Filomat 35:11 (2021), 3703–3712
https://doi.org/10.2298/FIL2111703A

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Applications of Berezin $\theta-$sequences and Berezin symbols

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Abstract. We consider the so-called Berezin $\theta-$ sequence, where θ is an operator valued inner function, for operators on the vector valued Hardy space $H_E^2(\mathbb{D})$, and study the invertibility of some operators on the model space $K_\theta = H_E^2 \ominus \theta H_E^2$ via Berezin $\theta-$ sequence. By applying of Berezin symbols technique the Toeplitz corona problem in the Bergman space $L_a^2(\mathbb{D})$ is studied. Moreover, $C-$ invertibility and $C-$unitarity of operators are also defined and studied.

Notation

$\mathbb{D}$	Open unit disc in the complex plane $\mathbb{C}$, $\mathbb{D} := \{z \in \mathbb{C} : \lvert z\rvert < 1\}$;
$\mathbb{T}$	unit circle, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : \lvert z\rvert = 1\}$;
H^p	Hardy classes of analytic functions, $H^p := \left\{ f \in L^p(\mathbb{T}) : \widehat{f}(k) := \int_{\mathbb{T}} f(\zeta)\,\zeta^{-k}\frac{\lvert d\zeta\rvert}{2\pi} = 0 \text{ for } k < 0 \right\}$. Hardy classes can be identified with spaces of analytic functions on the unit disc $\mathbb{D}$: in particular, H^∞ is the space of all bounded analytic functions on $\mathbb{D}$;
E, E_*	separable Hilbert spaces;
H_E^2	vector-valued Hardy space $H^2$with values in E;
$L^\infty_{E\to E_*}$	class of bounded functions on the unit circle $\mathbb{T}$ whose values are bounded operators from E to E_*;
$H^\infty_{E\to E_*}$	operator Hardy class of bounded analytic functions whose values are bounded operators from E to E_*, $\lVert F\rVert_\infty := \sup_{z\in\mathbb{D}} \lVert F(z)\rVert = ess\sup_{\zeta\in\mathbb{T}} \lVert F(\zeta)\rVert$;
K_θ	the model space, $K_\theta := H_E^2 \ominus \theta H_E^2$ for the inner function $\theta \in H^\infty_{E\to E}$;
T_Φ	Toeplitz operator with symbol $\Phi \in L^\infty_{E\to E}$, $T_\Phi f := P_+(\Phi f)$, $f \in H^2_{E\to E}$, where P_+ is an orthogonal projection onto $H^2_{E\to E}$.

Throughout the paper all Hilbert spaces are assumed to be separable, and also we always assume that in any Hilbert space an orthonormal basis is fixed, so any operator $A : E \to E_*$ can be identified with its matrix.

2020 *Mathematics Subject Classification*. Primary 47A12
Keywords. Berezin symbol, Berezin $\theta-$ sequence, model operator, Toeplitz operator, Riesz basis, Toeplitz corona problem, $C-$unitary operator
Received: 14 August 2020; Accepted: 16 December 2020
Communicated by Fuad Kittaneh
Research supported by Researchers Supporting Project Number (RSP-2021/187), King Saud University, Riyadh, Saudi Arabia
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1. Introduction and Preliminaries

Let E and E_* be two separable Hilbert spaces, and H_E^2 be the vector-valued Hardy space H^2 with values in E. The function $\theta, \theta \in H_{E\to E_*}^{\infty}$ is called inner if its angular limiting values $\theta(\zeta)$ are unitary operators in E for almost all ζ in $\mathbb{T}$. We consider the unilateral shift operator S in the space H_E^2

$$Sf(z) = zf(z), \; S^* f(z) = \frac{f(z) - f(0)}{z},$$

the backward shift operator S^* (the adjoint of S) and its restriction T to S^*-invariant subspace K_θ,

$$T = S^* \mid K_\theta. \tag{1}$$

By the well-known Sz.-Nagy and Foias theorem [21], any contraction A in a Hilbert space H such that

$$s - \lim_n A^n = s - \lim A^{*n} = 0,$$

is unitarily equivalent to an operator T of the form (1), and the function θ turns out to be the characteristic operator function of the contraction A^*.

Following [11], and its references, we shall assume the following:

(i) That portion of the spectrum $\sigma(T)$ of the operator T which lies within the unit disc $\mathbb{D}$ consist of characteristic values which are simple poles of the resolvent.

(ii) The system of characteristic functions of operator (1), which correspond to characteristic values lying inside the unit disc $\mathbb{D}$, is complete in K_θ.

