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Module Decompositions by Images of Fully Invariant Submodules

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Abstract. Let R be a ring with identity, M be a right R-module and F be a fully invariant submodule of M. The concept of an F-inverse split module M has been investigated recently. In this paper, we approach to this concept with a different perspective, that is, we deal with a notion of an F-image split module M, and study various properties and obtain some characterizations of this kind of modules. By means of F-image split modules M, we focus on modules M in which fully invariant submodules F are dual Rickart direct summands. In this way, we contribute to the notion of a T-dual Rickart module M by considering $\overline{Z}^2(M)$ as the fully invariant submodule F of M. We also deal with a notion of relatively image splitness to investigate direct sums of image split modules. Some applications of image split modules to rings are given.

1. Introduction

Throughout this paper R denotes an associative ring with identity and modules are unitary right R-modules unless otherwise stated. For a module M, $S = \mathrm{End}_R(M)$ is the ring of all right R-module endomorphisms of M and F stands for a fully invariant submodule of M (i.e., $f(F) \subseteq F$ for every $f \in S$). Maeda [8] and Hattori [5] studied Rickart rings (or principally projective rings), independently. A ring is called *right Rickart* if every principal right ideal is projective, equivalently, the right annihilator of any single element is generated by an idempotent as a right ideal. A left Rickart ring is defined similarly. Recently, the notion of Rickart rings was generalized to the module theoretic version and investigated in [1] and [6]. A module M is said to be *Rickart* if the right annihilator in M of any single element of S is generated by an idempotent of S, that is, for any $f \in S$, $r_M(f) = \mathrm{Ker} f = eM$ for some $e^2 = e \in S$. In [2], a concept of T-Rickart modules was defined by considering the second singular (or Goldie torsion) submodule of a module, namely, a module M is called *T-Rickart* if $t_M(f) = \{m \in M \mid f(m) \in Z_2(M)\}$ is a direct summand of M for every $f \in S$. On the other hand, in [15], a module M is said to be *F-inverse split* if $f^{-1}(F)$ is a direct summand of M for every $f \in S$. There are some interesting connections between these classes of modules. For example, in [15], it is proved that M is F-inverse split if and only if M has a decomposition $M = F \oplus N$ where N is a Rickart module. Since the second singular submodule $Z_2(M)$ of M is fully invariant in M, being a T-Rickart module and being a $Z_2(M)$-inverse split module are the same. Some applications of the notion of an F-inverse split module M are presented in [4], [14], [15] and [16] by considering certain fully invariant submodules aside from the second singular submodule.

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As a dual version of Rickart property for modules, in [7], a module M is called *dual Rickart* if $\mathrm{Im} f$ is a direct summand of M for every endomorphism f of M. Motivated by the concepts of dual Rickart modules and T-Rickart modules, T-dual Rickart modules were introduced in [3], that is, a module M is called *T-dual Rickart* if $f(\overline{Z}^2(M))$ is a direct summand of M for every $f \in S$ where $\overline{Z}^2(M) = \overline{Z}(\overline{Z}(M))$ and $\overline{Z}(M) = \cap\{\mathrm{Ker} f : f \in \mathrm{Hom}_R(M, N)$ where N is small in its injective hull$\}$ which was defined in [13]. With the inspiration of these works, it is of interest to present the notion of F-image split modules in a sense of a dual version of F-inverse split modules. We say that a module M is *F-image split* if $f(F)$ is a direct summand of M for every $f \in S$.

In the light of aforementioned concepts, it is a reasonable question that what kind of properties does F gain when a module M is splitted by the images of F? This question is one of the motivations to deal with the notion of an F-image split module M. We answer this question in Theorem 2.2, that is, F becomes a dual Rickart module in addition to be a direct summand of M. The concept of T-dual Rickart modules produces dual Rickart modules by employing the submodule $\overline{Z}^2(M)$ of M. By using the fully invariant submodule F of a module M, we produce much more dual Rickart modules for this general setting. Therefore, the concept of F-image splitness is more general than that of T-dual Rickart modules. These connections make the concept of an F-image split module M more attractive to study.

In Section 2, we give some properties and characterizations of F-image split modules. We get some results by considering the singular submodule as a fully invariant submodule. We also deal with an F-image split module concept for rings and we present some applications about these rings. In Section 3, we focus on when the direct sums of F-image split modules M satisfy the same property. In this direction, we study relatively F-image splitness. Lastly, in Section 4, we introduce strongly F-image split modules and observe a main characterization of these modules.

