb{N}^+$ implying $\limsup_{n\to\infty} d(f^n(x), f^n(y)) = 0$. Therefore we must have $x = y$.

Conversely, suppose that f is positively expansive injective map with expansivity constant $\mathfrak{c}$. If $x \neq y$ in X, then by the positive expansivity of f there exists an $n_1 \in \mathbb{N}$ such that $d(f^{n_1}(x), f^{n_1}(y)) > \mathfrak{c}$. By injectivity and positive expansivity of f, we can choose an $n_2 \in \mathbb{N}^+$ such that $d(f^{n_2+n_1+1}(x), f^{n_2+n_1+1}(y)) > \mathfrak{c}$. Continuing in this way, we can choose a strictly increasing sequence $\{m_i\}_{i=1}^{\infty}$ such that $d(f^{m_i}(x), f^{m_i}(y)) > \mathfrak{c}$. Therefore we have $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \mathfrak{c}$. Since x and y are chosen arbitrarily, we get the result. □

Proposition 2.9. *Let $F \in N(X)$ and $G \in N(Y)$. Then F and G are recurrently expansive (resp. mean expansive) if and only if $F \times G$ is recurrently expansive (resp. mean expansive).*

Proof. Proof is similar to the proof of [13, Theorem 2.4]. □

Proposition 2.10. *Let $F \in N(X)$ be equicontinuous. Then F is recurrently expansive if and only if F^k is recurrently expansive, for each $k \in \mathbb{N}^+$.*

Proof. Proof is similar to the proof of [13, Theorem 2.2]. □

Example 2.11. If f is given as in Example 2.2 (iii), then $F = \{f, f^{-1}, f^{-2}, f^2, f, f^{-1}, f^{-2}, xf^2, f^3, f^{-3}, f^{-4}, f^4, f, f^{-1}, f^{-2}, f^2, f^3, f^{-3}, f^{-4}, f^4, f^5, f^{-5}, f^{-6}, f^6, f^7, f^{-7}, f^{-8}, f^8, ...\}$ is recurrently expansive with expansivity constant $0 < \alpha < \frac{1}{2}$ but F^2 is not recurrently expansive. Thus Proposition 2.10 need not hold if NAS is not equicontinuous.

Proposition 2.12. *Let $F \in N(X)$. If F^k is mean expansive for some $k \in \mathbb{N}^+$, then F is mean expansive.*

Proof. Let us fix $k \in \mathbb{N}^+$. Suppose that F^k is mean expansive with expansivity constant $\mathfrak{c}$. For each distinct pair $x, y \in X$ and $n \in \mathbb{N}^+$, we have that $\frac{1}{n}\sum_{i=0}^{n-1} d((F^k)_i(x), (F^k)_i(y)) \leq k\frac{1}{nk}\sum_{i=0}^{(n-1)k} d(F_i(x), F_i(y)) \leq k\frac{1}{nk}\sum_{i=0}^{nk-1} d(F_i(x), F_i(y))$. From this we conclude that F is mean expansive with expansivity constant $\frac{\mathfrak{c}}{k}$. □

The following example justifies that the converse of the Proposition 2.12 need not be true. We further give sufficient condition under which the converse holds.

Example 2.13. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$, for each $x \in \mathbb{R}$. Then $F = \{f, f^{-1}, f^2, f^{-2}, f^3, f^{-3}, ...\}$ is mean expansive but F^2 is not mean expansive.

Proposition 2.14. *Let $F \in N_m(X)$ be ME. If F is mean expansive, then F^k is mean expansive for each $k \in \mathbb{N}^+$.*

Proof. Suppose that $F \in N_m(X)$ is mean expansive with expansivity constant $\mathfrak{c}$. Fix $k \in \mathbb{N}^+$ and choose a $\mathfrak{d} > 0$ such that for each pair of sequences $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$, $\limsup\limits_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(x_i, y_i) < m\mathfrak{d}$ implies that $\limsup\limits_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(F_j(x_i), F_j(y_i)) < \frac{\mathfrak{c}}{mk}$, for all $0 \le j \le (mk-1)$.

Choose a pair $x, y \in X$ such that $\limsup\limits_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d((F^k)_i(x), (F^k)_i(y)) < \mathfrak{d}$. Note that $\limsup\limits_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(F_{mki}(x), F_{mki}(y)) < m\mathfrak{d}$.

