



New Conformable Fractional Operator and Some Related Inequalities

Deniz Uçar^a

^aDepartment of Mathematics, Faculty of Sciences and Arts, Uşak University, Uşak, Turkey

Abstract. In this study, we introduce a new conformable derivative, namely the beta-conformable derivative. We derive Taylor's theorem for this derivative. We also investigate some new properties of Taylor's theorem and some useful related theorems for the beta-conformable derivative. In the light of the new operator, we extend some recent and classical integral inequalities including Steffensen and Hermite-Hadamard inequality.

1. Introduction

Since L'Hospital asked "What does $\frac{d^n f}{dx^n}$ mean if $n = \frac{1}{2}$ " to Leibniz, many researchers tried to define a fractional derivative. Most of them defined integral form for the fractional derivative. The most popular ones are:

(i) The Riemann-Liouville fractional derivative of a function f is defined as

$$D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\alpha-1} f(t) dt, \quad n-1 < \alpha \leq n.$$

(ii) Caputo's definition of fractional derivative is illustrated as follows

$${}^c D_t^\alpha (f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha \leq n.$$

(iii) The modified Liouville fractional derivative of a function f is defined as

$$D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\alpha-1} (f(t) - f(0)) dt, \quad n-1 < \alpha \leq n.$$

(iv) [11] The conformable fractional derivative of f of order α is defined by

$$f^{(\alpha)}(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

2020 *Mathematics Subject Classification.* Primary 34A08; Secondary 26A33, 26D15
Keywords. Taylor's formula, beta-integral, Steffensen inequality, Hermite-Hadamard inequality
Received: 03 August 2020; Revised: 30 January 2021; Accepted: 14 February 2021
Communicated by Miodrag Spalević
Email address: deniz.ucar@usak.edu.tr (Deniz Uçar)

for $t > 0$, $\alpha \in (0, 1)$.

(v) [14] The modified conformable fractional derivative is defined as

$$D^\alpha (f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}$$

for $t > 0$, $\alpha \in (0, 1)$.

For a review of this topic we direct the reader to the monograph [1]. However those fractional derivatives have some inconsistencies. In instance, if α is not a natural number, most of the defined fractional derivatives do not satisfy $D_a^\alpha(1) = 0$. Some of the fractional derivatives do not satisfy product rule for two functions. The conformable and modified conformable fractional derivatives satisfy the common properties of the standard rules but they have some limitations. We can see the weakness of the defined fractional derivatives in [14].

A. Atangana et al in [2] proposed a suitable derivative called the Beta-derivative that allowed us to escape the lack of the fractional derivatives. We use beta-fractional derivative introduced by Abdon Atangana in [2] to obtain our results.

Definition 1.1. Let f be a function, such that $f: [a, \infty) \rightarrow \mathbb{R}$. Then, the beta-derivative of a function f is defined as

$${}^A D_x^\beta (f(x)) = \lim_{\varepsilon \rightarrow 0} \frac{f\left(x + \varepsilon \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(x)}{\varepsilon},$$

for all $x \geq a$, $\beta \in (0, 1]$. Then if the limit exists, f is said to be β -differentiable.

There is a relation between β -derivative and usual derivative.

$${}^A D_x^\beta (f(x)) = \left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} f'(x),$$

where $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

Definition 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function on the opened interval (a, b) , then the β -integral of f is given as:

$${}^A I_t^\beta (f(t)) = \int_0^t \left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} f(x) dx.$$

This integral was recently referred to as the Atangana beta-integral.

Definition 1.3. Let $\beta \in (0, 1]$ and $0 \leq a < b$. A function $f : [a, b] \rightarrow \mathbb{R}$ is β -fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(t) d_\beta t := \int_a^b \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} f(t) dt$$

exists and finite.

We assume the reader is familiar with the notation and basic results for fractional calculus. For a review of this topic we direct the reader to the monograph [2, 4, 5, 12].