In what follows, $\mathrm{Soc}(M)$ and $Z(M)$ stand for the socle and the singular submodule of a module M, also, $J(R)$ denotes the Jacobson radical of a ring R, respectively. For a positive integer n, $M_n(R)$ denotes the ring of $n \times n$ matrices over a ring R.

2. F-image split modules

Throughout this paper, F denotes a fully invariant submodule of a module M under consideration. In this section we study the concept of an F-image split module M and get properties about this class of modules. We investigate useful characterizations for this notion. Also, we obtain some results about the ring cases of F-image split modules as an application to the ring theory.

Definition 2.1. A module M is called *F-image split* if $f(F)$ is a direct summand of M for every $f \in S$.

It is clear that every semisimple module M is F-image split and so every module M over a semisimple ring is F-image split. Obviously, every module M is 0-image split. It can be obtained from the definition, a module M is dual Rickart if and only if it is M-image split.

We now give an efficient characterization for an F-image split module M. Thanks to this characterization we can get dual Rickart modules by means of fully invariant submodules.

Theorem 2.2. *The following are equivalent for a module M.*

1. *M is an F-image split module.*
2. *F is a dual Rickart direct summand of M.*

Proof. (2) ⇒ (1) Let $f \in S$. As F is a direct summand of M, there exists an idempotent $e \in S$ such that $F = eM$. Then, $\mathrm{End}_R(F) = eSe$. Since F is dual Rickart, $efe(F)$ is a direct summand of F. We claim that $efe(F) = f(F)$. For any $x \in F$, $efe(x) = ef(x) = f(x)$. Therefore, $efe(F) = f(F)$. The rest is clear.
(1) ⇒ (2) Let M be F-image split. Then, for $1_M \in S$, $1_M(F) = F$ is a direct summand of M. Hence, $F = eM$ for some $e^2 = e \in S$. To see that F is a dual Rickart module, let $f \in \mathrm{End}_R(F) = eSe$. Thus, there exists $g \in S$ such that $f = ege$. Since M is F-image split, $g(F)$ is a direct summand of M. As F is fully invariant, $f(F) = ege(F) = g(F)$. So $f(F)$ is a direct summand of M. By modularity condition, $f(F)$ is a direct summand of F and so F is a dual Rickart module. □

Corollary 2.3. *Let M be an F-image split module and N a fully invariant submodule which contains F. If every endomorphism of N can be extended to an endomorphism of M, then N is F-image split.*

Corollary 2.4. *Every indecomposable F-image split module M is either dual Rickart or $F = 0$.*

The following corollary is a direct consequence of Corollary 2.4 if we consider the singular submodule as a fully invariant submodule.

Corollary 2.5. *Every indecomposable $Z(M)$-image split module M is either nonsingular or singular dual Rickart.*

Proof. Let M be an indecomposable $Z(M)$-image split module. By Corollary 2.4, $Z(M) = 0$, i.e., M is nonsingular or M is dual Rickart. In the latter case, $Z(M) = M$. □

Example 2.6. Let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ where F is a field. Then, $Z(R_R) = 0$. Thus, R_R is $Z(R_R)$-image split.

We approach to Theorem 2.2 in terms of singular submodules.

Theorem 2.7. *Let M be a module. Then, the following are equivalent.*

1. *M is $Z(M)$-image split.*
2. *$M = Z(M) \oplus N$ where $Z(M)$ is dual Rickart and N is nonsingular.*
3. *$Z(M)$ is a dual Rickart direct summand of M.*

Proof. (1) ⇒ (2) By hypothesis and Theorem 2.2, M has a decomposition $M = Z(M) \oplus N$ where $Z(M)$ is a dual Rickart module. Since $Z(M)$ is an essential submodule of $Z_2(M)$ and $Z_2(M) = Z(M) \oplus (Z_2(M) \cap N)$, we have $Z(M) = Z_2(M)$. Hence, N is nonsingular since $N \cong M/Z_2(M)$.
(2) ⇒ (3) It is clear.
(3) ⇒ (1) M is $Z(M)$-image split by Theorem 2.2. □

In the next result we investigate that F-image split property for a module M is transferred to direct summands of M.