By ME, $\limsup\limits_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d((F_{mki+j})(x), (F_{mki+j})(y)) < \frac{\mathfrak{c}}{mk}$, for all $0 \le j \le (mk-1)$ and hence,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(F_i(x), F_i(y)) &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{nmk-1} d(F_i(x), F_i(y)) \\ &\le \sum_{j=0}^{mk-1} \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(F_{mki+j}(x), F_{mki+j}(y)) < \mathfrak{c}.\end{aligned}$$

By using mean expansivity of F, we get that $x = y$. Hence F^k is mean expansive with expansivity constant $\mathfrak{d}$. □

Let $F \in N(X)$ and $\gamma = \{x_n\}_{n=0}^{\infty}$ be a sequence of elements of X. Then γ is said to be a δ-pseudo orbit of F if $d(f_{i+1}(x_i), x_{i+1}) < \delta$, for each $i \in \mathbb{N}$. We say that γ is a δ-average-pseudo orbit of F if there exists $N_\delta \in \mathbb{N}^+$ such that $\frac{1}{n}\sum_{i=0}^{n-1} d(f_{i+k+1}(x_{i+k}), x_{i+k+1}) < \delta$, for each $n \ge N_\delta$ and for each $k \in \mathbb{N}$. We say that γ can be ϵ-shadowed by a point $z \in X$ if $d(F_n(z), x_n) < \epsilon$, for each $n \in \mathbb{N}$, and that γ can be eventually ϵ-shadowed by a point $z \in X$ if $d(z, x_0) < \epsilon$ and $\limsup\limits_{n\to\infty} d(F_n(z), x_n) < \epsilon$. We say that γ can be ϵ-shadowed in average by a point $z \in X$ if $\limsup\limits_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(F_i(z), x_i) < \epsilon$, and that γ can be strongly ϵ-shadowed in average if it is ϵ-shadowed in average by a point $z \in X$ such that $d(z, x_0) < \epsilon$.

The NAS $F \in N(X)$ has shadowing property if for every $\epsilon > 0$, there exists a $\delta > 0$ such that every δ-pseudo orbit of F can be ϵ-shadowed by a point of X [13].

Definition 2.15. Let $F \in N(X)$. Then we say that:

(i) F has eventual shadowing property (ESP) if for every $\epsilon > 0$, there exists a $\delta > 0$ such that every δ-pseudo orbit of F can be eventually ϵ-shadowed by a point of X.

(ii) F has strong average shadowing property (SASP) if for every $\epsilon > 0$, there exists a $\delta > 0$ such that every δ-average pseudo orbit of F can be strongly ϵ-shadowed in average by a point of X.

Proposition 2.16. *Let $F \in N(X)$ be equicontinuous. Then F has ESP if and only if F^k has ESP, for each $k \in \mathbb{N}^+$.*

Proof. Proof is similar to the proof of [13, Theorem 3.3] and [13, Theorem 3.5]. □

Proposition 2.17. *Let (X, d) and (Y, p) be metric spaces, and let $F \in N(X)$ and $G \in N(Y)$. Then F and G both have SASP (resp. ESP) if and only if $F \times G$ has SASP (resp. ESP).*

Proof. Suppose that F and G has SASP. Let $\epsilon > 0$ and $\delta > 0$ be given for $\frac{\epsilon}{2}$ by SASP of F and G. Let $\lambda = \{(x_n, y_n)\}_{n=0}^{\infty}$ be a δ-average pseudo orbit of $F\times G$. Since $d(f_{n+1}(x_n), x_{n+1}) \le q((f_{n+1}\times g_{n+1})(x_n, y_n), (x_{n+1}, y_{n+1}))$ and $p(g_{n+1}(y_n), y_{n+1}) \le q((f_{n+1}\times g_{n+1})(x_n, y_n), (x_{n+1}, y_{n+1}))$, we get that $\gamma = \{x_n\}_{n=0}^{\infty}$ and $\eta = \{y_n\}_{n=0}^{\infty}$ are δ-average pseudo orbits of F and G respectively. Now, proof follows from the observation that if γ and η can be strongly $\frac{\epsilon}{2}$-shadowed in average respectively by $x \in X$ and $y \in Y$, then λ can be strongly ϵ-shadowed in average by $(x, y) \in X \times Y$.

Conversely, suppose that $F \times G$ has SASP. Let $\epsilon > 0$ and $\delta > 0$ be given for $\frac{\epsilon}{2}$ by SASP of $F \times G$. Let $\gamma = \{x_n\}_{n=0}^{\infty}$ be a δ-average pseudo orbit of F. Let $\eta = \{y_n = G_n(y)\}_{n=0}^{\infty}$, for some $y \in Y$. Clearly,

$\lambda = \{(x_n, y_n)\}_{n=0}^{\infty}$ is a δ-average pseudo orbit of $F \times G$ and hence, it can be strongly ϵ-shadowed in average by a point $(x, z) \in X \times Y$. Clearly, γ can be strongly ϵ-shadowed in average by x through F and hence, F has SASP. Similarly, one can prove that G has SASP.