2. Main Results

In this section, we give the main theorem of the paper and obtain some results close to the results in classical calculus. We first introduce Taylor formula with a new parameter.

Theorem 2.1. *Let $\beta \in (0, 1]$ and $n \in \mathbb{N}$. If the function f is $(n + 1)$ order β – fractional differentiable on $[0, \infty)$ and $s, t \in [0, \infty)$, then we have*

$$\begin{aligned} f(t) &= \sum_{k=0}^n \frac{\beta^{-k}}{k!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta k} D_s^{k\beta} f(s) \\ &\quad + \frac{\beta^{-n}}{n!} \int_s^t \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta n} D_\tau^{(n+1)\beta} f(\tau) d_\beta \tau. \end{aligned} \quad (1)$$

Proof. Using integraton by parts, we have

$$\begin{aligned} &\frac{\beta^{-n}}{n!} \int_s^t \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta n} D_\tau^{(n+1)\beta} f(\tau) d_\beta \tau \\ &= -\frac{\beta^{-n}}{n!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta n} D_s^{n\beta} f(s) + \frac{\beta^{1-n}}{(n-1)!} \int_s^t \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta(n-1)} D_\tau^{n\beta} f(\tau) d_\beta \tau. \end{aligned}$$

By the same way, integrating the second part of the right of equality, we obtain

$$\begin{aligned} &\frac{\beta^{1-n}}{(n-1)!} \int_s^t \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta(n-1)} D_\tau^{n\beta} f(\tau) d_\beta \tau \\ &= -\frac{\beta^{1-n}}{(n-1)!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta(n-1)} D_s^{(n-1)\beta} f(s) \\ &\quad + \frac{\beta^{2-n}}{(n-2)!} \int_s^t \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta(n-2)} D_\tau^{(n-1)\beta} f(\tau) d_\beta \tau. \end{aligned}$$

If we continue integrating by this way, we have

$$\begin{aligned} &\frac{\beta^{-n}}{n!} \int_s^t \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta n} D_\tau^{(n+1)\beta} f(\tau) d_\beta \tau \\ &= -\frac{\beta^{-n}}{n!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta n} D_s^{n\beta} f(s) \\ &\quad - \frac{\beta^{1-n}}{(n-1)!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta(n-1)} D_s^{(n-1)\beta} f(s) \\ &\quad + \dots - \frac{\beta^{-1}}{n!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta} D_s^\beta f(s) + \int_s^t D_\tau^\beta f(\tau) d_\beta \tau \\ &= -\frac{\beta^{-n}}{n!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta n} D_s^{n\beta} f(s) \\ &\quad - \frac{\beta^{1-n}}{(n-1)!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta(n-1)} D_s^{(n-1)\beta} f(s) \\ &\quad + \dots - \frac{\beta^{-1}}{n!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta} D_s^\beta f(s) + f(t) - f(s). \end{aligned}$$

Thus we prove the equality (1). We call the integral in the last inequality β -Taylor Remainder of the function f . \square

Definition 2.2. Let $\beta \in (0, 1]$ and the function f is $(n + 1)$ times β -fractional differentiable on $[0, \infty)$. Using (1), we define the remainder function by

$$R_{n,f}(s, t) := f(s) - \sum_{k=0}^n \frac{\beta^{-k}}{k!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right] D_s^{k\beta} f(s),$$

and

$$R_{n,f}(s, t) = \frac{\beta^{-n}}{n!} \int_s^t \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^n D_\tau^{(n+1)\beta} f(\tau) d_\beta \tau, \quad (2)$$

for $n > -1$.

