Filomat 35:10 (2021), 3549–3556
https://doi.org/10.2298/FIL2110549V

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation of Non - Interpolatory Complex Parabolic Spline on the Unit Circle

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Abstract. In this paper we have constructed a non-interpolatory spline on the unit circle. The rate of convergence and the error in approximation corresponding to the complex valued function has been considered.

1. INTRODUCTION

Let K denote the unit circle $|z| = 1$ of the complex plane and let m and n be integers, $m \geq 1, n \geq 2$. Furthermore, let $\Delta = \{z_1, z_2, \cdots, z_n\}$ be a mesh of n distinct points of K arranged in cyclic counter-clockwise order. A complex valued function $\mathbb{S}_\Delta(z)$ defined on K is called a polynomial spline function of degree $m-1$, if it satisfies the conditions:

1. $\mathbb{S}_\Delta(z) \in C^{m-2}(K)$,

2. $\mathbb{S}_\Delta(z)$ agrees in values with a polynomial of degree at most $m-1$, on each arc in which the points z_j divide the circle K.

If $\mathbb{S}_1(z), \mathbb{S}_2(z) \cdots, \mathbb{S}_n(z)$ denote the polynomial components of $S_\Delta(z)$ on the arcs $K_j = \{(z_j, z_{j+1}),\ j = 1, 2, \cdots, n\}$ respectively, where $z_{n+1} = z_1$, then the condition (1) or more explicitly $\mathbb{S}_\Delta(e^{i\theta}) \in C^{m-2}(K)$, is equivalent to the conditions:

$$\mathbb{S}_j^{(\nu)}(z_{j+1}) = \mathbb{S}_{j+1}^{(\nu)}(z_{j+1}), \quad \nu = 0, 1, 2, \cdots, m-2, \quad j = 1, 2, \cdots, n \tag{1}$$

where $S_{n+1}(z) = S_1(z)$.

In 1971, the problem of complex spline interpolation was initiated by Schoenberg [10] and Ahlberg, Nilson and Walsh in a sequence of papers [1–3]. The solutions were completely different. A related problem on the trigonometric spline interpolation was beautifully studied by Schoenberg [11], connecting the study to the differential operators $\Delta_m = D(D^2 + 1^2) \cdots (D^2 + m^2),\ (D = d/dx)$. Micchelli [7] exploiting Schoenberg's

2020 *Mathematics Subject Classification*. Primary 41A10 ; Secondary 97N50, 41A05, 30E10
Keywords. Spline Interpolation, Rate of Convergence, Non-Interpolatory Spline, Convergence on unit circle, Splines on unit circle
Received: 11 September 2020; Accepted: 22 November 2020
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idea and using the cardinal $\mathcal{L}$-splines related to the differential operator $\mathscr{L} = \prod_{j=0}^{n}(D - \gamma_j)$ with γ_j as real numbers, gave a complete and systematic treatment to the interpolation problem. The works of Shevaldin [14], [15], Subbotin and Chernykh [24] also deserve a mention.

Schoenberg [12] revisited Micchelli's theory and extended it to the operator $\mathscr{L}$ with imaginary γ_j's. Sharma and Tzimbalario [13] and Tzimbalario [25] further extended the study for cardinal splines related to the operators Δ_m and $\mathscr{L} = \prod_{j=0}^{n}(D - i(j+\ell)\eta)$ for some $\eta > 0$ and ℓ real, respectively.

Kvasov [6], Subbotin [23] (with different conditions) and Shevaldin [17] (in a more general statement) constructed local parabolic splines for functions defined on the axis or on the segment of the axis that preserve linear functions with an arbitrary distinct setting of nodes with good approximative property and their own local preservation of the sign, monotonicity and convexity of approximate functions [16]. Recently in a joint paper, Subbotin and Shevaldin [20] developed a general scheme of constructing such structures, special cases of which are the splines of [17, 23]. These splines and their generalizations are widely used in computational mathematics. In other papers, Kostosov and Shevaldin [5], Shevaldin [18] and Strelkova [19] have extended the study to trigonometric, exponential and average interpolation splines respectively. Article [23] gave rise to a whole series of works by Subbotin and Telyakovskii [21, 22] on estimates of Lebesgue constants of interpolatory splines and trigonometric polynomials and Konovalov's diameters of differentiable classes of functions.