Proposition 2.8. *If M is an F-image split module and N is a direct summand of M, then N is an $(F \cap N)$-image split module.*

Proof. Assume that $M = N \oplus K$ for some submodule K of M and M is F-image split. By [12], $F = (F \cap N) \oplus (F \cap K)$. Let $e : M \to N$ be the canonical homomorphism and $g \in \mathrm{End}_R(N)$. Since M is an F-image split module, $ge(F)$ is a direct summand of M. Also, $ge(F) = ge(F \cap N) + ge(F \cap K) = ge(F \cap N)$. As $g(F \cap N) = ge(F \cap N) = ge(F)$, $g(F \cap N)$ is a direct summand of M. Hence, $g(F \cap N)$ is a direct summand of N, as asserted. □

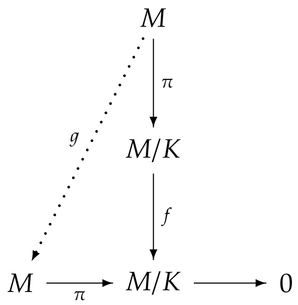
Corollary 2.9. *If M is a $Z(M)$-image split module, then any direct summand N of M is $Z(N)$-image split.*

Proof. Let M be a $Z(M)$-image split module and N a direct summand of M. Then, N is $(N \cap Z(M))$-image split by Proposition 2.8. Hence, N is $Z(N)$-image split since $N \cap Z(M) = Z(N)$. □

Proposition 2.10. *Let M be a quasi-projective module. Then, M is F-image split if and only if for every submodule K of M with $K \subseteq g(F)$ for each $0 \neq g \in End_R(M)$, M/K is F/K-image split.*

Proof. Let M be F-image split and $f \in \mathrm{End}_R(M/K)$. Since M is a quasi-projective module, there exists $g \in \mathrm{End}_R(M)$ such that the following diagram commutes. The module M being F-image split implies that $M = g(F) \oplus L$ for some submodule L of M. Then, $M/K = (g(F)/K) + ((L + K)/K)$ and this sum is direct since

$(g(F)/K) \cap ((L + K)/K) = \{0 + K\}$. Also, it can be shown that $f(F/K) = g(F)/K$. Hence, $f(F/K)$ is a direct summand of M/K, and so M/K is F/K-image split.



The converse is obvious. □

Recall that M has the *summand sum property* (*SSP*) if the sum of two direct summands is a direct summand of M. Also, M has the *strong summand sum property* (*SSSP*) if the sum of any number of direct summands is again a direct summand of M.

Proposition 2.11. *For an F-image split module M, the following statements hold.*

1. *Let K, L be direct summands of M and $K \subseteq F$. Then, $K + L$ is a direct summand of M.*
2. *M has SSP for direct summands which are contained in F.*

Proof. It is clear from the proof of [3, Proposition 3.14]. □

Theorem 2.12. *The following are equivalent for a module M.*

1. *M is F-image split.*
2. $\sum_{f\in I} f(F)$ *is a direct summand of M for every finitely generated right ideal I of S.*
3. $\sum_{f\in I} f(F)$ *is a direct summand of M for every finite subset I of S.*

Proof. (1) ⇒ (2) Let $I =< f_1, \dots, f_n >$ be a finitely generated right ideal of S. Since M is F-image split, $f_i(F)$ is a direct summand of M for each $1 \leq i \leq n$. Hence, $\sum_{f\in I} f(F)$ is a direct summand of M by Proposition 2.11(2).
(2) ⇒ (1) For every $f \in S$, $\sum_{g\in I} g(F) = f(F)$ for which $I = fS$. Hence, the proof is clear.
(1) ⇔ (3) It is obvious. □

Now we consider the concept of F-image splitness for rings. Note that I is an ideal of a ring R if and only if it is a fully invariant submodule of R_R.

Definition 2.13. Let I be an ideal of a ring R. Then, R is called *right I-image split* if for every $f \in \mathrm{End}_R(R_R)$, $f(I)$ is a direct summand of R_R, i.e., R is I-image split as a right R-module.

The left I-image splitness for a ring R can be defined similarly where I is an ideal of R. The right I-image split rings need not be left I-image split as the following example shows, therefore being an I-image split ring is not left-right symmetric.

Example 2.14. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is a field. Consider the ideal $I = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ of R. Then, $R = I \oplus J$ where $J = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ is a right ideal of R and I is dual Rickart. Hence, R is a right I-image split. Since I is essential in R as a left ideal, it is not a direct summand of R as a left ideal. Therefore, R is not left I-image split.

In the next theorem, we characterize a right I-image split ring R.