Using similar arguments as above, one can prove that F and G has ESP if and only if $F \times G$ has ESP □

We say that $F \in N(X)$ is transitive if for each pair of non-empty open sets U and V of X, there exists $n \in \mathbb{N}^+$ such that $F_{[i,i+(n-1)]}(U) \cap V \neq \phi$, for each $i \in \mathbb{N}^+$.

For given $x, y \in X$, we write $x\mathsf{R}_\delta y$ if there exists an $n \in \mathbb{N}^+$ such that for each $j \in \mathbb{N}^+$ there exists a finite sequence $x = x_0^j, x_1^j, ..., x_{n-1}^j, x_n^j = y$ satisfying $d(f_{j+i}(x_i^j), x_{i+1}^j) < \delta$, for all $0 \leq i < n$. We write $x\mathcal{R}_\delta y$ if $x\mathsf{R}_\delta y$ and $y\mathsf{R}_\delta x$ and we write $x\mathcal{R}y$ if $x\mathcal{R}_\delta y$, for each $\delta > 0$. We say that $F \in N(X)$ is chain transitive if $x\mathcal{R}y$, for each pair $x, y \in X$. Note that if $F = \langle f \rangle$, then this definition boils down to the following in case of autonomous systems.

$F = \langle f \rangle$ is said to be chain transitive if for every $\delta > 0$ and each pair $x, y \in X$, there exists a finite sequence $z_0 = x, z_1, z_2, ..., z_m = y$ of elements in X such that $d(f(z_i), z_{i+1}) < \delta$, for all $0 \leq i < m$.

Theorem 2.18. *Let $F \in N(X)$ be an equicontinuous NAS. If F is transitive, then F is chain transitive.*

Proof. For a given $\epsilon > 0$, choose a $\delta > 0$ by equicontinuity of F. By transitivity, choose $n \in \mathbb{N}^+$ such that $F_{[j,j+(n-1)]}(B(x,\delta)) \cap B(y,\delta) \neq \phi$, for each $j \in \mathbb{N}^+$. So, for each $j \in \mathbb{N}^+$, there exists $z^j \in B(x,\delta)$ such that the sequence $x = x_0^j, x_1^j = f_j(z^j), x_2^j = (f_{j+1} \circ f_j)(z^j), ..., x_{n-1}^j = (f_{j+(n-2)} \circ ... \circ f_{j+1} \circ f_j)(z^j), x_n^j = y$ satisfies $d(f_{j+i}(x_i^j), x_{i+1}^j) < \epsilon$, for all $0 \leq i \leq (n-1)$ and hence $x\mathsf{R}_\epsilon y$. Since j, x, y and ϵ are chosen arbitrarily, we conclude that F is chain transitive. □

Theorem 2.19. *Let $F \in N(X)$ be a surjective system with shadowing property. If F is chain transitive, then it is transitive.*

Proof. Choose a pair $x, y \in X$ and an $\epsilon > 0$. Let $\delta > 0$ be given for ϵ by the shadowing property of F. By chain transitivity of F, there exists $n \in \mathbb{N}^+$ such that for each $j \in \mathbb{N}^+$, there exists a finite sequence $x = x_0^j, x_1^j, ..., x_{n-1}^j, x_n^j = y$ satisfying $d(f_{j+i}(x_i^j), x_{i+1}^j) < \delta$, for all $0 \leq i \leq (n-1)$. Extend this sequence to a δ-pseudo orbit $\eta = \{z_0, ..., z_{j-3}, z_{j-2}, z_{j-1} = x = x_0^j, z_j = x_1^j, ..., z_{j+(n-2)} = x_{n-1}^j, z_{j+(n-1)} = x_n^j = y, z_{j+n} = f_{j+n}(y), z_{j+(n+1)} = f_{j+(n+1)}(z_{j+n}), ...\}$ of F, where $f_i(z_{i-1}) = z_i$, for all $1 \leq i \leq (j-1)$ and $z_{j+(n+k)} = f_{j+(n+k)}(z_{j+(n+k-1)})$, for each $k \geq 2$. By the shadowing property of F, there exists $w \in X$ such that $d(F_n(w), z_n) < \epsilon$, for each $n \in \mathbb{N}^+$. Therefore $F_{j-1}(w) \in B(x,\epsilon)$ and $F_{[j,j+(n-1)]}(F_{j-1}(w)) \in B(y,\epsilon)$. Hence $(F_{[j,j+(n-1)]}(B(x,\epsilon)) \cap B(x,\delta)) \neq \phi$. Since j, x, y and ϵ are chosen arbitrarily, we conclude that F is transitive. □

In the following two results, we study relations among shadowing property, eventual shadowing property, average shadowing property and strong average shadowing property in case of an autonomous system. Unfortunately, we do not know much in case of a nonautonomous system.

