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Calmness of the Solution Mapping of Parametric Variational Relation Problems

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Abstract. We consider a class of variational relation problems, depending on two parameters from metric spaces. The issue under investigation is the behaviour of the solution in a neighborhood of a fixed pair of parameters, more precisely, the Hölder calmness of the solution mapping. After establishing some sufficient condition for calmness in a general framework, we particularize the result for a variational inclusion and for an equilibrium problem.

1. Introduction and preliminaries

For problems that depend on parameters, an important and natural topic is the behavior of the solutions when the parameters are perturbed. One of the stability properties that is worth investigating is the calmness property, which is related to Lipschitz continuity and gives a bound on the distance between unperturbed and perturbed solutions.

Hölder continuity and calmness for the solution mappings of parametric equilibrium problems (in scalar or vector settings) have been studied in numerous papers, for instance in [17], [2], [3], [4] and [7]. In the case of variational inequalities, such results were obtained even earlier, see for instance [20] and [21].

Variational relation problems, introduced in [16], gather in the same model equilibrium problems, variational inclusions or optimization problems. Many existence results for variational relation problems, appeared in papers [1], [5], [6], [12], [11], [14], [15] and [18]. Mainly, two approaches are used to obtain such existence results: intersection theorems or fixed points theorems. Properties of the solution set, such as uniqueness, closedness, convexity, boundedness, were investigated in [13] and [19]. Also, in [13], a parametric problem was considered, and several types of continuity properties for the solution mapping were studied: inner continuity (lower semicontinuity), outer continuity, inner openness, outer openness. For an even more general model, continuity of solutions for parametric generalized variational relation problems was obtained in [10]. The literature dedicated to variational relation problems is substantial, yet, to the best of our knowledge, the Lipschitz or Hölder continuity and calmness of the solution mapping has not been studied.

In this paper we want to obtain conditions for the Hölder calmness (in the sense of [8], pag. 197) of the solution mapping, in the case of a simple variational relation problem that depends on two parameters. The main theorem generalizes, for variational relation problems, several previous results dedicated to

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equilibrium problems, in [2], [3] and [4]. Subsequently, as a particular case, we will be able to obtain a calmness result for a variational inclusion, formulated for instance in [9]. The study of calmness, both in the general framework of variational relation problems and for variational inclusions is new. In the last part of the paper, as a consequence of the previous theorem, we retrieve a result from [4], for an equilibrium problem.

We present in what follows the problem under study and some notions used in the paper. Let X, M and Λ be metric spaces. We will denote by d the metrics on all of these spaces, unless there is possibility of confusion.

Let $K : \Lambda \to 2^X$ be a set-valued mapping with nonempty closed values. Let $\mathcal{R} \subseteq X \times X \times M$ be a relation connecting $x, y \in X$ and depending on a parameter $\mu \in M$. For each pair of parameters (λ, μ), consider the variational relation problem

$$(VRP)(\lambda, \mu) \quad \text{Find } x(\lambda, \mu) \in K(\lambda) \text{ such that } \mathcal{R}(x(\lambda, \mu), y; \mu) \text{ holds for every } y \in K(\lambda).$$

This is a particular case of the problem formulated and studied in [16]. Denote by $S(\lambda, \mu)$ the set of solutions of the previous problem and suppose, in the whole paper, that it is nonempty. Our purpose is to investigate the calmness of the mapping $(\lambda, \mu) \mapsto S(\lambda, \mu)$, extending existing results from the case of equilibrium problems to variational relation problems.

For some nonempty sets A, B in the metric space (X, d) and $a \in X$, denote by $d(a, B) := \inf_{b \in B} d(a, b)$ the distance between the point a and the set B, by $e(A, B) := \sup_{a \in A} d(a, B)$ the excess of A beyond B and by $H(A, B) = \max\{e(A, B), e(B, A)\}$ the Hausdoff-Pompeiu distance between A and B. By convention, $e(\emptyset, B) = 0$ for $B \neq \emptyset$.

If a is a real number, we denote $a_- := d(a, [0, +\infty[) = \max\{-a, 0\}$ and $a_+ := d(a,]-\infty, 0]) = \max\{a, 0\}$.