Theorem 2.15. *Let R be a ring and I be an ideal of R. Then, the following are equivalent.*

1. *For every positive integer n, $M_n(R)$ is right $M_n(I)$-image split.*
2. *R is right I-image split.*
3. *I is a direct summand of R as a right ideal and $End_R(I)$ is a von Neumann regular ring.*
4. *For every $e^2 = e \in R$, eRe is right eIe-image split.*
5. *For every finitely generated free R-module M, M is I-image split.*

Proof. (1) ⇒ (2), (4) ⇒ (2) and (5) ⇒ (2) are clear.
(2) ⇒ (3) Let R be a right I-image split ring. By Theorem 2.2, I is a direct summand of R as a right ideal and it is also a dual Rickart module. Then, for every $f \in \mathrm{End}_R(I)$, $\mathrm{Im}f$ is a direct summand of I. Since $I/\mathrm{Ker}f \cong \mathrm{Im}f$ and I is projective, $\mathrm{Ker}f$ is a direct summand of I. Thus, $\mathrm{End}_R(I)$ is a von Neumann regular ring by [17, Corollary 3.2].
(2) ⇒ (1) Let n be a positive integer. Since I is a direct summand of R as a right ideal, there exists a right ideal J of R such that $R = I \oplus J$. Hence, $M_n(J)$ is a right ideal of $M_n(R)$ such that $M_n(R) = M_n(I) \oplus M_n(J)$. Note that $\mathrm{End}_{M_n(R)}(M_n(I)) \cong M_n(\mathrm{End}_R(I))$ is a von Neumann regular ring because $\mathrm{End}_R(I)$ is a von Neumann regular ring as in the proof of (2) ⇒ (3). Thus, $M_n(R)$ is right $M_n(I)$-image split by [7, Theorem 3.8].
(2) ⇒ (4) Let R be right I-image split and $e^2 = e \in R$. Then, eR is eI-image split. We claim that for every $f \in \mathrm{End}_{eRe}(eRe)$, $f(eIe)$ is a direct summand of eRe as a right ideal. Since $\mathrm{End}_{eRe}(eRe) \cong eRe \cong \mathrm{End}_R(eR)$, $f(eI)$ is a direct summand of eR. Hence, there exists a right ideal J of eR such that $eR = f(eI) \oplus J$. Thus, $eRe = f(eIe) + Je$. Since $f(eIe) \cap Je \subseteq f(eI) \cap J = 0$, $eRe = f(eIe) \oplus Je$. Therefore, $f(eIe)$ is a direct summand of eRe as a right ideal.
(3) ⇒ (5) Let K be a finitely generated free R-module. By (3), I is a direct summand of R as a right ideal, and so I is also a direct summand of K. Since $\mathrm{End}_R(I)$ is von Neumann regular, $\mathrm{Im}f$ is a direct summand of I for every $f \in \mathrm{End}_R(I)$ by [17, Corollary 3.2]. Hence, I is dual Rickart. Thus, K is I-image split by Theorem 2.2. □

Theorem 2.16. *Let R be a ring and I be an ideal of R. Then, the following are equivalent.*

1. *$\bigoplus_{n=1}^{\infty} R_n$ is $(\bigoplus_{n=1}^{\infty} I_n)$-image split where $R_n = R$ and $I_n = I$ for all n.*
2. *I is a direct summand of R as a right ideal and $End_R(I)$ is a semisimple ring.*

Proof. (1) ⇒ (2) Suppose that $\bigoplus_{n=1}^{\infty} R_n$ is $(\bigoplus_{n=1}^{\infty} I_n)$-image split where $R_n = R$ and $I_n = I$ for all n. By Proposition 2.8, R is right I-image split. In particular, I is a direct summand of R as a right ideal. As in the proof of (2) ⇒ (3) in Theorem 2.15, one can see that $\mathrm{End}_R(\bigoplus_{n=1}^{\infty} I_n)$ is a von Neumann regular ring. Let $K = \mathrm{End}_R(I)$. Note that $\mathrm{End}_R(\bigoplus_{n=1}^{\infty} I_n) \cong \mathrm{End}_K(\bigoplus_{n=1}^{\infty} K_n)$, where $K_n = K$ for all n. By [17, Theorem 3.5], K is a semisimple ring.
(2) ⇒ (1) Suppose that I is a direct summand of R as a right ideal and $\mathrm{End}_R(I)$ is a semisimple ring. Then, $\bigoplus_{n=1}^{\infty} I_n$ is a direct summand of $\bigoplus_{n=1}^{\infty} R_n$ where $I_n = I$ and $R_n = R$ for all n. Let $K = \mathrm{End}_R(I)$. Since $\mathrm{End}_R(\bigoplus_{n=1}^{\infty} I_n) \cong \mathrm{End}_K(\bigoplus_{n=1}^{\infty} K_n)$ where $K_n = K$ for all n, and K is a semisimple ring, we have that $\mathrm{End}_R(\bigoplus_{n=1}^{\infty} I_n)$ is a von Neumann regular ring. Hence, $\mathrm{Im}f$ is a direct summand of $\bigoplus_{n=1}^{\infty} I_n$ for every $f \in \mathrm{End}_R(\bigoplus_{n=1}^{\infty} I_n)$ by [17, Corollary 3,2]. Thus, $\bigoplus_{n=1}^{\infty} I_n$ is a dual Rickart module. Consequently, $\bigoplus_{n=1}^{\infty} R_n$ is $(\bigoplus_{n=1}^{\infty} I_n)$-image split by Theorem 2.2. □

We close this section by giving some applications about I-image split rings R.