In terms of the characteristic functions θ, condition (i) may be written as follows: if $\lambda_k \in \sigma(T^*) \cap \mathbb{D}$, then

$$\theta(z) = \theta_k(z) B_k(z), |z| < 1, \tag{2}$$

where θ_k, B_k are inner functions,

$$B_k(z) := \frac{\lambda_k - z}{1 - \overline{\lambda}_k z} \frac{\overline{\lambda}_k}{|\lambda_k|} \pi_k + (I - \pi_k), |z| < 1,$$

π_k is the orthogonal projector of E onto the subspace $\ker \theta(\lambda_k)$ of zeros of $\theta(\lambda_k)$, and $\theta_k(\lambda_k)$ is a bounded invertible operator in E. Condition (ii) may also be expressed in terms of the characteristic function of the operator T^*.

Let $\left\{\overline{\lambda}_k\right\}_{k=1}^{\infty} = \sigma(T) \cap \mathbb{D}$ be a sequence of eigenvalues of operator T of the form (1). It can be proved that the system of eigenvectors of operator T is formed by the functions

$$\Phi_k(z) := \left(1 - |\lambda_k|^2\right)^{1/2} \frac{\theta_k^{*-1}(\lambda_k) e}{1 - \overline{\lambda}_k z}, e \in \pi_k E, \|e\| = 1, \tag{3}$$

where π_k is the orthogonal projector onto $\ker \theta(\lambda_k)$. The biorthogonal system consisting of eigenvectors of the operator T^* has the form

$$\Psi_k(z) := \left(1 - |\lambda_k|^2\right)^{1/2} \theta_k(z) \frac{e}{1 - \overline{\lambda}_k z}, e \in \pi_k E, \|e\| = 1. \tag{4}$$

(Note that when speaking of system (3) and (4), we mean any of the systems $\{\Phi_{k,i}\}$ and $\{\Psi_{k,i}\}$ obtained from (3) and (4), respectively, by a choice of an orthonormal basis $\left\{e_k^{(i)}\right\}_i$ in each $\pi_k E$.) The following lemma has been originally proved by Katsnelson [16], see also Nikolski and Pavlov [18].

Lemma 1.1. *If the Carleson condition*

$$\inf_k \left| \prod_{n \neq k} \frac{\lambda_k - \lambda_n}{1 - \overline{\lambda}_k \lambda_n} \frac{\lambda_n}{|\lambda_n|} \right| > 0 \tag{C}$$

is satisfied, then the system $\{\Psi_k(z)\}_{k=1}^{\infty}$ *is a Riesz basis in* K_θ.

Let $\mathcal{B}(H)$ denote the Banach algebra of all bounded linear operators on H. The following definition is introduced in [11].

Definition 1.2. *For any operator* $A \in \mathcal{B}(K_\theta)$, *its Berezin* $\theta-$*sequence is defined as*

$$\widetilde{A^\theta}(\lambda_k) := \langle A\Psi_k(z), \Psi_k(z) \rangle, k \geq 1.$$

By considering that θ_k *is an inner function and* $\|e\| = 1$, *it is easy to verify that* $\|\Psi_k\|_{H_E^2} = 1$ *for all* $k \geq 1$, *and hence* $\left|\widetilde{A^\theta}(\lambda_k)\right| \leq \|A\|$ *for all* $k \geq 1$, *that is* $\left\{\widetilde{A^\theta}(\lambda_k)\right\}_{k\geq 1}$ *is a bounded sequence.*

For any function $F \in H^\infty_{E\to E}$, the model operator $F(M_\theta)$ on K_θ is defined by $F(M_\theta) f = P_\theta(Ff)$, $f \in K_\theta$, where $P_\theta : H_E^2 \to K_\theta, P_\theta = I - T_\theta T_\theta^*$, is the orthogonal projector onto K_θ.

In this paper, we will study invertibility of operators $F(M_\theta), F \in H^\infty_{E\to E}$, via Berezin $\theta-$sequences.

Note that this paper was mainly motivated with a question of Treil from his paper [22] concerning to Operator Corona Problem, which is stated as follows: *does there exist* $\delta > 0$ *(close to 1) such that for any* $F \in H^\infty_{E\to E_*}$ *the inequality* $I \geq F(z)^* F(z) \geq \delta^2 I$ *implies that there exists* $G \in H^\infty_{E_*\to E}$ *such that* $GF = I$ *?*

(Remark that the counterexample constructed in [22] works only $\delta < \frac{1}{3}$, the method from [23] gives counterexample $\delta < \frac{1}{\sqrt{2}}$).

In the present paper, we investigate the similar question for the model operators $F(M_\theta)$, $F \in H^\infty_{E\to E}$ (see Section 2). We also study in Section 3 the so-called Toeplitz corona problem in the Bergman-Hilbert space $L_a^2 = L_a^2(\mathbb{D})$. Namely, we prove a necessary condition for its solvability. In Section 4, we introduce the notions $C-$ invertible and $C-$ unitary operator which are extensions of invertible, unitary and essentially unitary operators, and prove some necessary and sufficient conditions for $C-$ unitarity.