Theorem 2.3. Let f and g are continuous on the closed interval $[a, b]$ and also $g \geq 0$. Then there exists a point $c \in [a, b]$ where

$$\int_a^b f(t) g(t) d_\beta t = f(c) \int_a^b g(t) d_\beta t.$$

Proof. We define $m := \min_{a \leq t \leq b} f(t)$, $M := \max_{a \leq t \leq b} f(t)$. So we have

$$m \leq f(t) \leq M. \quad (3)$$

Multiplying both sides of the inequality (3) by the function $\left(t + \frac{1}{\Gamma(\beta)} \right)^{\beta-1} g(t)$ and integrating over (a, b) with respect to t , we get

$$mg(t) \leq f(t) g(t) \leq Mg(t)$$

$$m \int_a^b g(t) d_\beta t \leq \int_a^b f(t) g(t) d_\beta t \leq M \int_a^b g(t) d_\beta t$$

$$m \leq \frac{\int_a^b f(t) g(t) d_\beta t}{\int_a^b g(t) d_\beta t} \leq M,$$

where $\int_a^b g(t) d_\beta t \neq 0$. If $\int_a^b g(t) d_\beta t = 0$, then we can choose any point c . Since the function f is continuous on $[a, b]$, f takes each value over $[m, M]$ at least once. So we have

$$f(c) = \frac{\int_a^b f(t) g(t) d_\beta t}{\int_a^b g(t) d_\beta t}$$

for some $c \in [a, b]$ and hence the proof is completed. \square

Remark 2.4. If we apply Theorem 2.3 to the β -Taylor Remainder (2), we have

$$\begin{aligned} R_{n,f}(s,t) &= D_t^{(n+1)\beta} f(c) \int_s^t \frac{\beta^{-n}}{n!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^n} d_\beta \tau \\ &= D_t^{(n+1)\beta} f(c) \frac{\beta^{-n}}{(n+1)!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(s + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^{n+1}}. \end{aligned}$$

We call this form of remainder β -Lagrange Remainder.

Lemma 2.5. Let $\beta \in (0, 1]$ and the function f is $(n+1)$ times β -fractional differentiable on $[0, \infty)$. If β -Taylor's remainder is defined as in (2), then the following inequality holds.

$$\begin{aligned} &\int_a^b \frac{\beta^{-n-1}}{(n+1)!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^{n+1}} D_\tau^{(n+1)\beta} f(\tau) d_\beta \tau \\ &= \int_a^t R_{n,f}(a, \tau) d_\beta \tau + \int_t^b R_{n,f}(b, \tau) d_\beta \tau. \end{aligned} \quad (4)$$

Proof. We use mathematical induction to prove (4). For $n = -1$, we get

$$\begin{aligned} &\int_a^b \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^{n+1}} D_\tau^{(n+1)\beta} f(\tau) d_\beta \tau \\ &= \int_a^b f(\tau) d_\beta \tau = \int_a^t f(\tau) d_\beta \tau + \int_t^b f(\tau) d_\beta \tau. \end{aligned}$$

Assume that (4) holds for $n = k - 1$,

$$\begin{aligned} &\int_a^b \frac{\beta^{-k}}{k!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^k} D_\tau^{k\beta} f(\tau) d_\beta \tau \\ &= \int_a^t R_{k-1,f}(a, \tau) d_\beta \tau + \int_t^b R_{k-1,f}(b, \tau) d_\beta \tau. \end{aligned}$$

Let $n = k$. Using the integration by parts, we obtain

$$\begin{aligned} &\int_a^b \frac{\beta^{-k-1}}{(k+1)!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^{k+1}} D_\tau^{(k+1)\beta} f(\tau) d_\beta \tau \\ &= \frac{\beta^{-k-1}}{(k+1)!} \left\{ \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(b + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^{k+1}} D_\tau^{k\beta} f(b) \right. \\ &\quad \left. - \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(a + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^{k+1}} D_\tau^{k\beta} f(a) \right\} \\ &\quad + \int_a^b \frac{\beta^{-k}}{k!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^k} D_\tau^{k\beta} f(\tau) d_\beta \tau \\ &= \frac{\beta^{-k-1}}{(k+1)!} \left\{ \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(b + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^{k+1}} D_\tau^{k\beta} f(b) \right. \\ &\quad \left. - \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(a + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{\beta^{k+1}} D_\tau^{k\beta} f(a) \right\} + \int_a^t R_{k-1,f}(a, \tau) d_\beta \tau + \int_t^b R_{k-1,f}(b, \tau) d_\beta \tau. \end{aligned} \quad (5)$$