Proposition 2.17. *If R is a right $Z(R_R)$-image split ring, then it is right nonsingular.*

Proof. Let R be a right $Z(R_R)$-image split ring and $x \in Z(R_R)$. Assume that $x \neq 0$ and we reach a contradiction. By definition, $xZ(R_R)$ is a direct summand of R. It entails that $xZ(R_R)$ has an idempotent e. Hence there exists $t \in Z(R_R)$ such that $e = xt$. Since $x, t \in Z(R_R)$, we have $e \in Z(R_R)$. This is the required contradiction since $r_R(e) = (1 - e)R$ is not essential in R. It follows $Z(R_R) = 0$. Therefore, R is right nonsingular. □

Recall that in [10], a right module M is called *mininjective* if for every simple right ideal K of R, each homomorphism $K \to M$ extends to a homomorphism $R \to M$. The next result shows that every module over Soc(·)-image split ring is mininjective.

Proposition 2.18. *If R is a right $Soc(R_R)$-image split ring, then every right R-module is mininjective.*

Proof. Let R be a right $\mathrm{Soc}(R_R)$-image split ring. Then, $R = \mathrm{Soc}(R_R) \oplus K$ for some right ideal K of R. Hence, $J(R) = \mathrm{Rad}(\mathrm{Soc}(R_R)) \oplus \mathrm{Rad}(K)$. This yields $J(R) = \mathrm{Rad}(K)$ since $\mathrm{Rad}(\mathrm{Soc}(R_R)) = 0$. Thus, $\mathrm{Soc}(R_R) \cap J(R) = 0$ and so every right R-module is mininjective by [11, Theorem 2.36]. □

3. Direct sums of *F*-image split modules

A direct sum of F_i-image split modules M_i where $i \in \mathcal{I}$ for some index set $\mathcal{I}$ need not satisfy image splitness as shown in [3, Example 4.1]. In this section, we investigate under which conditions direct sums of F_i-image split modules M_i have the same property.

Proposition 3.1. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R-modules for an arbitrary index set $\mathcal{I}$. If for every $i \in \mathcal{I}$, M_i is a fully invariant submodule of $\bigoplus_{i \in \mathcal{I}} M_i$, then M_i is F_i-image split for every $i \in \mathcal{I}$ if and only if $\bigoplus_{i \in \mathcal{I}} M_i$ is $\bigoplus_{i \in \mathcal{I}} F_i$-image split.*

Proof. Let $\bigoplus_{i \in \mathcal{I}} M_i$ be $\bigoplus_{i \in \mathcal{I}} F_i$-image split. Then, by Proposition 2.8, M_i is F_i-image split for every $i \in \mathcal{I}$. For the necessity, let $M = \bigoplus_{i \in \mathcal{I}} M_i$, $F = \bigoplus_{i \in \mathcal{I}} F_i$ and $f = (f_{ij}) \in S$ where $f_{ij} \in \mathrm{Hom}_R(M_j, M_i)$. Since for every $i \in \mathcal{I}$, M_i is a fully invariant submodule of $\bigoplus_{i \in \mathcal{I}} M_i$, $\mathrm{Hom}_R(M_j, M_i) = 0$ for every $i, j \in \mathcal{I}$ with $i \neq j$. By hypothesis, $f_{ii}(F_i)$ is a direct summand of M_i for each $i \in \mathcal{I}$. On the other hand, we have $f(F) = \bigoplus_{i \in \mathcal{I}} f_{ii}(F_i)$. Hence, $f(F)$ is a direct summand of M, as asserted. □

Recall that a module is said to be *abelian* if every idempotent element of its endomorphism ring is central. By the fact that a module M is abelian if and only if every direct summand of M is fully invariant in M, the following result is an immediate consequence of Proposition 3.1.

Corollary 3.2. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R-modules for an arbitrary index set $\mathcal{I}$ and $\bigoplus_{i \in \mathcal{I}} M_i$ be an abelian module. Then, M_i is F_i-image split for all $i \in \mathcal{I}$ if and only if $\bigoplus_{i \in \mathcal{I}} M_i$ is $\bigoplus_{i \in \mathcal{I}} F_i$-image split.*

In the following, we introduce relatively F-image splitness in order to a more comprehensively study on direct sums of F_i-image split modules M_i where $i \in \mathcal{I}$ for some index set $\mathcal{I}$.