Before giving our results, note that the sequence $X := \{x_n\}_{n\geq 1}$, where $x_n \in H, n \geq 1$ is called a Riesz basis in H if there exists an isomorphism V mapping X onto an orthonormal basis $\{x_n : n \geq 1\}$; the operator V will be called the orthogonalizer of X. It is well known that X is a Riesz basis in H if there are positive constants c and C such that

$$c\left(\sum_{n\geq 1} |a_n|^2\right)^{1/2} \leq \left\|\sum_{n\geq 1} a_n x_n\right\| \leq C\left(\sum_{n\geq 1} |a_n|^2\right)^{1/2} \tag{5}$$

for all finite complex sequence $\{a_n\}_{n\geq 1}$. Note that if V is orthogonalizer of the family X, then the product $r(X) := \|V\| \left\|V^{-1}\right\|$ characterizes the deviation of the basis X from an orthonormal one and $\left\|V^{-1}\right\|$ and $\|V\|^{-1}$ are the best constants in the inequality (5); $r(X)$ will be referred to as the Riesz constant of the family X. Obviously, $r(X) \geq 1$.

2. Invertibility of model operators via Berezin $\theta-$sequences

The main result of this section is the following theorem in which the invertibility of some operators on K_θ via Berezin $\theta-$sequences, is proved.

Theorem 2.1. *Let* $\Psi := \{\Psi_k(z)\} = \left\{\theta_k(z) \frac{e}{1-\overline{\lambda}_k z}\left(1 - |\lambda_k|^2\right)^{1/2}\right\}$ *be a biorthogonal system (of system (4)) consisting of eigenfunctions of the operator* T^*, *where* $\Lambda := \{\lambda_k\}_{k=1}^{\infty}$ *satisfies Carleson condition (C), and let* $r(\Psi) := \|V\| \left\|V^{-1}\right\|$

be a corresponding Riesz constant of the Riesz basis Ψ (see Lemma 1.1), where V is an orthogonalizer of the system Ψ. Let $F(M_\theta)$ be a model operator acting in $K_\theta = H^2_E \ominus \theta H^2_E$ and satisfying the following conditions:

$$(i) \sum_{k=1}^{\infty} \left(\|F(M_\theta)\|^2 - \left| F\widetilde{(M_\theta)}^{\theta}(\lambda_k) \right|^2 \right) < +\infty;$$

$$(ii) \inf_{k\geq 1} \left| F\widetilde{(M_\theta)}^{\theta}(\lambda_k) \right| =: \delta > r(\Psi)\|V\| \, \mathrm{b}_{F(M_\theta)},$$

where

$$\mathrm{b}_{F(M_\theta)} := \left(\sum_{k=1}^{\infty} \left(\|F(M_\theta)\|^2 - \left| F\widetilde{(M_\theta)}^{\theta}(\lambda_k) \right|^2 \right) \right)^{1/2}.$$

Then, $F(M_\theta)$ is invertible in K_θ and

$$\left\| F(M_\theta)^{-1} \right\| \leq \left(\frac{\delta}{r(\Psi)} - \|V\| \, \mathrm{b}_{F(M_\theta)} \right)^{-1}.$$

Proof. By considering that $\Lambda \in$(C), it follows from Lemma 1.1 that Ψ is a Riesz basis in the model space K_θ. Therefore we have that

$$\|V\|^{-1} \left(\sum_k |a_k|^2 \right)^{1/2} \leq \left\| \sum_k a_k \Psi_k \right\| \leq \left\| V^{-1} \right\| \left(\sum_k |a_k|^2 \right)^{1/2} \tag{6}$$

for all finite sequences $\{a_k\}$ of complex numbers. By considering (6) and condition (i) of the theorem, we obtain :

$$\begin{aligned}
\left\| \sum_{k=1}^{N} a_k F\widetilde{(M_\theta)}^{\theta}(\lambda_k) \Psi_k(z) \right\| &\geq \|V\|^{-1} \left(\sum_{k=1}^{N} \left| a_k F\widetilde{(M_\theta)}^{\theta}(\lambda_k) \right|^2 \right)^{1/2} \\
&\geq \delta \|V\|^{-1} \left(\sum_{k=1}^{N} |a_k|^2 \right)^{1/2} \\
&\geq \frac{\delta}{\|V\| \left\| V^{-1} \right\|} \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\| \\
&= \frac{\delta}{r(\Psi)} \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\|,
\end{aligned}$$

hence

$$\left\| \sum_{k=1}^{N} a_k F\widetilde{(M_\theta)}^{\theta}(\lambda_k) \Psi_k(z) \right\| \geq \frac{\delta}{r(\Psi)} \left\| \sum_{k=1}^{N} a_k \Psi_k(z) \right\| \tag{7}$$

for all finite sequences $\{a_k\}_{k=1}^N$. Using condition (ii) of the theorem and inequalities (6), (7), we have for every sequence $\{a_k\}_{k=1}^N$ that