Lemma 2.20. *Let $F = \langle f \rangle$ be a chain transitive system on a compact metric space X. Then F has ESP if and only if F has shadowing property.*

Proof. As the converse implication follows from the definition, we only need to prove the forward implication. Suppose that F has ESP but does not have shadowing property. For given $\epsilon > 0$ and each $n \in \mathbb{N}^+$, we can choose a finite $\frac{1}{n}$-pseudo orbit α_n of F which cannot be ϵ-shadowed. Let $\delta > 0$ be given for this ϵ by ESP and fix $k \in \mathbb{N}^+$ such that $\frac{1}{k} < \delta$. By chain transitivity of F, choose a finite $\frac{1}{m}$-pseudo orbits γ_m of F such that $\alpha_m \gamma_m \alpha_{m+1}$ forms a finite $\frac{1}{m}$-pseudo orbits of F, for each $m \geq k$. Clearly, $\alpha_k \gamma_k \alpha_{k+1} \gamma_{k+1} \alpha_{k+2}...$ forms a δ-pseudo orbit of F and hence it can be eventually ϵ-shadowed by a point of X. So, there exists $p \in \mathbb{N}^+$ such that for each $j \geq p$, α_j can be ϵ-shadowed by a point of X, which is a contradiction. □

Theorem 2.21. *Let $F = \langle f \rangle$ be chain transitive with ESP on a compact metric space X. Then f has average shadowing property if and only if f has SASP.*

Proof. Since the converse implication follows from the definition, we only need to prove the forward implication. By Theorem 2.20, f has shadowing property and hence by [8, Theorem 1], f has specification property. Using similar arguments as given in the proof of [8, Lemma 12], we can prove that f has SASP. □

3. Sufficient Conditions For Topological Stability

For a metric space (X, d), define the bounded metric d_1 by $d_1(x, y) = \min\{d(x, y), 1\}$, for each pair $x, y \in X$. Let $(C(X), \eta)$ be the metric space of set of all continuous self maps of X, where the metric η is defined by $\eta(f, g) = \sup_{x\in X} d_1(f(x), g(x))$. We define the metric γ on $N(X)$ by $\gamma(F, G) = \sup_{i\in\mathbb{N}} \eta(f_i, g_i)$, where $F = \{f_i\}_{i\in\mathbb{N}^+}, G = \{g_i\}_{i\in\mathbb{N}^+}$.

Definition 3.1. We say that $F \in NC(X)$ is topologically stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for any $G \in NC(X)$ satisfying $\gamma(F, G) < \delta$, there exists a continuous map $h : X \to X$ such that $f_i \circ h = h \circ g_i$, for each $i \in \mathbb{N}$ and $d(h(x), x) < \epsilon$, for all $x \in X$.

We say that $F \in N(X)$ and $G \in N(Y)$ are uniformly conjugate if there exists a uniform equivalence $h : Y \to X$ such that $f_i \circ h = h \circ g_i$, for each $i \in \mathbb{N}$. The map h is called uniform conjugacy between X and Y. A property of a NAS is called uniform dynamical property if it is preserved under uniform conjugacy.

Theorem 3.2. *Let (X, d) and (Y, p) be two metric spaces and let $F \in NC(X)$, $H \in NC(Y)$. If F and H are uniformly conjugate, then F is topologically stable if and only if H is topologically stable. In other words, topological stability of a NAS is uniform dynamical property.*

Proof. Suppose that F is topologically stable. Let $j : Y \to X$ be a uniform conjugacy between F and H i.e. $f_i \circ j = j \circ h_i$, for each $i \in \mathbb{N}$. We claim that H is also topologically stable. For a given $\epsilon \in (0, 1)$, choose a $\beta \in (0, 1)$ by the uniform continuity of j^{-1}. For this β, choose an $\alpha \in (0, 1)$ by the topological stability of F. For this α, choose a $\delta \in (0, 1)$ by the uniform continuity of j. Let $G \in NC(Y)$ satisfies $\gamma^Y(H, G) < \delta$. Since $\sup_{i\in\mathbb{N}} \eta^Y(h_i, g_i) < \delta$ implies that $\eta^Y(j^{-1} \circ f_i \circ j, g_i) < \delta$, for each $i \in \mathbb{N}$, we get that $d_1^Y(j^{-1} \circ f_i \circ j(y), g_i(y)) < \delta$, for each $i \in \mathbb{N}$ and for each $y \in Y$. By the uniform continuity of j, we get that $d_1^X((f_i \circ j)(y), (j \circ g_i \circ j^{-1})(j(y))) < \alpha$, for each $i \in \mathbb{N}$ and for each $y \in Y$. Set $G' = \{g'_i = j \circ g_i \circ j^{-1}\}_{i\in\mathbb{N}^+}$. By topological stability of F, there exists a continuous map $k : X \to X$ such that $f_i \circ k = k \circ g'_i$, for each $i \in \mathbb{N}$ and $d(k(x), x) < \beta$, for each $x \in X$. If we set $k' = j^{-1} \circ k \circ j$, then $h_i \circ k' = h_i \circ j^{-1} \circ k \circ j = j^{-1} \circ f_i \circ k \circ j = j^{-1} \circ k \circ g'_i \circ j = j^{-1} \circ k \circ j \circ g_i \circ j^{-1} \circ j = k' \circ g_i$, for each $i \in \mathbb{N}$. Also, by uniform continuity of j^{-1}, we have $p(k'(y), y) < \epsilon$, for each $y \in Y$, which completes the proof of forward implication. Using similar arguments, one can prove that if H is topologically stable, then F is topologically stable. □