Let X and Y be metric spaces. A function $g : X \to Y$ is said to be *globally Hölder continuous* if there exist some constants $k, \gamma > 0$ such that

$$d(g(x), g(x')) \leq k d^{\gamma}(x, x') \text{ for any } x, x' \in X.$$

The function g is said to be *Hölder calm* at a point $\bar{x} \in X$ if there exist some constants $k, \gamma > 0$ and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that

$$d(g(x), g(\bar{x})) \leq k d^{\gamma}(x, \bar{x}) \text{ for any } x \in N(\bar{x}).$$

A set-valued mapping $T : X \to 2^Y$ is said to be *globally Hölder continuous* if there exist some constants $k, \gamma > 0$ such that

$$e(T(x), T(x')) \leq k d^{\gamma}(x, x') \text{ for any } x, x' \in X.$$

The mapping T is said to be *Hölder calm* at a point $(\bar{x}, \bar{y})$ with $\bar{x} \in X$, $\bar{y} \in T(\bar{x})$, if there exist $k, \gamma > 0$ and some neighborhoods $N(\bar{x})$ of $\bar{x}$, $N(\bar{y})$ of $\bar{y}$, such that

$$e(T(x) \cap N(\bar{y}), T(\bar{x})) \leq k d^{\gamma}(x, \bar{x}) \text{ for any } x \in N(\bar{x}).$$

For $\gamma = 1$, one gets the notions of Lipschitz continuity and calmness (see for instance [8]).

2. Hölder calmness of the solution mapping

The next result gives sufficient conditions for the Hölder calmness of the solution mapping of the variational relation problem (VRP)(λ, μ). Given initial values $\bar{\lambda}, \bar{\mu}$ for both parameters and a solution in $S(\bar{\lambda}, \bar{\mu})$, we are interested in the behaviour of the mapping $(\lambda, \mu) \mapsto S(\lambda, \mu)$ when the pair (λ, μ) is in a neighborhood of $(\bar{\lambda}, \bar{\mu})$.

Theorem 2.1. *Let $\bar{\mu} \in M$, $\bar{\lambda} \in \Lambda$ and $x(\bar{\lambda}, \bar{\mu}) \in S(\bar{\lambda}, \bar{\mu})$ be fixed. Suppose that there exist some neighborhoods $U(\bar{\lambda})$, $V(\bar{\mu})$ and $\bar{W}$ of $\bar{\lambda}$, $\bar{\mu}$ and $x(\bar{\lambda}, \bar{\mu})$ respectively, for which the following conditions hold:*

(i) there exist $l, \alpha > 0$ such that, for all $\lambda \in U(\bar{\lambda})$,

$$H(K(\lambda), K(\bar{\lambda})) \leq l d^{\alpha}(\lambda, \bar{\lambda}) \tag{1}$$

(ii) there exist $m, \beta > 0$ such that, if $\mathcal{R}(x, y; \bar{\mu})$ and $\mathcal{R}(y, x; \mu)$ hold for $x \in X$, $y \in \bar{W}$ and $\mu \in V(\bar{\mu})$ it follows that

$$d(x, y) \leq m d^{\beta}(\mu, \bar{\mu}) \tag{2}$$

(iii) there exist $n, \delta, \psi > 0$ such that, if $\mathcal{R}(x_1, y_1; \bar{\mu})$ and $\mathcal{R}(x_2, y_2; \bar{\mu})$ hold for $x_1, x_2, y_1, y_2 \in X$ it follows that

$$d^{\psi}(x_1, x_2) \leq n(d^{\delta}(x_1, y_2) + d^{\delta}(x_2, y_1)). \tag{3}$$

Then:

(a) the set $S(\bar{\lambda}, \bar{\mu}) \cap \bar{W}$ has a unique element;

(b) for every $\lambda \in U(\bar{\lambda})$ and $\mu \in U(\bar{\mu})$,

$$e(S(\lambda, \mu) \cap \bar{W}, S(\bar{\lambda}, \bar{\mu})) \leq k_1 d^{\gamma_1}(\lambda, \bar{\lambda}) + k_2 d^{\gamma_2}(\mu, \bar{\mu})$$

where $k_1 = (2nl^{\delta})^{\frac{1}{\psi}}$, $k_2 = m$, $\gamma_1 = \frac{\alpha\delta}{\psi}$ and $\gamma_2 = \beta$.