$$\begin{aligned}
\left\|F(M_\theta)\sum_{k=1}^{N}a_k\Psi_k(z)\right\| &= \left\|\sum_{k=1}^{N}a_kF(M_\theta)\Psi_k(z)\right\| \\
&= \left\|\sum_{k=1}^{N}a_k\left(F(M_\theta)\Psi_k - F\widetilde{(M_\theta)}^\theta(\lambda_k)\Psi_k + F\widetilde{(M_\theta)}^\theta(\lambda_k)\Psi_k\right)\right\| \\
&\geq \left\|\sum_{k=1}^{N}a_kF\widetilde{(M_\theta)}^\theta(\lambda_k)\Psi_k(z)\right\| \\
&\quad - \left\|\sum_{k=1}^{N}a_k\left(F(M_\theta)\Psi_k - F\widetilde{(M_\theta)}^\theta(\lambda_k)\Psi_k\right)\right\| \\
&\geq \frac{\delta}{r(\Psi)}\left\|\sum_{k=1}^{N}a_k\Psi_k(z)\right\| \\
&\quad - \sum_{k=1}^{N}|a_k|\left\|F(M_\theta)\Psi_k - F\widetilde{(M_\theta)}^\theta(\lambda_k)\Psi_k\right\| \\
&\geq \frac{\delta}{r(\Psi)}\left\|\sum_{k=1}^{N}a_k\Psi_k(z)\right\| \\
&\quad - \left(\sum_{k=1}^{N}|a_k|^2\right)^{1/2}\left(\sum_{k=1}^{N}\left\|F(M_\theta)\Psi_k - F\widetilde{(M_\theta)}^\theta(\lambda_k)\Psi_k\right\|^2\right)^{1/2} \\
&\geq \frac{\delta}{r(\Psi)}\left\|\sum_{k=1}^{N}a_k\Psi_k(z)\right\| \\
&\quad - \|V\|\left\|\sum_{k=1}^{N}a_k\Psi_k(z)\right\|\left(\sum_{k=1}^{\infty}\left\|F(M_\theta)\Psi_k - F\widetilde{(M_\theta)}^\theta(\lambda_k)\Psi_k\right\|^2\right)^{1/2} \\
&\geq \frac{\delta}{r(\Psi)}\left\|\sum_{k=1}^{N}a_k\Psi_k(z)\right\| \\
&\quad - \|V\|\left\|\sum_{k=1}^{N}a_k\Psi_k(z)\right\|\left(\sum_{k=1}^{\infty}\left\langle F(M_\theta)\Psi_k - F\widetilde{(M_\theta)}^\theta(\lambda_k)\Psi_k, F(M_\theta)\Psi_k - F\widetilde{(M_\theta)}^\theta(\lambda_k)\Psi_k\right\rangle\right)^{1/2} \\
&\geq \frac{\delta}{r(\Psi)}\left\|\sum_{k=1}^{N}a_k\Psi_k(z)\right\| \\
&\quad - \|V\|\left\|\sum_{k=1}^{N}a_k\Psi_k\right\|\left(\sum_{k=1}^{\infty}\left(\|F(M_\theta)\|^2 - \left|F\widetilde{(M_\theta)}^\theta(\lambda_k)\right|^2 - \left|F\widetilde{(M_\theta)}^\theta(\lambda_k)\right|^2 + \left|F\widetilde{(M_\theta)}^\theta(\lambda_k)\right|^2\right)\right)^{1/2} \\
&= \frac{\delta}{r(\Psi)}\left\|\sum_{k=1}^{N}a_k\Psi_k(z)\right\| - \|V\|\left(\sum_{k=1}^{\infty}\left(\|F(M_\theta)\|^2 - \left|F\widetilde{(M_\theta)}^\theta(\lambda_k)\right|^2\right)\right)^{1/2}\left\|\sum_{k=1}^{N}a_k\Psi_k\right\| \\
&= \left(\frac{\delta}{r(\Psi)} - \|V\|\,b_{F(M_\theta)}\right)\left\|\sum_{k=1}^{N}a_k\Psi_k(z)\right\|.
\end{aligned}$$

Thus

$$\left\|F(M_\theta)\sum_{k=1}^{N}a_k\Psi_k(z)\right\| \geq \left(\frac{\delta}{r(\Psi)} - \|V\|\, b_{F(M_\theta)}\right)\left\|\sum_{k=1}^{N}a_k\Psi_k\right\|.$$

Now, since Ψ is a complete system in K_θ, we have from the latter that

$$\left\|F(M_\theta) f\right\| \geq \left(\frac{\delta}{r(\Psi)} - b_{F(M_\theta)}\|V\|\right)\left\|f\right\| \tag{8}$$

for all $f \in K_\theta$.