The aim of this paper is to construct a non - interpolatory complex parabolic spline $S_\Delta(z)$ on a unit circle K, study its rate of convergence and error in approximation corresponding to an analytic function $f(z) \in W_K^2 = \{f : \max|f''(z)| \le 1\}$ on K.

2. CONSTRUCTION OF COMPLEX PARABOLIC SPLINE

We are interested to construct a non-interpolatory spline $\mathbb{S}_\Delta(z)$ for the subdivision Δ, on the unit circle K, composed of complex quadratics $\mathbb{S}_j(z)$ on the arc K_j from z_j to z_{j+1}, where $z_j = \exp\left(\frac{2j\pi i}{n}\right)$. For this purpose, we follow the scheme of works [17, 23]. Obviously,

$$z_{j+1} = \exp\left(\frac{2(j+1)\pi i}{n}\right) = \exp(ih)\, z_j,$$

where $h = \frac{2\pi}{n}$. Let $f : \mathbb{C} \rightarrowtail \mathbb{C}$ and $y_j = f(z_j)$. Associate operator Λ on the space of sequences $\{y_j\}$, as

$$\Lambda(y_{j-1}) := y_{j+1} - (e^{ih}+1)y_j + e^{ih}y_{j-1}.$$

For $z \in K_j$, the spline $\mathbb{S}_j(z)$, can be represented in the form

$$\mathbb{S}_j(z) = C_0^{(j)} + C_1^{(j)}\left(z - z_j\right) + C_2^{(j)}\left(z - z_j\right)^2 + C_3^{(j)}\left(z - z_{j+\frac{1}{2}}\right)_+^2, \tag{2}$$

where

$$\left(z - z_{j+\frac{1}{2}}\right)_+ = \begin{cases} z - z_{j+\frac{1}{2}} & , \quad \arg z > \arg z_{j+\frac{1}{2}} \\ 0 & , \quad \arg z \le \arg z_{j+\frac{1}{2}} \end{cases} \tag{3}$$

and $C_0^{(j)}, C_1^{(j)}, C_2^{(j)}, C_3^{(j)}$ are complex constants, given by

$$C_0^{(j)} = y_j + \frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}} - 1)\Lambda(y_{j-1})}{2(e^{2ih} - 1)}, \tag{4}$$

$$C_1^{(j)} = \frac{e^{ih}(y_{j+1} - y_{j-1})}{(e^{2ih} - 1)z_j}, \tag{5}$$

$$C_2^{(j)} = \frac{\Lambda(y_{j-1})}{(e^{ih} - 1)(e^{2ih} - 1)z_j^2} \tag{6}$$

and

$$C_3^{(j)} = \frac{\Lambda(y_j) - \Lambda(y_{j-1})}{e^{\frac{ih}{2}}(e^{\frac{ih}{2}} - 1)(e^{2ih} - 1)z_j^2}. \tag{7}$$

Theorem 2.1. *For $z \in K_j$, the spline $\mathbb{S}_j(z)$, satisfies the following properties:*

1. $\mathbb{S}_j(z_{j+1}) = y_{j+1} + b\,\Lambda(y_j)$, *where*
$$b = \frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}} - 1)}{2(e^{2ih} - 1)}.$$
2. $\mathbb{S}_j(z)$ *has a continuous derivative on K_j, such that*
$$\mathbb{S}_j'(z_j) = \frac{e^{ih}(y_{j+1} - y_{j-1})}{(e^{2ih} - 1)z_j}.$$
3. *For* $\arg z \le \arg z_{j+\frac{1}{2}}$
$$\mathbb{S}_j''(z_j) = \frac{2\Lambda(y_{j-1})}{(e^{ih} - 1)(e^{2ih} - 1)z_j^2}$$
and for $\arg z > \arg z_{j+\frac{1}{2}}$
$$\mathbb{S}_j''(z_{j+1}) = \frac{2(e^{\frac{ih}{2}} + 1)\Lambda(y_j) - 2\Lambda(y_{j-1})}{e^{\frac{ih}{2}}(e^{ih} - 1)(e^{2ih} - 1)z_j^2}.$$