Proof: We prove first that the problem $(VRP)(\bar{\lambda}, \bar{\mu})$ has at most one solution in the neighborhood $\bar{W}$. Let $x, x' \in S(\bar{\lambda}, \bar{\mu}) \cap \bar{W}$. This means that $x, x' \in K(\bar{\lambda}) \cap \bar{W}$ and the relations $\mathcal{R}(x, x'; \bar{\mu})$ and $\mathcal{R}(x', x; \bar{\mu})$ hold. According to hypothesis (ii), it follows that $d(x, x') \leq m d^{\beta}(\bar{\mu}, \bar{\mu}) = 0$, so $x = x'$.

Step I: Let $\lambda \in U(\bar{\lambda})$ and $x(\lambda, \bar{\mu}) \in S(\lambda, \bar{\mu})$. We want to estimate the distance between $x(\bar{\lambda}, \bar{\mu})$ and $x(\lambda, \bar{\mu})$. Since $x(\bar{\lambda}, \bar{\mu}) \in K(\bar{\lambda})$ and $x(\lambda, \bar{\mu}) \in K(\lambda)$, condition (i) implies that there exist $x_1 \in K(\lambda)$ and $x_2 \in K(\bar{\lambda})$ such that

$$d(x(\bar{\lambda}, \bar{\mu}), x_1) \leq l d^{\alpha}(\lambda, \bar{\lambda}) \text{ and } d(x(\lambda, \bar{\mu}), x_2) \leq l d^{\alpha}(\lambda, \bar{\lambda}). \tag{4}$$

Since $x(\lambda, \bar{\mu})$ is a solution of the variational problem with the parameters $\lambda, \bar{\mu}$ and $x_1 \in K(\lambda)$ we get that $\mathcal{R}(x(\lambda, \bar{\mu}), x_1; \bar{\mu})$ holds. In the same way, $\mathcal{R}(x(\bar{\lambda}, \bar{\mu}), x_2; \bar{\mu})$ holds too. Hypothesis (iii) and (4) lead then to the estimation:

$$d^{\psi}(x(\bar{\lambda}, \bar{\mu}), x(\lambda, \bar{\mu})) \leq n(d^{\delta}(x(\lambda, \bar{\mu}), x_2) + d^{\delta}(x(\bar{\lambda}, \bar{\mu}), x_1)) \leq 2nl^{\delta} d^{\alpha\delta}(\lambda, \bar{\lambda}),$$

$$d(x(\bar{\lambda}, \bar{\mu}), x(\lambda, \bar{\mu})) \leq (2nl^{\delta})^{\frac{1}{\psi}} d^{\frac{\alpha\delta}{\psi}}(\lambda, \bar{\lambda}). \tag{5}$$

Step II: For $\mu \in V(\bar{\mu})$ and $x(\lambda, \mu) \in S(\lambda, \mu) \cap \bar{W}$, we estimate the distance between $x(\lambda, \bar{\mu})$ and $x(\lambda, \mu)$. From the definition of the variational problem follows that the relations $\mathcal{R}(x(\lambda, \bar{\mu}), x(\lambda, \mu); \bar{\mu})$ and $\mathcal{R}(x(\lambda, \mu), x(\lambda, \bar{\mu}); \mu)$ hold. According to hypothesis (ii), this implies

$$d(x(\lambda, \bar{\mu}), x(\lambda, \mu)) \leq m d^{\beta}(\mu, \bar{\mu}). \tag{6}$$

Step III: If $S(\lambda, \mu) \cap \bar{W} = \emptyset$, then $e(S(\lambda, \mu) \cap \bar{W}, S(\bar{\lambda}, \bar{\mu})) = 0$ and the conclusion of the theorem is direct. If $S(\lambda, \mu) \cap \bar{W} \neq \emptyset$, using (5) and (6) from the previous two steps we have, for every $x(\lambda, \mu) \in S(\lambda, \mu) \cap \bar{W}$

$$\begin{aligned} d(x(\lambda, \mu), x(\bar{\lambda}, \bar{\mu})) &\leq d(x(\lambda, \mu), x(\lambda, \bar{\mu})) + d(x(\lambda, \bar{\mu}), x(\bar{\lambda}, \bar{\mu})) \\ &\leq (2nl^{\delta})^{\frac{1}{\psi}} d^{\frac{\alpha\delta}{\psi}}(\lambda, \bar{\lambda}) + m d^{\beta}(\mu, \bar{\mu}), \end{aligned}$$

which implies

$$e(S(\lambda, \mu) \cap \bar{W}, S(\bar{\lambda}, \bar{\mu})) \leq (2nl^{\delta})^{\frac{1}{\psi}} d^{\frac{\alpha\delta}{\psi}}(\lambda, \bar{\lambda}) + m d^{\beta}(\mu, \bar{\mu}). \ \square$$