Definition 3.3. A module M is called *F-image split module relative to N* (or shortly, *N-F-image split*) if for each $f \in \mathrm{Hom}_R(M, N)$, $f(F)$ is a direct summand of N.

Theorem 3.4. *Let M and N be R-modules. Then, M is an F-image split module relative to N if and only if for every direct summand L of M and every submodule K of N, L is $(L \cap F)$-image split relative to K.*

Proof. Let L be a direct summand of M, K a submodule of N and $f \in \mathrm{Hom}_R(L, K)$. Then, $L = eM$ for some $e^2 = e \in S$ and $fe \in \mathrm{Hom}_R(M, N)$. Since M is N-F-image split, $fe(F)$ is a direct summand of N. As $fe(F) \subseteq K$, $fe(F)$ is a direct summand of K. We claim that $fe(F) = f(L \cap F)$. To see that let $x \in f(L \cap F)$. Then, there exists $y \in L \cap F$ such that $f(y) = x$. Since $y \in L = eM$, $x = f(y) = fe(y) \in fe(F)$. Hence, $f(L \cap F) \subseteq fe(F)$. It is clear that $fe(F) \subseteq f(L \cap F)$. Thus, $f(L \cap F)$ is a direct summand of K, as asserted. The converse is clear. □

The next result is obtained as an immediate consequence of Theorem 3.4 if we take into account of the fully invariant submodule $\overline{Z}^2(M)$ of a module M.

Corollary 3.5. [3, Corollary 3.13] *Let M be an R-module. Then, the following are equivalent.*

1. *M is T-dual Rickart.*
2. *For any submodule N of M, each direct summand L of M is T-dual Rickart relative to N.*
3. *If L and N are direct summands of M, then for any $\varphi \in Hom_R(L, N)$, $\varphi(\overline{Z}^2(L))$ is a direct summand of N.*

Proposition 3.6. *Let $\{M_i\}_{i\in\mathcal{I}}$ be a class of R-modules for an index set $\mathcal{I}$ and N an R-module with a fully invariant submodule F of $\bigoplus_{i\in\mathcal{I}} M_i$. Then, the following hold.*

1. *Let N have SSP and $\mathcal{I}$ be finite. Then, $\bigoplus_{i\in\mathcal{I}} M_i$ is N-F-image split if and only if M_i is N-$(F \cap M_i)$-image split for all $i \in I$.*
2. *Let N have SSSP and $\mathcal{I}$ be arbitrary. Then,*
   (a) *$\bigoplus_{i\in\mathcal{I}} M_i$ is N-F-image split if and only if M_i is N-$(F \cap M_i)$-image split for all $i \in I$.*
   (b) *$\prod_{i\in\mathcal{I}} M_i$ is N-F-image split if and only if M_i is N-$(F \cap M_i)$-image split for all $i \in I$.*

Proof. (1) Let $\bigoplus_{i\in\mathcal{I}} M_i$ is N-F-image split, then M_i is N-$(F \cap M_i)$-image split for all $i \in I$ by Theorem 3.4. To see the converse statement let $f : \bigoplus_{i\in\mathcal{I}} M_i \to N$ and $\iota_i : M_i \to \bigoplus_{i\in\mathcal{I}} M_i$ be a inclusion. Then, $f_i = f\iota_i \in \mathrm{Hom}(M_i, N)$. It can be seen that $f(F) = \sum_{i\in\mathcal{I}} f_i(F \cap M_i)$. By hypothesis, $f_i(F \cap M_i)$ is a direct summand of N for all $i \in I$. Since N has SSP, $\sum_{i\in\mathcal{I}} f_i(F \cap M_i)$ is a direct summand of N. Hence, $f(F)$ is a direct summand of N as asserted.
The proof of (2) is similar to that of (1). □

Corollary 3.7. *Let $\{M_i\}_{i\in\mathcal{I}}$ be R-modules where $\mathcal{I} = \{1, 2, \ldots, n\}$ and F a fully invariant submodule of $\bigoplus_{i\in\mathcal{I}} M_i$. Then, for each $j \in \mathcal{I}$, $\bigoplus_{i\in\mathcal{I}} M_i$ is M_j-F-image split if and only if M_i is M_j-$(F \cap M_i)$-image split for all $i \in \mathcal{I}$.*