Proof. 1. Let $z \in K_j$, then putting $z = z_j$ in (2), we have

$$\mathbb{S}_j(z_j) = C_0^{(j)} = y_j + \frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}} - 1)\Lambda(y_{j-1})}{2(e^{2ih} - 1)}$$

and

$$\begin{aligned}\mathbb{S}_j(z_{j+1}) &= C_0^{(j)} + C_1^{(j)}(z_{j+1} - z_j) + C_2^{(j)}(z_{j+1} - z_j)^2 + C_3^{(j)}(z_{j+1} - z_{j+\frac{1}{2}})_+^2 \\ &= C_0^{(j)} + C_1^{(j)}(e^{ih} - 1)z_j + C_2^{(j)}(e^{ih} - 1)^2 z_j^2 + C_3^{(j)} e^{ih}(e^{\frac{ih}{2}} - 1)^2 z_j^2,\end{aligned}$$

which due to (4), (5), (6) and (7) implies

$$\mathbb{S}_j(z_{j+1}) = y_{j+1} + \frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}} - 1)\Lambda(y_j)}{2(e^{2ih} - 1)}.$$

2. The continuity of $\mathbb{S}_j'(z)$ is obvious on K except at the points z_j of the spline. On differentiating (2) w.r.t z, we get

$$\mathbb{S}_j'(z) = C_1^{(j)} + 2C_2^{(j)}(z - z_j) + 2C_3^{(j)}(z - z_{j+\frac{1}{2}})_+, \tag{8}$$

which on substituting $z = z_{j+1}$, due to (5), (6) and (7), gives

$$\begin{aligned}\mathbb{S}_j'(z_{j+1}) &= C_1^{(j)} + 2C_2^{(j)}(e^{ih} - 1)z_j + 2C_3^{(j)} e^{\frac{ih}{2}}(e^{\frac{ih}{2}} - 1)z_j \\ &= \frac{e^{ih}(y_{j+2} - y_j)}{(e^{2ih} - 1)z_{j+1}}.\end{aligned}$$

Also for $z \in K_{j+1}$, due to (5), we have

$$\mathbb{S}'_{j+1}(z_{j+1}) = C_1^{(j+1)} = \frac{e^{ih}(y_{j+2} - y_j)}{(e^{2ih} - 1)z_{j+1}},$$

which implies the continuity of $\mathbb{S}'_j(z)$ at the grid points z_{j+1}.

3. Lastly on differentiating (8) w.r.t z and putting $z = z_j$, due to (6), we get

$$\mathbb{S}''_j(z_j) = 2C_2^{(j)} = \frac{2\Lambda(y_{j-1})}{(e^{ih} - 1)(e^{2ih} - 1)z_j^2}.$$

Similarly, differentiating (8) w.r.t z and putting $z = z_{j+1}$, due to (6) and (7), we have

$$\begin{aligned} \mathbb{S}''_j(z_{j+1}) &= 2C_2^{(j)} + 2C_3^{(j)} \\ &= \frac{2e^{\frac{ih}{2}}\Lambda(y_j) + 2(\Lambda(y_j) - \Lambda(y_{j-1}))}{e^{\frac{ih}{2}}(e^{ih} - 1)(e^{2ih} - 1)z_j^2}, \end{aligned}$$

which proves the theorem.
□

3. RATE OF CONVERGENCE

Convergence on the boundary. To study the convergence properties of the complex spline $\mathbb{S}_\Delta(z)$, we follow the ideas of Ahlberg, Nilson and Walsh [2]. We consider the convergence of $\{\mathbb{S}_{\Delta_k}(t)\}$ for the sequence of meshes $\Delta_k = \{z_{k,1}, z_{k,2}, \cdots, z_{k,n}\}$ with $\|\Delta_k\| = \max_j |z_{k,j+1} - z_{k,j}| \to 0$, as $k \to \infty$. Let $\{S_{k,j}(z)\}_{j=1}^n$ be the complex quadratic splines on the arcs $K_{k,j}$ from $z_{k,j}$ to $z_{k,j+1}$[1)]. Then, we shall prove the following:

Theorem 3.1. *Let $f(z)$ be continuous on K. Let $\{\Delta_k\}$ be a sequence of subdivisions of K with $\lim_{k\to\infty} \|\Delta_k\| = 0$. Let $\mathbb{S}_{\Delta_k}(z)$ be the complex quadratic spline on Δ_k, then $\left\{\mathbb{S}_{\Delta_k}(z)\right\} \to f(z)$ uniformly as $\|\Delta_k\| \to 0$. Further, if $f(z)$ satisfies a Hölder's condition of order α $(0 < \alpha \le 1)$ on K, then*

$$|\ \mathbb{S}_{\Delta_k}(z) - f(z)\ | = O(\|\Delta_k\|^\alpha).$$

Proof. Let $f(z)$ be continuous on K. Then on K_j, by setting $z = (z_j + z_{j+1})/2 + \epsilon$, where ϵ is a complex number such that $0 < |\epsilon/h| \le 1/2$, we have

$$\arg(z) - \arg(z_{j+\frac{1}{2}}) = \arg\left(\frac{z_{j+1} + z_j + 2\epsilon}{2}\right) - \arg(z_{j+\frac{1}{2}}) < 0$$

and

$$(z - z_j) = \left(\frac{z_j + z_{j+1}}{2} + \epsilon - z_j\right) = \left(\frac{z_j(e^{ih} - 1)}{2} + \epsilon\right).$$

[1)]For the sake of convenience we shall drop the index "k" from the subscript

Due to (3), for $z \in K_j$, it follows that

$$\begin{aligned} |\mathbb{S}_j(z) - f(z)| &\leq |f(z_j) - f(z)| + \left|\frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}} - 1)}{2(e^{2ih} - 1)}\right| [|f(z_{j+1}) - f(z_j)| + |e^{ih}|\,|f(z_j) - f(z_{j-1})|] \\ &+ \left[\frac{|e^{ih}|\,(|f(z_{j+1}) - f(z_j)| + |f(z_j) - f(z_{j-1})|)}{|(e^{2ih} - 1)|\,|z_j|}\right]\left|\frac{z_j(e^{ih} - 1)}{2} + \epsilon\right| \\ &+ \left[\frac{|f(z_{j+1}) - f(z_j)| + |e^{ih}||(f(z_j) - f(z_{j-1}))|}{|(e^{ih} - 1)||(e^{2ih} - 1)||z_j^2|}\right]\left|\frac{z_j(e^{ih} - 1)}{2} + \epsilon\right|^2 \\ &\leq \omega(f, \|\Delta_k\|)\Bigg[1 + \left|\frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}} - 1)}{2(e^{2ih} - 1)}\right|(2) + \frac{2}{|e^{2ih} - 1|}\left|\frac{z_j(e^{ih} - 1)}{2} + \epsilon\right| \\ &+ \frac{2}{|e^{ih} - 1||e^{2ih} - 1|}\left|\frac{z_j(e^{ih} - 1)}{2} + \epsilon\right|^2\Bigg]. \end{aligned} \tag{9}$$

where $\omega(f, \|\Delta_k\|)$ is the modulus of continuity of f on K. Further, we need $|e^{ih}| = 1$ and $|e^{ih} - 1| = \sqrt{(\cos h - 1)^2 + \sin^2 h} = 2\sin(h/2)$. From [9], we have for $0 \leq |h| \leq \pi/2$

$$|e^{ih} - 1| \geq 2|h|/\pi \tag{10}$$

and for $h \geq 0$

$$|e^{ih} - 1| \leq h. \tag{11}$$

Using (10) and (11) in the last inequality of (9), we get

$$|\mathbb{S}_j(z) - f(z)| \leq \omega(f, \|\Delta_k\|)\Big[1 + \frac{5\pi}{8} + \frac{3\pi}{4}\left|\frac{\epsilon}{h}\right| + \frac{\pi^2}{4}\left|\frac{\epsilon^2}{h^2}\right|\Big].$$

Since $0 < |\epsilon/h| \leq 1/2$, therefore

$$|\, \mathbb{S}_j(z) - f(z) \,| = C\omega(f, \|\Delta_k\|), \tag{12}$$

where C is a constant, from which the Theorem follows. □

In order to obtain the convergence properties of the complex spline $\mathbb{S}_\Delta(z)$, it is necessary to show that $\mathbb{S}_\Delta(t) - f(t)$ or its derivatives satisfy suitable Hölder's conditions.