Since

$$\begin{aligned} & \frac{\beta^{-k}}{k!} D_{\tau}^{k\beta} f(b) \int_b^t \left[\left(\tau + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(b + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^k d_{\beta} \tau \\ &= \frac{\beta^{-k-1}}{(k+1)!} D_{\tau}^{k\beta} f(b) \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(b + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^{k+1}, \end{aligned}$$

and

$$\begin{aligned} & \frac{\beta^{-k}}{k!} D_{\tau}^{k\beta} f(a) \int_a^t \left[\left(\tau + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(a + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^k d_{\beta} \tau \\ &= \frac{\beta^{-k-1}}{(k+1)!} D_{\tau}^{k\beta} f(a) \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(a + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^{k+1}. \end{aligned}$$

Then, we can write (5) to obtain

$$\begin{aligned} & \int_a^b \frac{\beta^{-k-1}}{(k+1)!} \left[\left(t + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^{k+1} D_{\tau}^{(k+1)\beta} f(\tau) d_{\beta} \tau \\ &= \int_a^t R_{k-1,f}(a, \tau) d_{\beta} \tau + \int_t^b R_{k-1,f}(b, \tau) d_{\beta} \tau \\ & \quad + \frac{\beta^{-k}}{k!} D_{\tau}^{k\beta} f(b) \int_b^t \left[\left(\tau + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(b + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^k d_{\beta} \tau \\ & \quad - \frac{\beta^{-k}}{k!} D_{\tau}^{k\beta} f(a) \int_a^t \left[\left(\tau + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(a + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^k d_{\beta} \tau \\ &= \int_a^t \left\{ R_{k-1,f}(a, \tau) - \frac{\beta^{-k}}{k!} D_{\tau}^{k\beta} f(a) \left[\left(\tau + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(a + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^k \right\} d_{\beta} \tau \\ & \quad + \int_t^b \left\{ R_{k-1,f}(b, \tau) - \frac{\beta^{-k}}{k!} D_{\tau}^{k\beta} f(b) \left[\left(\tau + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(b + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^k \right\} d_{\beta} \tau. \end{aligned}$$

This completes the proof. \square

Corollary 2.6. Let $\beta \in (0, 1]$ in Lemma 2.5, then we have

$$\begin{aligned} & \int_a^b \frac{\beta^{-n-1}}{(n+1)!} \left[\left(a + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^{n+1} D_{\tau}^{(n+1)\beta} f(\tau) d_{\beta} \tau \\ &= \int_a^b R_{n,f}(b, \tau) d_{\beta} \tau \int_a^b \frac{\beta^{-n-1}}{(n+1)!} \left[\left(b + \frac{1}{\Gamma(\beta)} \right)^{\beta} - \left(\tau + \frac{1}{\Gamma(\beta)} \right)^{\beta} \right]^{n+1} D_{\tau}^{(n+1)\beta} f(\tau) d_{\beta} \tau \\ &= \int_a^b R_{n,f}(a, \tau) d_{\beta} \tau. \end{aligned}$$

2.1. Steffensen Inequality

We prove a new β – fractional version of Steffensen inequality and of Hayashi’s inequality. We need the following lemma to prove our results.