Similarly, it is easy to show that

$$\sum_{k=1}^{\infty}\left\|F(M_\theta)^*\Psi_k - F\widetilde{(M_\theta)}^{*\theta}(\lambda_k)\Psi_k\right\|^2 \leq \sum_{k=1}^{\infty}\left(1-\left|F\widetilde{(M_\theta)}^{\theta}(\lambda_k)\right|^2\right).$$

Then, by similar arguments it can be proved that

$$\left\|F(M_\theta)^* f\right\| \geq \left(\frac{\delta}{r(\Psi)} - b_{F(M_\theta)}\|V\|\right)\left\|f\right\| \tag{9}$$

for all $f \in K_\theta$.

Hence, we deduce from (8), (9) and condition (ii) of the theorem that $F(M_\theta)$ is an invertible operator on K_θ and

$$\left\|F(M_\theta)^{-1}\right\| \leq \left(\frac{\delta}{r(\Psi)} - b_{F(M_\theta)}\|V\|\right)^{-1},$$

which proves the theorem. □

We remark that it easy to see from the proof of Theorem 2.1 that by the same method it can be proved invertibility of more general operators on K_θ including, in particular, truncated Toeplitz operators T_F with symbols F in $L^\infty_{E\to E}$ (for the related results, see [11, 12]).

3. On the Toeplitz Corona Problem

Let dA denote Lebesgue area measure on $\mathbb{D}$, normalized so that the measure of the disc $\mathbb{D}$ is 1. The Bergman space $L_a^2 := L_a^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are square integrable with respect to the measure dA. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_φ with symbol φ is defined on L_a^2 by $T_\varphi f = P(\varphi f)$, where $P : L^2(\mathbb{D}, dA) \to L_a^2$ is the orthogonal projector. Using the concrete form of the reproducing kernel $k_{z,L_a^2}(w), z, w \in \mathbb{D}$, we can express the Toeplitz operator to be the integral operator:

$$\begin{aligned} T_\varphi f &= \int_{\mathbb{D}} \varphi(w) f(w)\overline{k_{z,L_a^2}(w)}dA(w) \\ &= \int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1-\overline{w}z)^2}dA(w) \end{aligned}$$

for f in L_a^2.

In this section, we will consider the solvability of the operator equation in the set of Toeplitz operators on the Bergman space L_a^2 (i.e., the Toeplitz Corona Problem in the Bergman space):

$$X_1T_{\varphi_1} + X_2T_{\varphi_2} + \ldots + X_nT_{\varphi_n} = I,$$

where $T_{\varphi_i}, i = 1, 2, ..., n$, are given Toeplitz operators on the Bergman space L_a^2.

Note that for $\varphi \in L^\infty(\mathbb{D})$, the Berezin symbol $\widetilde{T_\varphi}$ of the Toeplitz operators T_φ on L_a^2 is defined as (see Englis [6] and Zhu [24])

$$\begin{aligned}\widetilde{T_\varphi}(z) \quad : \quad &= \left\langle T_\varphi \widehat{k}_{\lambda,L_a^2}, \widehat{k}_{\lambda,L_a^2}\right\rangle = \left\langle T_\varphi \frac{1-|z|^2}{(1-\overline{w}z)^2}, \frac{1-|z|^2}{(1-\overline{w}z)^2}\right\rangle \\ &= \int_{\mathbb{D}} \varphi(w) \left|k_{z,L_a^2}(w)\right|^2 dA(w),\end{aligned}$$

where $\widehat{k}_{z,L_a^2}$ is the normalized Bergman reproducing kernel of $L_a^2(\mathbb{D})$ given by

$$\widehat{k}_{z,L_a^2}(w) = \frac{1-|z|^2}{(1-\overline{w}z)^2} \quad (z \in \mathbb{D}).$$

For simplicity, we will denote $\widetilde{\varphi} := \widetilde{T_\varphi}$. It is well known that $\widetilde{\varphi} = \varphi$ for any bounded harmonic function φ (see Ahern, Flores and Rudin [1] and Englis [5]). For other results related with Berezin symbols and Toeplitz operators the reader can be found in [2, 5–7, 10, 17, 24, 25]. The following lemma belongs to Axler and Zheng [2].

Lemma 3.1. *If φ is bounded harmonic function, then the function $\lambda \to \left\|T_{\varphi-\varphi(\lambda)}\widehat{k}_{\lambda,L_a^2}\right\|_{L_a^2}$ has nontangential limit 0 at almost every point of $\partial\mathbb{D}$.*

Now we are ready to formulate and prove our next result.