We shall prove the following:

Corollary 3.2. *Under the conditions of Theorem 3.1 with $f(z)$ satisfying a Hölder condition of order $\alpha (0 < \alpha \leq 1)$, the function $[\mathbb{S}_{\Delta_k}(z) - f(z)]/\|\Delta_k\|^{\alpha-\delta}$ satisfies a Hölder's condition of order δ, $0 < \delta \leq \alpha$, uniformly with respect to k.*

Proof. For z and τ on K_j, we have

$$\begin{aligned} \mathbb{S}_j(z) - \mathbb{S}_j(\tau) &= \left[\frac{e^{ih}[f(z_{j+1}) - f(z_j) + f(z_j) - f(z_{j-1})]}{(e^{2ih} - 1)z_j}\right](z - z_j - (\tau - z_j)) \\ &+ \left[\left(\frac{[f(z_{j+1}) - f(z_j) - e^{ih}(f(z_j) - f(z_{j-1}))]}{(e^{ih} - 1)(e^{2ih} - 1)z_j^2}\right)\right]\left[(z - z_j)^2 - (\tau - z_j)^2\right] \\ &+ \frac{1}{2}\Bigg[\frac{[f(z_{j+2}) - f(z_{j+1}) - e^{ih}(f(z_{j+1}) - f(z_j))]}{e^{\frac{ih}{2}}(e^{\frac{ih}{2}} - 1)(e^{2ih} - 1)z_j^2} \\ &+ \frac{[f(z_{j+1}) - f(z_j) - e^{ih}(f(z_j) - f(z_{j-1}))]}{e^{\frac{ih}{2}}(1 - e^{\frac{ih}{2}})(e^{2ih} - 1)z_j^2}\Bigg]\left[(z - z_{j+\frac{1}{2}})_+^2 - (\tau - z_{j+\frac{1}{2}})_+^2\right]. \end{aligned}$$

Let us consider two cases-:
Case(*i*) If $\arg(z) \le \arg(z_{j+\frac{1}{2}})$ and $\arg(\tau) \le \arg(z_{j+\frac{1}{2}})$,
Case(*ii*) If $\arg(z) > \arg(z_{j+\frac{1}{2}})$ and $\arg(\tau) > \arg(z_{j+\frac{1}{2}})$.

Case (*i*) implies that $(z - z_{j+\frac{1}{2}})_+^2 = (\tau - z_{j+\frac{1}{2}})_+^2 = 0$, then

$$\begin{aligned}\mathbb{S}_j(z) - \mathbb{S}_j(\tau) \quad &= \quad (z-\tau)\Bigg\{\left[\frac{e^{ih}[f(z_{j+1}) - f(z_j) + f(z_j) - f(z_{j-1})]}{(e^{2ih}-1)z_j}\right] \\ &+ \left[\frac{[f(z_{j+1}) - f(z_j) - e^{ih}(f(z_j) - f(z_{j-1}))]}{(e^{ih}-1)(e^{2ih}-1)z_j^2}\right](z+\tau-2z_j)\Bigg\}.\end{aligned}$$

If $f(z)$ satisfies Hölder's condition of order α and if $\exists$ a δ such that $0 < \delta \le \alpha$, then

$$\begin{aligned}&\Big|\mathbb{S}_j(z) - \mathbb{S}_j(\tau) + f(\tau) - f(z)\Big| \le |z-\tau|\Bigg\{\left[\frac{|f(z_{j+1}) - f(z_j)| + |f(z_j) - f(z_{j-1})|}{|e^{2ih}-1|}\right] \\ &+\left[\frac{|f(z_{j+1}) - f(z_j)| + |f(z_j) - f(z_{j-1})|}{|(e^{ih}-1)(e^{2ih}-1)|}\right]\Big(|z-\tau| + 2|z_j-\tau|\Big)\Bigg\} + |f(\tau) - f(z)| \\ &\le |z-\tau|\Bigg\{\left[\frac{|z_{j+1}-z_j|^\alpha + |z_j - z_{j-1}|^\alpha}{|e^{2ih}-1|}\right] + \left[\frac{|z_{j+1}-z_j|^\alpha + |z_j - z_{j-1}|^\alpha}{|(e^{ih}-1)(e^{2ih}-1)|}\right]\Big(|z-\tau| + 2|z_j-\tau|\Big)\Bigg\} \\ &+|\tau - z|^\alpha \\ &\le |z-\tau|\Bigg\{\left[\frac{2|e^{ih}-1|^\alpha}{|e^{2ih}-1|}\right] + \left[\frac{2|e^{ih}-1|^{\alpha-1}}{|(e^{2ih}-1)|}\right]\Big(|z-\tau| + 2|z_j-\tau|\Big)\Bigg\} + |\tau - z|^\alpha.\end{aligned}$$