Lemma 2.7. Let $\beta \in (0, 1]$ and $a, b \in \mathbb{R}$ with $0 \leq a < b$. We assume $M > 0$ and $f : [a, b] \rightarrow [0, M]$ be an β -fractional integrable function on $[a, b]$. Then the inequalities

$$\int_{b-l}^b M d_{\beta}t \leq \int_a^b f(t) d_{\beta}t \leq \int_a^{a+l} M d_{\beta}t \quad (6)$$

hold where

$$l := \frac{\beta(b-a)}{M \left[\left(b + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \left(a + \frac{1}{\Gamma(\beta)}\right)^{\beta} \right]} \int_a^b f(t) d_{\beta}t, \quad t \in [0, b-a]. \quad (7)$$

Proof. Since $f(t) \in [0, M]$ for all $t \in [a, b]$, using (7) we have

$$\begin{aligned} 0 &\leq l = \frac{\beta(b-a)}{M \left[\left(b + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \left(a + \frac{1}{\Gamma(\beta)}\right)^{\beta} \right]} \int_a^b f(t) d_{\beta}t \\ &\leq \frac{\beta(b-a)}{\left(b + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \left(a + \frac{1}{\Gamma(\beta)}\right)^{\beta}} \int_a^b d_{\beta}t = b-a. \end{aligned}$$

We can easily see that $\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1}$ is a decreasing function on $[a, b]$ or $(a, b]$ for $a = 0$. Thus using the fact that $d_{\beta}t = \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1}$, we obtain the following inequalities

$$\frac{1}{l} \int_{b-l}^b d_{\beta}t \leq \frac{1}{b-a} \int_a^b d_{\beta}t \leq \frac{1}{l} \int_a^{a+l} d_{\beta}t.$$

So that using (7), we have

$$\int_{b-l}^b M d_{\beta}t \leq \frac{l}{b-a} \int_a^b M d_{\beta}t \leq \int_a^{a+l} M d_{\beta}t,$$

which ends the proof. \square

We call the following theorem as Steffensen's inequality. If we choose $M = 1$ and $A > 0$ for generality it is called Hayashi's inequality.

Theorem 2.8. Let $\beta \in (0, 1]$ and $a, b \in \mathbb{R}$ with $0 \leq a < b$ and $M > 0$. Assume that the functions $f : [a, b] \rightarrow [0, M]$ and $g : [a, b] \rightarrow [0, M]$ be β -fractional integrable functions on $[a, b]$. If f is nonnegative and nonincreasing, then

$$M \int_{b-l}^b f(t) d_{\beta}t \leq \int_a^b f(t) g(t) d_{\beta}t \leq M \int_a^{a+l} f(t) d_{\beta}t, \quad (8)$$

where l is defined in (7).

Proof. Assume that f is nonnegative and nonincreasing, we first prove the left side of the inequality. By the definition of l in (7) and the conditions on function g , we have inequality (6). Using the left hand side of the

inequality (8), we obtain

$$\begin{aligned} & \int_a^b f(t)g(t) d_{\beta}t - M \int_{b-l}^b f(t) d_{\beta}t \\ &= \int_a^{b-l} f(t)g(t) d_{\beta}t + \int_{b-l}^b f(t)g(t) d_{\beta}t - M \int_{b-l}^b f(t) d_{\beta}t \\ &= \int_a^{b-l} f(t)g(t) d_{\beta}t - \int_{b-l}^b (M - g(t)) f(t) d_{\beta}t. \end{aligned}$$

Since f is nonincreasing and for $t \in [b-l, b]$, we have

$$f(t) \leq f(b-l). \quad (9)$$

Using the inequality (9), we get

$$\begin{aligned} & \int_a^{b-l} f(t)g(t) d_{\beta}t - \int_{b-l}^b (M - g(t)) f(t) d_{\beta}t \\ &\geq \int_a^{b-l} f(t)g(t) d_{\beta}t - f(b-l) \int_{b-l}^b (M - g(t)) d_{\beta}t \\ &\geq \int_a^{b-l} f(t)g(t) d_{\beta}t - f(b-l) \int_{b-l}^b g(t) d_{\beta}t \\ &\geq \int_a^{b-l} (f(t) - f(b-l))g(t) d_{\beta}t \geq 0. \end{aligned}$$

Thus we prove the left hand side of the inequality (8). The proof of the right hand side of the inequality is similar and one can easily prove the inequality by using (6). \square

Theorem 2.9. If f is nonpositive and nondecreasing function and $g : [a, b] \rightarrow [0, M]$, the inequalities in Theorem 2.8 are reversed.