Remark 2.2. *1) Conclusion (b) of the theorem is a calmness property of the solution mapping. Hypothesis (i) of the previous result is stronger than the Hölder calmness of the mapping K, but weaker than the Hölder continuity. The condition is used in this form in paper [4].*

2) For the particular value $\mu := \bar{\mu}$, hypothesis (ii) is equivalent with the anti-symmetry of relation $\mathcal{R}$. This property is used in [13], Proposition 3.1, to prove the uniqueness of the solution.

Example 2.3. *Let $X = \mathbb{R}$ and $M = [-1, 1]$ be endowed with the euclidean distance and $m, \beta > 0$ be some fixed constants. The relation $\mathcal{R}$ is defined by: $\mathcal{R}(x, y; \mu)$ holds if and only if $y - m|\mu|^{\beta} \leq x \leq 1 - |\mu|$. $\mathcal{R}$ verifies both hypotheses (ii) and (iii) for the parameter $\bar{\mu} = 0$.*

Indeed, let x, y and μ be such that $\mathcal{R}(x, y; 0)$ and $\mathcal{R}(y, x; \mu)$ hold, that is $y \leq x \leq 1$ and $x - m|\mu|^{\beta} \leq y \leq 1 - |\mu|$. Then

$$|x - y| = x - y \leq m|\mu|^{\beta} = md^{\beta}(\mu, \bar{\mu}),$$

so (2) is checked.

Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ be such that $\mathcal{R}(x_1, y_1; 0)$ and $\mathcal{R}(x_2, y_2; 0)$ hold, that is $y_1 \leq x_1 \leq 1$ and $y_2 \leq x_2 \leq 1$. If $x_1 \leq x_2$, then $0 \leq x_2 - x_1 \leq x_2 - y_1$. If $x_2 \leq x_1$, then $0 \leq x_1 - x_2 \leq x_1 - y_2$. These inequalities imply that $|x_1 - x_2| \leq |x_1 - y_2| + |x_2 - y_1|$, so (3) is satisfied.

Let $K = [0, 1 - \lambda]$, with the parameter $\lambda \in [0, \frac{1}{2}]$. For $\bar{\lambda} = 0$, condition (i) in the previous Theorem is verified.

For a pair of parameters (λ, μ) the variational relation problem is:

Find $x \in [0, 1 - \lambda]$ such that $y - m|\mu|^{\beta} \leq x \leq 1 - |\mu|$ holds for every $y \in [0, 1 - \lambda]$.

In the case $\bar{\lambda} = \bar{\mu} = 0$, the solution of the problem is unique, $S(0, 0) = \{1\}$. For arbitrary parameters (λ, μ) the solution is the interval

$$S(\lambda, \mu) = [\{1 - \lambda - m|\mu|^{\beta}\}_{+}, \min\{1 - \lambda, 1 - |\mu|\}].$$

This is nonempty if and only if $|\mu| - m|\mu|^{\beta} \leq \lambda$. If $S(\lambda, \mu) \neq \emptyset$, choosing $\bar{W} = \mathbb{R}$, we have the estimation

$$e(S(\lambda, \mu) \cap \bar{W}, S(0.0)) = \begin{cases} 1, \text{ if } 1 - \lambda - m|\mu|^{\beta} \leq 0 \\ \lambda + m|\mu|^{\beta}, \text{ otherwise.} \end{cases}$$

In both situations it follows that $e(S(\lambda, \mu) \cap \bar{W}, S(0.0)) \leq \lambda + m|\mu|^{\beta}$, which gives the conclusion of Theorem 2.1.