We say that a metric space is Mandelkern locally compact [9] if every bounded subset of the space is contained in a compact subset of the space. Observe that, a metric space is Mandelkern locally compact if and only if every closed ball of a finite radius is a compact subset of the space. From now onwards, we assume that X is a Mandelkern locally compact metric space. Without loss of generality, we also assume that $0 < \epsilon, \delta, \alpha, \beta, \mathfrak{c}, \mathfrak{c}' < 1$.

Theorem 3.3. *Let $F \in NC(X)$ be an equicontinuous recurrently expansive NAS with expansivity constant $\mathfrak{c}$. If F has ESP, then F is topologically stable. Moreover, for each $\epsilon \in (0, \frac{\mathfrak{c}}{3})$, there exists a $\delta > 0$ such that if $G \in NC(X)$ satisfies $\gamma(F, G) < \delta$, then there exists a unique continuous map $h : X \to X$ such that $F_n \circ h = h \circ G_n$, for each $n \in \mathbb{N}$ and $d(h(x), x) < \epsilon$, for each $x \in X$. In addition, if G is expansive with expansivity constant $\mathfrak{c}' \geq 3\epsilon$, then the conjugating map h is injective.*

Lemma 3.4. *Let $F \in N(X)$ be recurrently expansive with expansivity constant $\mathfrak{c}$. If F has ESP, then for each $\epsilon \in (0, \frac{\mathfrak{c}}{3})$ there exists $\delta \in (0, \frac{\mathfrak{c}}{3})$ given for this ϵ by ESP of F such that each δ-pseudo orbit of F can be eventually ϵ-shadowed by exactly one point.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a δ-pseudo orbit of F and suppose that $x, y \in X$ eventually ϵ-shadows a δ-pseudo orbit $\{x_n\}_{n\in\mathbb{N}}$ of F. Since $d(F_n(x), F_n(y)) \leq d(F_n(x), x_n) + d(x_n, F_n(y))$, for each $n \in \mathbb{N}$, we get that,

$$\begin{aligned}\limsup_{n\to\infty} d(F_n(x), F_n(y)) &\leq \limsup_{n\to\infty}(d(F_n(x), x_n) + d(x_n, F_n(y))) \\ &\leq \limsup_{n\to\infty} d(F_n(x), x_n) + \limsup_{n\to\infty} d(x_n, F_n(y)) \\ &\leq 2\epsilon < \mathfrak{c}.\end{aligned}$$

By recurrent expansivity of F, we get that $x = y$ which completes the proof. □

Lemma 3.5. *Let F be recurrently expansive with expansivity constant $\mathfrak{c}$. For any $x_0 \in X$ and $\lambda > 0$, there exists an $N > 0$ such that if $x \in X$ satisfies $d(F_n(x_0), F_n(x)) \leq \mathfrak{c}$, for all $0 \leq n \leq N$, then $d(x_0, x) < \lambda$.*

Proof. Fix $x_0 \in X$ and choose a sequence $\{x_N\}_{N \in \mathbb{N}^+}$ in X such that $d(F_n(x_0), F_n(x_N)) \leq \mathfrak{c}$, for all $0 \leq n \leq N$ and $d(x_0, x_N) \geq \lambda$. Since $B[x_0, \mathfrak{c}]$ is a compact subset of X, we can assume that x_N converges to x, for some $x \in X$. By using continuity of each F_n, we get that $d(F_n(x_0), F_n(x)) \leq \mathfrak{c}$, for each $n \in \mathbb{N}$ and $d(x_0, x) \geq \lambda$, a contradiction to the recurrent expansivity of F. □