Proof. If for each $j \in \mathcal{I}$, $\bigoplus_{i\in\mathcal{I}} M_i$ is M_j-F-image split, then M_i is M_j-$(F\cap M_i)$-image split for all $i \in \mathcal{I}$ by Theorem 3.4. To see the converse statement, let M_i be M_j-$(F \cap M_i)$-image split for all $i \in \mathcal{I}$. Then, M_j is $F \cap M_j$-image split. Hence, M_j has SSP for direct summands which are contained in $F \cap M_j$ by Proposition 2.11. Thus, the rest can be proved similar to the proof of Proposition 3.6(1). □

Theorem 3.8. *Let $\{M_i\}_{i\in\mathcal{I}}$ be a class of R-modules, N an R-module where $\mathcal{I} = \{1, 2, \ldots, n\}$ and assume that M_i is M_j-projective for all $i \geq j \in \mathcal{I}$. Then, an R-module N is $\bigoplus_{i\in\mathcal{I}} M_i$-$F$-image split if and only if N is M_j-F-image split for all $j \in \mathcal{I}$.*

Proof. The sufficiency is clear by Theorem 3.4. Let N be an M_j-F-image split for all $j \in \mathcal{I}$. We use induction on n. Let $n = 2$, f be a homomorphism from N to $M_1 \oplus M_2$ and $\pi_i : M_1 \oplus M_2 \to M_i$ be a natural projection where $i = 1, 2$. Since N is M_2-F-image split, $\pi_2 f(F)$ is a direct summand of M_2. Hence, $M_1 \oplus \pi_2 f(F)$ is a direct summand of $M_1 \oplus M_2$. To see $M_1 + f(F) = M_1 \oplus \pi_2 f(F)$, let $z + y \in M_1 + f(F)$. Then, $y = \pi_1 y + \pi_2 y$. Hence,

$z+y = z+\pi_1 y+\pi_2 y \in M_1 + \pi_2 f(F)$. For the reverse inclusion, let $x+y \in M_1 \oplus \pi_2 f(F)$. Then, there exists $z \in F$ such that $y = \pi_2 f(z)$. Hence, $x + y = x + \pi_2 f(z) + \pi_1 f(z) - \pi_1 f(z) = x - \pi_1 f(z) + f(z) \in M_1 + f(F)$, as asserted. By hypothesis, M_2 is M_1-projective and so $\pi_2 f(F)$ is M_1-projective. Then, there exists a submodule K of $f(F)$ such that $M_1 + f(F) = M_1 \oplus K$ by [9, Lemma 4.47]. Hence, $f(F) = K \oplus (M_1 \cap f(F))$. Thus, $\pi_1 f(F) = M_1 \cap f(F)$ since $K \cap M_1 = 0$. Hence, $f(F) = K \oplus \pi_1 f(F)$ which is a direct summand of $K \oplus M_1$. Since $K \oplus M_1 = M_1 \oplus \pi_2 f(F)$, $f(F)$ is a direct summand of $M_1 \oplus M_2$. Thus, N is $(M_1 \oplus M_2)$-F-image split. Consequently, we complete the rest of the proof by induction on n. □

Corollary 3.9. *Let $\{M_i\}_{i\in\mathcal{I}}$ be a class of R-modules where $\mathcal{I} = \{1, 2, \ldots, n\}$ and assume that M_i is M_j-projective for all $i \geq j \in \mathcal{I}$. Then, $\bigoplus_{i\in\mathcal{I}} M_i$ is F-image split if and only if M_i is M_j-$(F \cap M_i)$-image split for all $i, j \in \mathcal{I}$.*

Proof. The sufficiency is clear by Theorem 3.4. For the necessity, let M_i be M_j-$(F \cap M_i)$-image split for all $i, j \in \mathcal{I}$. Then, $\bigoplus_{i\in\mathcal{I}} M_i$ is M_j-F-image split by Corollary 3.7. Hence, $\bigoplus_{i\in\mathcal{I}} M_i$ is F-image split by Theorem 3.8. □

4. Strongly *F*-image split modules

In this section, we deal with a module M for which $f(F)$ is not only a direct summand but also a fully invariant submodule for every $f \in S$.

Definition 4.1. An R-module M is called *strongly F-image split* if for every $f \in S$, $f(F)$ is a fully invariant direct summand of M.

It is obvious that M being a strongly F-image split module implies that it is F-image split. We now investigate when the converse holds.