Remark 2.4. *The calmness property of the mapping $(\lambda, \mu) \mapsto S(\lambda, \mu)$ can be still obtained if in Theorem 2.1 hypotheses (ii) and (iii) are replaced by the condition:*

There exist $m, n, \beta, \delta > 0$ such that, if $\mathcal{R}(x_1, y_1; \mu_1)$ and $\mathcal{R}(x_2, y_2; \mu_2)$ hold for $x_1, x_2, y_1, y_2 \in X$, $\mu_1, \mu_2 \in M$ it follows that

$$d(x_1, x_2) \leq n(d^{\delta}(x_1, y_2) + d^{\delta}(x_2, y_1)) + md^{\beta}(\mu_1, \mu_2).$$

Indeed, let $\lambda \in U(\bar{\lambda})$, $\mu \in V(\bar{\mu})$, $x(\lambda, \mu) \in S(\lambda, \mu) \cap \bar{W}$ (which we suppose is nonempty). Since $x(\bar{\lambda}, \bar{\mu}) \in K(\bar{\lambda})$ and $x(\lambda, \mu) \in K(\lambda)$, from (i) there exist $x \in K(\lambda)$ and $x' \in K(\bar{\lambda})$ such that

$$d(x(\lambda, \mu), x') \leq ld^{\alpha}(\bar{\lambda}, \lambda) \text{ and } d(x(\bar{\lambda}, \bar{\mu}), x) \leq ld^{\alpha}(\bar{\lambda}, \lambda).$$

For these, $\mathcal{R}(x(\bar{\lambda}; \bar{\mu}), x'.\bar{\mu})$ and $\mathcal{R}(x(\lambda, \mu), x; \mu)$ hold. So,

$$d(x(\bar{\lambda}, \bar{\mu}), x(\lambda, \mu)) \leq n(d^{\delta}(x(\bar{\lambda}, \bar{\mu})), x) + d^{\delta}(x(\lambda, \mu)), x')) + md^{\beta}(\bar{\mu}, \mu) \leq 2nl^{\delta}d^{\alpha\delta}(\bar{\lambda}, \lambda) + md^{\beta}(\bar{\mu}, \mu).$$

3. Applications

I. Let $F : X \times X \times M \to 2^Y$ and $G : X \times X \times M \to 2^Y$ be two mappings with nonempty values. Consider the following variational inclusion problem (studied in [9]):

$(VIP)(\lambda,\mu)$ Find $x(\lambda,\mu) \in K(\lambda)$ such that $F(x(\lambda,\mu), y; \mu) \subseteq G(x(\lambda,\mu), y; \mu)$ for every $y \in K(\lambda)$.

Denote by $S(\lambda,\mu)$ the set of solutions and suppose that it is nonempty.

Theorem 3.1. *Let $\bar{\mu} \in M$, $\bar{\lambda} \in \Lambda$ and $x(\bar{\lambda},\bar{\mu}) \in S(\bar{\lambda},\bar{\mu})$ be fixed. Suppose that there exist some neighborhoods $U(\bar{\lambda})$, $V(\bar{\mu})$ and $\bar{W}$ of $\bar{\lambda}$, $\bar{\mu}$ and $x(\bar{\lambda},\bar{\mu})$ respectively, such that:*
(i') there exist $l, \alpha > 0$ such that, for all $\lambda \in U(\bar{\lambda})$,

$$H(K(\lambda), K(\bar{\lambda})) \leq l d^{\alpha}(\lambda, \bar{\lambda})$$

(ii') there exist $h, \psi > 0$ such that, for all $x, y \in X$

$$hd^{\psi}(x,y) \leq e(F(x,y;\bar{\mu}), G(x,y;\bar{\mu})) + e(F(y,x;\bar{\mu}), G(y,x;\bar{\mu})) \tag{7}$$

(iii') there exist $a, \varepsilon > 0$ and $\theta \geq 0$ such that, for all $\mu \in V(\bar{\mu})$, $x \in \bar{W}$ and $y \in X$, with $x \neq y$,

$$\max\{e(F(x,y;\bar{\mu}), F(x,y;\mu)), e(G(x,y;\mu), G(x,y;\bar{\mu}))\} \leq a d^{\varepsilon}(\mu,\bar{\mu}) d^{\theta}(x,y) \tag{8}$$

(iv') there exist $n, \delta > 0$ such that, for any $x, y_1, y_2 \in X$,

$$\max\{e(F(x,y_1;\bar{\mu}), F(x,y_2;\bar{\mu})), e(G(x,y_1;\bar{\mu}), G(x,y_2;\bar{\mu}))\} \leq \frac{n}{2} d^{\delta}(y_1,y_2) \tag{9}$$