Proof of Theorem 3.3. For a given $\epsilon \in (0, \frac{\mathfrak{c}}{3})$, choose a $\beta \in (0, \epsilon)$ by the equicontinuity of F. Let $\delta \in (0, \beta)$ be given for β by the ESP of F. Let $G \in NC(X)$ be such that $\gamma(F, G) < \delta$ i.e. $\eta(f_i(x), g_i(x)) < \delta$, for each $i \in \mathbb{N}$ and for each $x \in X$. Therefore $\{G_n(x)\}_{n \in \mathbb{N}}$ forms a δ-pseudo orbit of F, for each $x \in X$. By Lemma 3.4, define $h : X \to X$ such that $h(x)$ is a unique eventually β-tracing point of the δ-pseudo orbit $\{G_n(x)\}_{n \in \mathbb{N}}$ i.e. $\limsup_{n\to\infty} d(F_n(h(x)), G_n(x)) < \beta$, for each $x \in X$, where $d(h(x), x) < \epsilon$, for each $x \in X$. Note that for each $x \in X$ and for each $i \in \mathbb{N}$, we have

$$\begin{aligned}\limsup_{n\to\infty} d(F_n(f_i(h(x))), F_n(h(g_i(x)))) &\leq \limsup_{n\to\infty} d(f_i(F_n(h(x))), f_i(G_n(x))) \\ &+ \limsup_{n\to\infty} d(f_i(G_n(x)), g_i(G_n(x))) \\ &+ \limsup_{n\to\infty} d(G_n(g_i(x)), F_n(h(g_i(x)))) \\ &< 3\epsilon < \mathfrak{c}.\end{aligned}$$

By the recurrent expansivity of F, we get that$(f_i \circ h)(x) = (h \circ g_i)(x)$, for each $i \in \mathbb{N}$. Hence $F_n \circ h = h \circ G_n$, for each $n \in \mathbb{N}$.

Now, we show that h is a continuous map. Let $x_0 \in X$ and $\lambda > 0$. By Lemma 3.5, there exists an $N > 0$ such that if $y \in X$ satisfies $d(F_n(h(x_0)), F_n(h(y))) \leq \mathfrak{c}$, for each $n \leq N$, then $d(h(x_0), h(y)) < \lambda$. Choose an $\alpha > 0$ such that if $y \in X$ satisfies $d(x_0, y) < \alpha$, then $d(G_n(x_0), G_n(y)) < \frac{\mathfrak{c}}{3}$, for each $n \leq N$. Therefore if $y \in X$ satisfies $d(x_0, y) < \alpha$, then for each $n \leq N$, we have,

$$\begin{aligned}&d(F_n(h(x_0)), F_n(h(y))) = d(h(G_n(x_0)), h(G_n(y))) \\ &\leq d(h(G_n(x_0)), G_n(x_0)) + d(G_n(x_0), G_n(y)) + d(G_n(y), h(G_n(y))) \\ &< \epsilon + \tfrac{\mathfrak{c}}{3} + \epsilon < \mathfrak{c}.\end{aligned}$$

Thus, we get that $d(h(x_0), h(y)) < \lambda$. Since λ is chosen arbitrarily, we get that h is continuous at x_0. Since x_0 is chosen arbitrarily, we get that h is a continuous map.

Assume that there exists another continuous map $h' : X \to X$ such that $F_n \circ h' = h' \circ F_n$, for each $n \in \mathbb{N}$ and $d(h'(x), x) < \epsilon$, for each $x \in X$. Therefore for each $n \in \mathbb{N}$ and for each $x \in X$, we have

$$\begin{aligned}&d(F_n(h(x)), F_n(h'(x))) \leq d(F_n(h(x)), G_n(x)) + d(G_n(x), F_n(h'(x))) = d(h(G_n(x)), G_n(x)) \\ &+ d(G_n(x), h'(G_n(x))) < 2\epsilon < \mathfrak{c}.\end{aligned}$$

Hence by recurrent expansivity of F, we get that $h(x) = h'(x)$, for each $x \in X$.

Now, assume that $h(x) = h(y)$. Note that for each $n \in \mathbb{N}$, we have,

$$\begin{aligned}&d(G_n(x), G_n(y)) \leq d(G_n(x), h(G_n(x))) + d(h(G_n(x)), h(G_n(y))) + d(h(G_n(y)), G_n(y)) \\ &< \epsilon + 0 + \epsilon = 2\epsilon < \mathfrak{c}'.\end{aligned}$$

Therefore by the expansivity of G, we get that $x = y$.