Proof. Assume f is nonpositive and nondecreasing. In this case, we prove the right hand side of the inequality (8). Using the inequality (8) and the inequality (6), we have

$$\begin{aligned} & \int_a^b f(t)g(t) d_{\beta}t - M \int_a^{a+l} f(t) d_{\beta}t \\ &= \int_a^{a+l} f(t)g(t) d_{\beta}t + \int_{a+l}^b f(t)g(t) d_{\beta}t - M \int_a^{a+l} f(t) d_{\beta}t \\ &= \int_{a+l}^b f(t)g(t) d_{\beta}t + \int_a^{a+l} (g(t) - M) f(t) d_{\beta}t \\ &\geq \int_{a+l}^b f(t)g(t) d_{\beta}t + f(a+l) \int_a^{a+l} (g(t) - M) d_{\beta}t \\ &\geq \int_{a+l}^b f(t)g(t) d_{\beta}t - f(a+l) \int_a^{a+l} g(t) d_{\beta}t \\ &= \int_{a+l}^b (f(t) - f(a+l))g(t) d_{\beta}t \geq 0. \end{aligned}$$

Thus the right hand side of the inequality (8) holds. The proof of the left hand side of the reversed inequality is similar as in the proof of the Theorem 2.8. \square

In the following we obtain some results by using β – fractional Steffensen inequality.

Theorem 2.10. Let $\beta \in (0, 1]$ and the function $f : [a, b] \rightarrow \mathbb{R}$ be $(n + 1)$ times β – fractional differentiable. We assume that $D^{(n+1)\beta} f$ is increasing and $D^{n\beta} f$ is decreasing on $[a, b]$. Then inequalities

$$\begin{aligned} D^{n\beta} f(a+l) - D^{n\beta} f(a) &\leq (n+1)! \beta^{n+1} \left[\left(b + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(a + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{-n-1} \int_a^t R_{n,f}(a, \tau) d_\beta \tau \\ &\leq D^{n\beta} f(b) - D^{n\beta} f(b-l) \end{aligned}$$

hold where

$$l := \frac{b-a}{n+2}.$$

Proof. We define the function $F := -D^{(n+1)\beta} f$. Since $D^{(n+1)\beta} f$ is increasing and $D^{n\beta} f$ is decreasing on $[a, b]$, we have $D^{(n+1)\beta} f \leq 0$. So that $F \geq 0$ and decreasing on $[a, b]$. For the assumptions of Steffensen's inequality we define

$$g(t) := \frac{\left[\left(b + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{n+1}}{\left[\left(b + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(a + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{n+1}},$$

for $t \in [a, b]$, $n \geq -1$. We apply $M = 1$ in Theorem 2.8, the functions F and g satisfy the assumptions of Steffensen's inequality. We can write

$$-\int_{b-l}^b D_t^{(n+1)\beta} f(t) d_\beta t \leq -\int_a^b D_t^{(n+1)\beta} f(t) g(t) d_\beta t \leq -\int_a^{a+l} D^{(n+1)\beta} f(t) d_\beta t. \quad (10)$$

If we simplify (10) using Corollary 2.6, we obtain

$$\begin{aligned} D^{n\beta} f(a+l) - D^{n\beta} f(a) &\leq (n+1)! \beta^{n+1} \left[\left(b + \frac{1}{\Gamma(\beta)} \right)^\beta - \left(a + \frac{1}{\Gamma(\beta)} \right)^\beta \right]^{-n-1} \int_a^t R_{n,f}(a, \tau) d_\beta \tau \\ &\leq D^{n\beta} f(b) - D^{n\beta} f(b-l). \end{aligned}$$

This completes the proof. \square

Remark 2.11. Let $D^{(n+1)\beta} f$ is decreasing and $D^{n\beta} f$ is increasing on $[a, b]$. If we define the function $F := D^{(n+1)\beta} f$, one can easily show that the above inequalities in Theorem 2.10 are reversed.