Theorem 4.2. *The following are equivalent for a module M.*

1. *M is strongly F-image split.*
2. *M is F-image split and each direct summand of M which is contained in F is fully invariant.*
3. *F is a dual Rickart and abelian direct summand of M.*

Proof. (1) ⇒ (2) Let N be a direct summand of M with $N \subseteq F$. Then, there exists $e^2 = e \in S$ such that $N = eM$. It can be shown that $e(F) = N$. By hypothesis, $e(F)$ is fully invariant in M. Thus, N is fully invariant in M.
(2) ⇒ (3) By Theorem 2.2 and (2), F is a dual Rickart direct summand of M. To see that F is an abelian module, let L be a direct summand of F. Then, L is a direct summand of M. Hence, L is a fully invariant submodule of M by hypothesis. We show that L is fully invariant in F. We have $F = eM$ for some $e^2 = e \in S$, and so $\mathrm{End}_R(F) = eSe$. Let $f \in \mathrm{End}_R(F)$. Then, there exists $g \in \mathrm{End}_R(M)$ such that $f = ege$. Hence, $f(L) = ege(L) \subseteq L$ since L is fully invariant in M. Thus, we have every direct summand of F is fully invariant in F. So, F is abelian.
(3) ⇒ (1) Let $f \in S$. By Theorem 2.2, M is F-image split. Hence, $f(F)$ is a direct summand of M. We need to show that $f(F)$ is fully invariant in M. We have $F = eM$ for some $e^2 = e \in S$, and so $\mathrm{End}_R(F) = eSe$. Thus, $efe(F)$ is a fully invariant direct summand of F by hypothesis. Since F is fully invariant in M, $efe(F) = f(F)$. Therefore, $f(F)$ is a fully invariant submodule of M because F and $f(F)$ is fully invariant in M and F, respectively. So, M is strongly F-image split, as claimed. □

The following example shows that an F-image split module M need not be strongly F-image split in general.

Example 4.3. Let n be a positive integer with $n \geq 2$ and M a vector space over a field K of dimension n. Then, M is semisimple and so it is dual Rickart. Hence, M is M-image split. But it is not abelian. Thus, M is not strongly M-image split.

We end the paper by observing some basic results concerning direct summands and direct sums of strongly F-image split modules.

Proposition 4.4. *Let M be a strongly F-image split module. Then, every direct summand N of M is strongly $(N \cap F)$-image split.*

Proof. Let $K \subseteq N \cap F$ be a direct summand of N. Since N is a direct summand of M, $M = N \oplus T$ for some submodule T of M. By Proposition 2.8, T is $(T \cap F)$-image split. Then, there exists a submodule T' of T such that $T = (T \cap F) \oplus T'$. Hence, $M = K \oplus K' \oplus (T \cap F) \oplus T'$ for some submodule K' of N. Also, $K \oplus (T \cap F)$ is contained in F since $K \subseteq N \cap F$. Thus, $K \oplus (T \cap F)$ is fully invariant in M by hypothesis and Theorem 4.2. To see K is fully invariant in N, let $f \in \mathrm{End}_R(N)$. Then, $(f \oplus 1_T)(K \oplus (T \cap F)) \subseteq K \oplus (T \cap F)$. Hence, $f(K) \subseteq K$, as asserted. □

Theorem 4.5. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R-modules for an arbitrary index set $\mathcal{I}$ and $M = \bigoplus_{i \in I} M_i$. Then, M is strongly F-image split if and only if for each $i \in \mathcal{I}$, M_i is strongly $(F \cap M_i)$-image split and $Hom_R(F \cap M_i, F \cap M_j) = 0$ for every $i, j \in \mathcal{I}$ with $i \neq j$.*

Proof. Let M be a strongly F-image split module. Then, for every $i \in \mathcal{I}$, M_i is strongly $(F \cap M_i)$-image split by Proposition 4.4. Since F is fully invariant in M, $F = \bigoplus_{i \in \mathcal{I}} (F \cap M_i)$. Also, F is a dual Rickart and abelian module by hypothesis and Theorem 4.2. Hence, for every $i \in \mathcal{I}$, $F \cap M_i$ is fully invariant in F. Thus, $\mathrm{Hom}_R(F \cap M_i, F \cap M_j) = 0$ for every $i, j \in \mathcal{I}$ with $i \neq j$. To see the converse statement, let $f = [f_{ij}] \in \mathrm{End}_R(\bigoplus_{i \in I} M_i)$ where $f_{ij} \in \mathrm{Hom}_R(M_j, M_i)$. Then, $f(F) = \bigoplus_{i \in I} f_{ii}(F \cap M_i)$ by hypothesis, and so $f(F)$ is a direct summand of M. Since for each $i \in \mathcal{I}$, $f_{ii}(F \cap M_i)$ is fully invariant in M_i, $f(F)$ is fully invariant in M. □

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