Theorem 3.6. *Let $F \in NC(X)$ be an equicontinuous expansive system with expansivity constant $\mathfrak{c}$. If F has shadowing property, then F is topologically stable. Moreover, for each $\epsilon \in (0, \frac{\mathfrak{c}}{3})$, there exists a $\delta > 0$ such that if $G \in NC(X)$ satisfies $\gamma(F, G) < \delta$, then there exists a unique continuous map $h : X \to X$ such that $F_n \circ h = h \circ G_n$, for each $n \in \mathbb{N}$ and $d(h(x), x) < \epsilon$, for each $x \in X$. In addition, if G is expansive with expansivity constant $\mathfrak{c}' \geq 3\epsilon$, then h is injective.*

Proof. Proof is similar to the proof of Theorem 3.3 □

Theorem 3.7. *Let $F \in NC(X)$ be mean equicontinuous and mean expansive with expansivity constant $\mathfrak{c}$. If F has SASP, then F is topologically stable. Moreover, for each $\epsilon \in (0, \frac{\mathfrak{c}}{3})$, there exists a $\delta > 0$ such that if $G \in NC(X)$ satisfies $\gamma(F, G) < \delta$, then there exists a unique continuous map $h : X \to X$ satisfying $F_n \circ h = h \circ G_n$, for each $n \in \mathbb{N}$ and $d(h(x), x) < \epsilon$, for each $x \in X$. In addition, if G is expansive with expansivity constant $\mathfrak{c}' \geq 3\epsilon$, then h is injective.*

Lemma 3.8. *Let F be mean expansive with expansivity constant $\mathfrak{c}$. Suppose that F has SASP and for a given $\epsilon \in (0, \frac{\mathfrak{c}}{3})$, choose a $\delta > 0$ by SASP of F. Then each δ-average pseudo orbit of F can be strongly ϵ-shadowed in average uniquely.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a δ-average pseudo orbit of F and let it can be strongly ϵ-shadowed in average by $x, y \in X$. Then $d(F_n(x), F_n(y)) \leq d(F_n(x), x_n) + d(x_n, F_n(y))$, for each $n \in \mathbb{N}$, which implies that,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x), F_i(y)) &\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (d(F_i(x), x_i) + d(x_i, F_i(y))) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x), x_i) + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(x_i, F_i(y)) \\
&< 2\epsilon < \mathfrak{c}.
\end{aligned}$$

By mean expansivity of F, we get that $x = y$ which completes the proof. □

Lemma 3.9. *Let F be mean expansive with expansivity constant $\mathfrak{c}$. For any $x_0 \in X$ and $\lambda > 0$, there exists an $N > 0$ such that if $x \in X$ satisfies $\frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x_0), F_i(x)) \leq \mathfrak{c}$, for each $n \leq N$, then $d(x_0, x) < \lambda$.*

Proof. Fix an $x_0 \in X$ and choose a sequence $\{x_N\}_{N\in\mathbb{N}^+}$ in X such that $\frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x_0), F_i(x_N)) \leq \mathfrak{c}$, for each $n \leq N$ and $d(x_0, x_N) \geq \lambda$. Since $B[x_0, \mathfrak{c}]$ is a compact subset of X, we can assume that x_N converges to $x \in X$. For each $n \in \mathbb{N}$ and for a given $\epsilon > 0$, there exists an $N' \geq n$ such that $d(F_i(x), F_i(x_N)) < \epsilon$, for all $0 \leq i \leq n$ and for each $N \geq N'$. For such an N', $\frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x), F_i(x_{N'})) < \epsilon$ and hence,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x_0), F_i(x)) \leq \frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x_0), F_i(x_{N'})) + \frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x_{N'}), F_i(x)) \leq \mathfrak{c} + \epsilon.$$

Since n and ϵ are chosen arbitrarily, we get that $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x_0), F_i(x)) \leq \mathfrak{c}$ and $d(x_0, x) \geq \lambda$ which is a contradiction to the mean expansivity of F. □

Proof of Theorem 3.7 Let $\beta \in (0, \epsilon)$ be given for $\epsilon \in (0, \frac{\mathfrak{c}}{3})$ by mean equicontinuity of F. For β, choose a $\delta \in (0, \beta)$ such that each δ-average pseudo orbit of F can be strongly β-shadowed in average by a point of X. Let $G \in NC(X)$ be such that $\gamma(F, G) < \delta$ i.e. $\eta(f_i(x), g_i(x)) < \delta$, for each $x \in X$ and for each $i \in \mathbb{N}$. Thus, the sequence $\{G_n(x)\}_{n\in\mathbb{N}}$ forms a δ-average pseudo orbit of F, for each $x \in X$. Define the map $h : X \to X$ such that $h(x)$ is a unique strongly β-tracing point in average of the δ-average pseudo orbit $\{G_n(x)\}_{n\in\mathbb{N}}$ i.e. $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F_i(h(x)), G_i(x)) < \beta$, for each $x \in X$ and $d(h(x), x) < \beta$, for each $x \in X$. Note that for each $x \in X$ and for each $m \in \mathbb{N}$, we have,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d(F_i(f_m(h(x))),F_i(h(g_m(x)))) &\leq \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d(f_m(F_i(h(x))),f_m(G_i(x)))\\ &+\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d(f_m(G_i(x)),g_m(G_i(x)))\\ &+\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d(G_i(g_m(x)),F_i(h(g_m(x))))\\ &< 3\epsilon < \mathfrak{c}.\end{aligned}$$