If we choose $n = 0$ in Theorem 2.10, we obtain the well-known inequality called Hermite-Hadamard. Since the discovery of the inequality of Hermite-Hadamard, a number of mathematicians have dedicated their efforts to obtain new proofs, generalizations, refinements variations and applications. In recent years, authors have studied the Hermite-Hadamard inequality for different classes of functions and for fractional calculus. For more details, we refer the reader to [3, 6–10, 13].

Definition 2.12. Let $\beta \in (0, 1]$ and the function $f : [a, b] \rightarrow \mathbb{R}$ be β -fractional differentiable. If $D^\beta f$ is increasing and f is decreasing on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\beta}{\left(b + \frac{1}{\Gamma(\beta)}\right)^\beta - \left(a + \frac{1}{\Gamma(\beta)}\right)^\beta} \int_a^b f(t) d_\beta t \leq f(b) + f(a) - f\left(\frac{a+b}{2}\right). \quad (11)$$

Remark 2.13. If $D^\beta f$ is decreasing and f is increasing on $[a, b]$, then the inequalities (11) are reversed.

Conclusion

In this paper, we have studied Taylor's formula with a new parameter and have obtained new results for beta-derivative. The main theorem improves previously results and this presents a new approach to β -version of Steffensen inequality and well known Hermite-Hadamard inequality.

References

- [1] A. Atangana, Derivative with a New Parameter, Theory, Methods and Applications, Elsevier, Oxford, 2016.
- [2] A. Atangana, E. F. D. Goufo, Extension of matched asymptotic method to fractional boundary layers problems, Mathematical Problems in Engineering 2014.
- [3] A. Fernandez, P. O. Mohammed, Hermite-Hadamard inequality in fractional calculus defined using Mittag-Leffler kernels, Mathematical Methods in the Applied Sciences (2020) 1–18.
- [4] B. Ross, Fractional Calculus and Its Applications, Springer-Verlag, Berlin, 1975.
- [5] D. R. Anderson, Taylor's formula and integral inequalities for conformable fractional derivatives, Contributions in Mathematics and Engineering, Springer International Publishing, Switzerland 25–43, 2016.
- [6] J. Han, P. O. Mohammed, H. Zeng, Generalized fractional integral inequalities of Hermite-Hadamard type for a convex function, Open Mathematics 18 (2020) 794–806.
- [7] P. O. Mohammed, I. Brevik, A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals, Symmetry 12 (4) (2020).
- [8] P. O. Mohammed, M. Z. Sarikaya, On generalized fractional integral inequalities for twice differentiable convex functions, Journal of Computational and Applied Mathematics 372 (2020).
- [9] P. O. Mohammed, T. Abdeljawad, Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel, Advances in Difference Equations 363 (2020).
- [10] P. O. Mohammed, T. Abdeljawad, Modification of certain fractional integral inequalities for convex functions, Advances in Difference Equations 69 (2020).
- [11] R. Khalil, M. A. Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, Journal of Computational and Applied Mathematics 264 (2014) 65–70.
- [12] T. Abdeljawad, On conformable fractional calculus, Journal of Computational and Applied Mathematics 279 (2015) 57–66.
- [13] T. Abdeljawad, P. O. Mohammed, A. Kashuri, New modified conformable fractional integral inequalities of Hermite-Hadamard type with applications, Journal of Function Spaces 2020 (2020).
- [14] U. N. Katugampola, A new fractional derivative with classical properties, arXiv:1410.6535, preprint.