Theorem 3.2. *Let $\varphi_1, \varphi_2, ..., \varphi_n$ be bounded harmonic functions on $\mathbb{D}$. If there exist the bounded harmonic functions $\psi_1, \psi_2, ..., \psi_n$ such that*

$$T_{\psi_1}T_{\varphi_1} + T_{\psi_2}T_{\varphi_2} + ... + T_{\psi_n}T_{\varphi_n} = I,$$

where $T_{\psi_i}, T_{\varphi_i}$ $(i = 1, 2, ..., n)$ are Toeplitz operators on L_a^2, then

$$ess \inf_{\mathbb{T}} \left(\left|\varphi_1(\zeta)\right| + ... + \left|\varphi_n(\zeta)\right|\right) > 0. \tag{10}$$

Proof. Let $T_{\psi_1}T_{\varphi_1} + T_{\psi_2}T_{\varphi_2} + ... + T_{\psi_n}T_{\varphi_n} = I$. Then, passing to the Berezin transform, we have:

$$\begin{aligned} 1 &= \widetilde{T_{\psi_1}T_{\varphi_1}}(\lambda) + \widetilde{T_{\psi_2}T_{\varphi_2}}(\lambda) + ... + \widetilde{T_{\psi_n}T_{\varphi_n}}(\lambda) \\ &= \left\langle T_{\psi_1}T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}, \widehat{k}_{\lambda,L_a^2}\right\rangle + ... + \left\langle T_{\psi_n}T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}, \widehat{k}_{\lambda,L_a^2}\right\rangle \\ &= \left\langle T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}, T_{\psi_1}^*\widehat{k}_{\lambda,L_a^2}\right\rangle + ... + \left\langle T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}, T_{\psi_n}^*\widehat{k}_{\lambda,L_a^2}\right\rangle \\ &= \left\langle T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi}_1}\widehat{k}_{\lambda,L_a^2}\right\rangle + ... + \left\langle T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi}_n}\widehat{k}_{\lambda,L_a^2}\right\rangle \\ &= \left\langle T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}, \widetilde{\overline{\psi}}_1(\lambda)\widehat{k}_{\lambda,L_a^2}\right\rangle + ... + \left\langle T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}, \widetilde{\overline{\psi}}_n(\lambda)\widehat{k}_{\lambda,L_a^2}\right\rangle \\ &\quad + \left\langle T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi}_1}\widehat{k}_{\lambda,L_a^2} - \widetilde{\overline{\psi}}_1(\lambda)\widehat{k}_{\lambda,L_a^2}\right\rangle + ... + \left\langle T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi}_n}\widehat{k}_{\lambda,L_a^2} - \widetilde{\overline{\psi}}_n(\lambda)\widehat{k}_{\lambda,L_a^2}\right\rangle \\ &= \widetilde{\psi_1}(\lambda)\widetilde{T_{\varphi_1}}(\lambda) + ... + \widetilde{\psi_n}(\lambda)\widetilde{T_{\varphi_n}}(\lambda) + \left\langle T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi}_1-\widetilde{\overline{\psi}}_1(\lambda)}\widehat{k}_{\lambda,L_a^2}\right\rangle + ... + \left\langle T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi}_n-\widetilde{\overline{\psi}}_n(\lambda)}\widehat{k}_{\lambda,L_a^2}\right\rangle \\ &= \widetilde{\psi_1}(\lambda)\widetilde{\varphi_1(\lambda)} + ... + \widetilde{\psi_n}(\lambda)\widetilde{\varphi_n(\lambda)} + \left\langle T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi_1-\widetilde{\psi_1}(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\rangle + ... + \left\langle T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi_n-\widetilde{\psi_n}(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\rangle \\ &= \psi_1(\lambda)\varphi_1(\lambda) + ... + \psi_n(\lambda)\varphi_n(\lambda) + \left\langle T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi_1-\widetilde{\psi_1}(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\rangle + ... + \left\langle T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi_n-\widetilde{\psi_n}(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\rangle \end{aligned}$$

and hence

$$1 = \psi_1(\lambda)\varphi_1(\lambda) + ... + \psi_n(\lambda)\varphi_n(\lambda) + \left\langle T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi_1-\psi_1(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\rangle + ... + \left\langle T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}, T_{\overline{\psi_n-\psi_n(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\rangle$$

for all $\lambda \in \mathbb{D}$. Therefore we obtain :

$$\begin{aligned} 1 &\leq \left|\psi_1(\lambda)\right|\left|\varphi_1(\lambda)\right| + ... + \left|\psi_n(\lambda)\right|\left|\varphi_n(\lambda)\right| \\ &\quad + \left\|T_{\varphi_1}\widehat{k}_{\lambda,L_a^2}\right\|\left\|T_{\overline{\psi_1-\psi_1(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\| + ... + \left\|T_{\varphi_n}\widehat{k}_{\lambda,L_a^2}\right\|\left\|T_{\overline{\psi_n-\psi_n(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\| \\ &\leq \left|\psi_1(\lambda)\right|\left|\varphi_1(\lambda)\right| + ... + \left|\psi_n(\lambda)\right|\left|\varphi_n(\lambda)\right| \\ &\quad + \left\|\varphi_1\right\|_{L^\infty(\mathbb{D})}\left\|T_{\overline{\psi_1-\psi_1(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\| + ... + \left\|\varphi_n\right\|_{L^\infty(\mathbb{D})}\left\|T_{\overline{\psi_n-\psi_n(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\|. \end{aligned}$$