Since $z, \tau \in K_j$, therefore, owing to (10) and (11), we have $|z-\tau| \le |z_{j+1}-z_j| \le |e^{ih}-1| \le h$ and $|z_j-\tau| \le |e^{ih}-1|$, which leads to

$$\begin{aligned}\Big|\mathbb{S}_j(z) - \mathbb{S}_j(\tau) + f(\tau) - f(z)\Big| \quad &\le \quad |z-\tau|^\delta ||\Delta_k||^{\alpha-\delta}\frac{|z-\tau|^{\alpha-\delta}}{||\Delta_k||^{\alpha-\delta}}\left\{\frac{8|z-\tau||e^{ih}-1|^\alpha}{|e^{2ih}-1||z-\tau|^\alpha} + 1\right\} \\ &\le \quad (2\pi+1)|z-\tau|^\delta ||\Delta_k||^{\alpha-\delta}\left(\frac{|z-\tau|}{||\Delta_k||}\right)^{\alpha-\delta} \\ &\le \quad (2\pi+1)|z-\tau|^\delta ||\Delta_k||^{\alpha-\delta}.\end{aligned}$$

Thus, we deduce that $(\mathbb{S}_j(z) - f(z))/||\Delta_k||^{\alpha-\delta}$ satisfies uniformly Hölder's condition of order δ. Working corresponding to Case (ii) has been omitted as a mutatis-mutandis approach leads to the above conclusion. □

For the proof of the following theorem, we adopt the scheme of works [17, 23].

Theorem 3.3. *Let $f \in \mathbb{C}$ be analytic on K and $f \in W_K^2$. Let Δ_k be a sequence of subdivisions of K with* $\lim_{k\to\infty} ||\Delta_k|| = 0$. *Let $\mathbb{S}_j(z)$ be the complex quadratic spline on K_j, then*

$$\sup_{f \in W_K^2} |f(z) - \mathbb{S}_j(z)|_K = O\left(\frac{1}{n^2}\right). \tag{13}$$

Proof. Without violating generality, taking a periodic case, we can accept that $z \in K_1$, where K_1 is the arc joining the points z_1 and z_2. Moreover, we can accept that z lies in the arc joining z_1 and $z_{3/2}$, that is where $\arg(z) - \arg(z_{3/2}) < 0$. Otherwise we can make a change in variable $z = z_2 - v$. Also, we can take $z_1 = e^{ih}, z_{3/2} = e^{3ih/2}$, where $h = \frac{2\pi}{n}$. Consider $z = z_1 e^{i\theta}$, where $0 \le \theta \le h$, hence

$$\begin{aligned}
f(z) - \mathbb{S}_1(z) &= \left\{ z_1 f'(z_1)(e^{i\theta}-1) + \int_{z_1}^{z} (z_1 e^{i\theta} - \tau)\, f''(z_1\tau)\, z_1 d\tau \right\} - \frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}}-1)\Lambda(y_0)}{2(e^{2ih}-1)} \\
&\quad - \left[\frac{e^{ih}(y_2 - y_0)}{(e^{2ih}-1)}\right](e^{i\theta}-1) + \left[\frac{\Lambda(y_0)}{(e^{ih}-1)(e^{2ih}-1)}\right](e^{i\theta}-1)^2 \\
&= \left\{ z_1 f'(z_1)(e^{i\theta}-1) + \int_{z_1}^{z} (z_1 e^{i\theta} - \tau)\, f''(z_1\tau)\, d\tau \right\} \\
&\quad + (f(z_2) - f(z_1))\left[-\frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}}-1)}{2(e^{2ih}-1)} - \frac{e^{ih}(e^{i\theta}-1)}{(e^{2ih}-1)} - \frac{(e^{i\theta}-1)^2}{(e^{ih}-1)(e^{2ih}-1)}\right] \\
&\quad + (f(z_1) - f(z_0))\left[\frac{e^{\frac{ih}{2}}e^{ih}(e^{\frac{ih}{2}}-1)}{2(e^{2ih}-1)} - \frac{e^{ih}(e^{i\theta}-1)}{(e^{2ih}-1)} + \frac{e^{ih}(e^{i\theta}-1)^2}{(e^{ih}-1)(e^{2ih}-1)}\right].
\end{aligned}$$