By mean expansivity of F, we get that $f_m \circ h = h \circ g_m$, for each $m \in \mathbb{N}$ and hence we have $F_n \circ h = h \circ G_n$, for each $n \in \mathbb{N}$.

We claim that h is continuous. Let $x_0 \in X$ and $\lambda > 0$. By Lemma 3.9, there exists an $N > 0$ such that if $y \in X$ satisfies $\frac{1}{n}\sum_{i=0}^{n-1} d(F_i(x_0), F_i(y)) \leq \mathfrak{c}$, for each $n \leq N$, then $d(x_0, y) < \lambda$. Choose an $\alpha > 0$ such that if $y \in X$ satisfies $d(x_0, y) < \alpha$, then $d(G_n(x_0), G_n(y)) < \frac{\mathfrak{c}}{3}$, for each $n \leq N$. Therefore if $y \in X$ satisfies $d(x_0, y) < \alpha$, then for each $n \leq N$, we have,

$\frac{1}{n}\sum_{i=0}^{n-1} d(F_i(h(x_0)), F_i(h(y))) = \frac{1}{n}\sum_{i=0}^{n-1} d(h(G_i(x_0)), h(G_i(y)))$
$\leq \frac{1}{n}\sum_{i=0}^{n-1} d(h(G_i(x_0)), G_i(x_0)) + \frac{1}{n}\sum_{i=0}^{n-1} d(G_i(x_0), G_i(y)) + \frac{1}{n}\sum_{i=0}^{n-1} d(G_i(y), h(G_i(y)))$
$< \mathfrak{c}$.

Therefore $d(h(x_0), h(y)) < \lambda$. Since λ is chosen arbitrarily, we get that h is continuous at x_0. Since x_0 is chosen arbitrarily, we get that h is a continuous map.

Assume that there exists another continuous map $h' : X \to X$ satisfying $F_n \circ h' = h' \circ G_n$, for each $n \in \mathbb{N}$ and $d(h'(x), x) < \epsilon$, for each $x \in X$. Therefore for each $n \in \mathbb{N}^+$ and for each $x \in X$, we have,

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1}d(F_i(h(x)),F_i(h'(x))) &\leq \frac{1}{n}\sum_{i=0}^{n-1}d(F_i(h(x)),G_i(x)) + \frac{1}{n}\sum_{i=0}^{n-1}d(G_i(x),G_i(h'(x)))\\ &= \frac{1}{n}\sum_{i=0}^{n-1}d(h(G_i(x)),G_i(x)) + \frac{1}{n}\sum_{i=0}^{n-1}d(G_i(x),h'(G_i(x)))\\ &< 2\epsilon < \mathfrak{c}.\end{aligned}$$

Hence by mean expansivity of F, we get that $h(x) = h'(x)$, for each $x \in X$.

Assume that $h(x) = h(y)$. Now, for each $n \in \mathbb{N}$, we have $d(G_n(x), G_n(y)) \leq d(G_n(x), h(G_n(x))) + d(h(G_n(x)), h(G_n(y))) + d(h(G_n(y)), G_n(y)) < \mathfrak{c}'$. Therefore by the expansivity of G we get that $x = y$.

Corollary 3.10. *Every commutative equicontinuous mean expansive NAS with ESP on a Mandelkern locally compact metric space is topologically stable.*

Example 3.11. Let $X = \mathbb{R}$ be given with the usual metric and choose an $m > 1$. Define $g_m : X \to X$ by $g_m(x) = mx$, for each $x \in X$. Consider $F_m = \{f_i\}_{i\in\mathbb{N}^+}$ such that for each $i \in \mathbb{N}^+$, exactly one of the triplet $f_{3i+1}, f_{3i+2}, f_{3i+3}$ is g_m and the other two are identity maps on X. Note that F_m need not be a periodic system. Also, F_m is equicontinuous, mean equicontinuous, recurrently expansive and mean expansive. Since $F_m^3 = \langle g_m \rangle$, by Proposition 2.16, we get that F_m has ESP. From Corollary 3.10, we get that F_m is topologically stable.

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