Note that every bounded harmonic function has nontangential limits at almost every point of $\mathbb{T}$, and by applying Lemma 3.1 we have that the functions $\lambda \to \left\|T_{\overline{\psi_i-\psi_i(\lambda)}}\widehat{k}_{\lambda,L_a^2}\right\|_{L_a^2}$ $(i = 1, 2, ..., n)$ have nontangential limits 0 at almost every point of $\mathbb{T}$. Then we have from the latter inequality that

$$\begin{aligned} 1 &\leq \left|\psi_1(\zeta)\right|\left|\varphi_1(\zeta)\right| + ... + \left|\psi_n(\zeta)\right|\left|\varphi_n(\zeta)\right| \\ &\leq \left\|\psi_1\right\|_{L^\infty(\mathbb{T})}\left|\varphi_1(\zeta)\right| + ... + \left\|\psi_n\right\|_{L^\infty(\mathbb{T})}\left|\varphi_n(\zeta)\right| \end{aligned}$$

for almost all $\zeta \in \mathbb{T}$. This implies that

$$ess \inf_{\mathbb{T}}\left(\left|\varphi_1(\zeta)\right| + ... + \left|\varphi_n(\zeta)\right|\right) \geq \frac{1}{\max\left\{\left\|\psi_i\right\|_{L^\infty(\mathbb{T})} : 1 \leq i \leq n\right\}} > 0,$$

which proves the theorem. □

We do not know: *is (10) a sufficient condition in Theorem 3.2?*

4. On the $C-$ unitarity of $C-$ invertible operators

Recall that the Berezin symbol $\widetilde{A}$ of an operator on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ with reproducing kernel $k_{\mathcal{H},\lambda}$ is defined by

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle, \lambda \in \Omega,$$

where $\widehat{k}_{\mathcal{H},\lambda} := \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|}$ is the normalized reproducing kernel for the space $\mathcal{H}$. It is clear that:

(i) $\widetilde{A}$ is a bounded function on Ω and $\sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right| =: ber(A) \leq \|A\|$; $ber(A)$ is called the Berezin number of operator A;

(ii) the Berezin set $Ber(A) := Range\left(\widetilde{A}\right)$ is contained in the numerical range $W(A) := \{\langle Ax, x\rangle : \|x\| = 1\}$ of operator A, and hence $ber(A) \leq w(A)$, where $w(A) := \sup\{|\langle Ax, x\rangle| : x \in H \text{ and } \|x\| = 1\}$ is the numerical radius of operator $A \in \mathcal{B}(H)$.

(iii) $\widetilde{A^*} = \overline{\widetilde{A}}$, and hence $\left|\widetilde{A^*}\right| = \left|\widetilde{A}\right|$.

For more detail about Berezin symbols, and their applications, see for instance Berezin [3, 4], Karaev [11, 13, 14], Zhu [24] and Zorboska [25].

Definition 4.1. *(i) Let $A, C \in \mathcal{B}(\mathcal{H})$. We say that A is a $C-$ invertible operator if there exists an operator $B \in \mathcal{B}(\mathcal{H})$ (which is called a $C-$ inverse of A)such that*

$$BA = AB = C. \tag{11}$$

(ii) We say that a $C-$ invertible operator A is $C-$ unitary if $B = A^*$.

Note that $C-$ unitary operator is a generalization of unitary ($C = I$) and essentially unitary ($C = I + K$, where K is compact) operators on a Hilbert space. So, the main result of this section (Theorem 4.1) improves some results of the works [9, 11, 14, 15].

It is easy to see that if $A \in \mathcal{B}(\mathcal{H}(\Omega))$ is a $C-$ unitary operator, then $\left\|A\widehat{k}_{\mathcal{H},\lambda}\right\|^2 = \widetilde{C}(\lambda)$ and $\left\|A^*\widehat{k}_{\mathcal{H},\lambda}\right\|^2 = \widetilde{C}(\lambda)$ for all $\lambda \in \Omega$ (which in particular shows that $\widetilde{C} \geq 0$, and hence $Ber(C) \subset [0, +\infty)$). Indeed, it follows from (11) that

$$\begin{aligned}
\widetilde{C}(\lambda) &= \widetilde{BA}(\lambda) = \left\langle BA\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle \\
&= \left\langle A^*A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{H,\lambda}\right\rangle = \left\langle A\widehat{k}_{\mathcal{H},\lambda}, A\widehat{k}_{\mathcal{H},\lambda}\right\rangle \\
&= \left\|A\widehat{k}_{\mathcal{H},\lambda}\right\|^2,
\end{aligned}$$

and similarly we obtain from $C = AB = AA^*$ that $\widetilde{C}(\lambda) = \left\|A^*\widehat{k}_{\mathcal{H},\lambda}\right\|^2$ for all $\lambda \in \Omega$.