As $\|\Delta_k\| \to 0$, we can use Taylor's theorem with integral form of the remainder, to get

$$\begin{aligned}
f(z) - \mathbb{S}_1(z) &= \left\{ z_1 f'(z_1)(e^{i\theta}-1) + \int_{z_1}^{z} (z_1 e^{i\theta} - \tau)\, f''(\tau)\, d\tau \right\} \\
&+ \left[-\frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}}-1)}{2(e^{2ih}-1)} - \frac{e^{ih}(e^{i\theta}-1)}{(e^{2ih}-1)} - \frac{(e^{i\theta}-1)^2}{(e^{ih}-1)(e^{2ih}-1)}\right]\left\{(z_2 - z_1)f'(z_1) + \int_{z_1}^{z_2}(z_2-\tau)f''(\tau)d\tau\right\} \\
&+ \left[\frac{e^{\frac{ih}{2}}e^{ih}(e^{\frac{ih}{2}}-1)}{2(e^{2ih}-1)} - \frac{e^{ih}(e^{i\theta}-1)}{(e^{2ih}-1)} + \frac{e^{ih}(e^{i\theta}-1)^2}{(e^{ih}-1)(e^{2ih}-1)}\right]\left\{(z_1 - z_0)f'(z_0) + \int_{z_0}^{z_1}(z_1-\tau)f''(\tau)d\tau\right\}
\end{aligned}$$

$$\begin{aligned}
f(z) - \mathbb{S}_1(z) &= \left[\frac{e^{\frac{ih}{2}}e^{ih}(e^{\frac{ih}{2}}-1)}{2(e^{2ih}-1)} - \frac{e^{ih}(e^{i\theta}-1)}{(e^{2ih}-1)} + \frac{e^{ih}(e^{i\theta}-1)^2}{(e^{ih}-1)(e^{2ih}-1)}\right]\int_{z_0}^{z_1}(z_0-\tau)f''(\tau)d\tau \\
&- \int_{z_1}^{z}\left[\frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}}-1)(z_2-\tau)}{2(e^{2ih}-1)} + \frac{e^{ih}(e^{i\theta}-1)(z_2-\tau)}{(e^{2ih}-1)} + \frac{(z_2-\tau)(e^{i\theta}-1)^2}{(e^{ih}-1)(e^{2ih}-1)} - (z_1 e^{i\theta} - \tau)\right]f''(\tau)d\tau \\
&- \left[\frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}}-1)}{2(e^{2ih}-1)} + \frac{e^{ih}(e^{i\theta}-1)}{(e^{2ih}-1)} + \frac{(e^{i\theta}-1)^2}{(e^{ih}-1)(e^{2ih}-1)}\right]\int_{z}^{z_2}(z_2-\tau)f''(\tau)d\tau.
\end{aligned}$$

Since $f \in W_K^2$, thus due to (10) and (11), we have

$$\begin{aligned}
\left|f(z) - \mathbb{S}_1(z)\right| &\le \left|\frac{e^{\frac{ih}{2}}e^{ih}(e^{\frac{ih}{2}}-1)}{2(e^{2ih}-1)} - \frac{e^{ih}(e^{i\theta}-1)}{(e^{2ih}-1)} + \frac{e^{ih}(e^{i\theta}-1)^2}{(e^{ih}-1)(e^{2ih}-1)}\right|\frac{|z_1 - z_0|^2}{2} \\
&\quad + \left|\frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}}-1)(z_2-\tau)^2}{4(e^{2ih}-1)} + \frac{e^{ih}(e^{i\theta}-1)(z_2-\tau)^2}{2(e^{2ih}-1)} + \frac{(z_2-\tau)^2(e^{i\theta}-1)^2}{2(e^{ih}-1)(e^{2ih}-1)} - \frac{(z_1e^{i\theta}-\tau)^2}{2}\right|_{z_1}^{z} \\
&\quad + \left|\frac{e^{\frac{ih}{2}}(e^{\frac{ih}{2}}-1)}{2(e^{2ih}-1)} + \frac{e^{ih}(e^{i\theta}-1)}{(e^{2ih}-1)} + \frac{(e^{i\theta}-1)^2}{(e^{ih}-1)(e^{2ih}-1)}\right|\frac{|z_2 - z|^2}{2} \\
&\le h^2\left(\frac{500\pi + 13\pi^2}{256} + \frac{1}{2}\right),
\end{aligned}$$

from which the theorem follows. □

4. Acknowledgement

Authors are thankful to the referee for his constructive suggestions.

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