In this section, we give sufficient conditions for $C-$ unitarity which are very close to the above mentioned necessary conditions. We denote by A^{-1C} the $C-$inverse of $C-$invertible operator, i.e, $A^{-1C} := B$ in (11).

Note that unitary operators can be characterized as invertible contractions with contractive inverses, i.e., as operators A with $\|A\| \leq 1$ and $\left\|A^{-1}\right\| \leq 1$ (see, for instance, Furuta [8]). Sano and Uchiyama [19] improved this result by proving that if A is an invertible operator on the Hilbert space H such that $w(A) \leq 1$ and $w\left(A^{-1}\right) \leq 1$, then A is unitary (see also Stampfli [20, Corollary 1]). Now the following question is natural: is it true that if A is invertible and $ber(A) \leq 1$ and $ber\left(A^{-1}\right) \leq 1$, then A is unitary?

In the following theorem we partially solve this question. Our proof is based mainly on a Furuta's argument contained in [8].

Theorem 4.2. *Let $C \in \mathcal{B}(\mathcal{H}(\Omega))$ be fixed such that $Re\left(\widetilde{C}\right) \geq 0$. If $A \in \mathcal{B}(\mathcal{H}(\Omega))$ is a $C-$invertible operator such that $\left\|A^*\widehat{k}_{\mathcal{H},\lambda}\right\|^2 \leq Re\left(\widetilde{C}(\lambda)\right)$ and $\left\|A^{-1C}\widehat{k}_{\mathcal{H},\lambda}\right\|^2 \leq Re\left(\widetilde{C}(\lambda)\right)$ for all $\lambda \in \Omega$, then A is a $C-$ unitary operator.*

Proof. Indeed, we have for every $\lambda \in \Omega$ that

$$\begin{aligned}
\left\|\left(A^* - A^{-1C}\right)\widehat{k}_{\mathcal{H},\lambda}\right\|^2 &= \left\langle\left(A^* - A^{-1C}\right)\widehat{k}_{\mathcal{H},\lambda}, \left(A^* - A^{-1C}\right)\widehat{k}_{\mathcal{H},\lambda}\right\rangle \\
&= \left\|A^*\widehat{k}_{\mathcal{H},\lambda}\right\|^2 + \left\|A^{-1C}\widehat{k}_{\mathcal{H},\lambda}\right\|^2 - \left\langle A^*\widehat{k}_{\mathcal{H},\lambda}, A^{-1C}\widehat{k}_{\mathcal{H},\lambda}\right\rangle \\
&\quad - \left\langle A^{-1C}\widehat{k}_{\mathcal{H},\lambda}, A^*\widehat{k}_{\mathcal{H},\lambda}\right\rangle \\
&= \left\|A^*\widehat{k}_{\mathcal{H},\lambda}\right\|^2 + \left\|A^{-1C}\widehat{k}_{\mathcal{H},\lambda}\right\|^2 \\
&\quad - \left\langle\widehat{k}_{\mathcal{H},\lambda}, AA^{-1C}\widehat{k}_{\mathcal{H},\lambda}\right\rangle - \left\langle AA^{-1C}\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle \\
&= \left\|A^*\widehat{k}_{\mathcal{H},\lambda}\right\|^2 + \left\|A^{-1C}\widehat{k}_{\mathcal{H},\lambda}\right\|^2 \\
&\quad - \left\langle\widehat{k}_{\mathcal{H},\lambda}, C\widehat{k}_{\mathcal{H},\lambda}\right\rangle - \left\langle C\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle \\
&= \left\|A^*\widehat{k}_{\mathcal{H},\lambda}\right\|^2 + \left\|A^{-1C}\widehat{k}_{\mathcal{H},\lambda}\right\|^2 - \left(\overline{\widetilde{C}(\lambda)} + \widetilde{C}(\lambda)\right) \\
&= \left\|A^*\widehat{k}_{\mathcal{H},\lambda}\right\|^2 + \left\|A^{-1C}\widehat{k}_{\mathcal{H},\lambda}\right\|^2 - 2Re\left(\widetilde{C}(\lambda)\right) \leq 0
\end{aligned}$$

(by conditions of the theorem). This shows that $\left(A^* - A^{-1C}\right)k_{\mathcal{H},\lambda} = 0$ for all $\lambda \in \Omega$ which implies that $A^{-1C} = A^*$ because the system $\left\{k_{\mathcal{H},\lambda} : \lambda \in \Omega\right\}$ is complete on $\mathcal{H}$. The theorem is proved. □

It is easy to see that the conditions of this theorem implies the inequalities $ber\left(AA^{*}\right) \leq \sup_{\lambda\in\Omega} Re\left(\widetilde{C}\left(\lambda\right)\right)$ and $ber\left(A^{-1C^{*}}A^{-1C}\right) \leq \sup_{\lambda\in\Omega} Re\left(\widetilde{C}\left(\lambda\right)\right)$.

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