(v') $\theta < \psi$.
Then:
(a) the set $S(\bar{\lambda},\bar{\mu}) \cap \bar{W}$ has only one element;
(b) for every $\lambda \in U(\bar{\lambda})$ and $\mu \in U(\bar{\mu})$,

$$e(S(\lambda,\mu) \cap \bar{W}, S(\bar{\lambda},\bar{\mu})) \leq k_1 d^{\gamma_1}(\lambda,\bar{\lambda}) + k_2 d^{\gamma_2}(\mu,\bar{\mu})$$

where $k_1 = (2nl^{\delta})^{\frac{1}{\psi}}$, $k_2 = \left(\frac{2a}{h}\right)^{\frac{1}{\psi-\theta}}$, $\gamma_1 = \frac{\alpha\delta}{\psi}$ and $\gamma_2 = \frac{\varepsilon}{\psi-\theta}$.

Proof: It is enough to check the conditions (ii) and (iii) of Theorem 2.1, for the relation $\mathcal{R} \subseteq X \times X \times M$ defined by:
$\mathcal{R}(x,y;\mu)$ holds if and only if $F(x,y;\mu) \subseteq G(x,y;\mu)$.
Let $\mu \in V(\bar{\mu})$, $x \in X$, $y \in \bar{W}$ such that $\mathcal{R}(x,y;\bar{\mu})$ and $\mathcal{R}(y,x;\mu)$ hold. This means

$$F(x,y;\bar{\mu}) \subseteq G(x,y;\bar{\mu}) \text{ and } F(y,x;\mu) \subseteq G(y,x;\mu) \tag{10}$$

which implies, by the definition of the excess $e(\cdot,\cdot)$

$$e(F(x,y;\bar{\mu}), G(x,y;\bar{\mu})) = e(F(y,x;\mu), G(y,x;\mu)) = 0. \tag{11}$$

From hypotheses (ii'), (iii') and (11) follows, in the case $x \neq y$,

$$\begin{aligned} hd^{\psi}(x,y) &\leq e(F(x,y;\bar{\mu}), G(x,y;\bar{\mu})) + e(F(y,x;\bar{\mu}), G(y,x;\bar{\mu})) = e(F(y,x;\bar{\mu}), G(y,x;\bar{\mu})) \\ &\leq e(F(y,x;\bar{\mu}), F(y,x;\mu)) + e(F(y,x;\mu), G(y,x;\mu)) + e(G(y,x;\mu), G(y,x;\bar{\mu})) \\ &= e(F(y,x;\bar{\mu}), F(y,x;\mu)) + e(G(y,x;\mu), G(y,x;\bar{\mu})) \leq 2ad^{\varepsilon}(\mu,\bar{\mu}) d^{\theta}(x,y). \end{aligned}$$

Subsequently,

$$d(x,y) \leq \left(\frac{2a}{h}\right)^{\frac{1}{\psi-\theta}} d^{\frac{\varepsilon}{\psi-\theta}}(\mu,\bar{\mu}),$$

that is condition (2) holds. If $x = y$ this is obvious.

To check the inequality (3), let $x_1, x_2, y_1, y_2 \in X$ be such that $\mathcal{R}(x_1, y_1; \bar{\mu})$ and $\mathcal{R}(x_2, y_2; \bar{\mu})$ hold, that is $F(x_1, y_1; \bar{\mu}) \subseteq G(x_1, y_1; \bar{\mu})$ and $F(x_2, y_2; \bar{\mu}) \subseteq G(x_2, y_2; \bar{\mu})$.

Using (8), (9) and the triangle inequality, follows

$$\begin{aligned} hd^{\psi}(x_1, x_2) &\le e(F(x_1, x_2; \bar{\mu}), G(x_1, x_2; \bar{\mu})) + e(F(x_2, x_1; \bar{\mu}), G(x_2, x_1; \bar{\mu})) \\ &\le e(F(x_1, x_2; \bar{\mu}), F(x_1, y_1; \bar{\mu})) + e(F(x_1, y_1; \bar{\mu}), G(x_1, y_1; \bar{\mu})) + e(G(x_1, y_1; \bar{\mu}), G(x_1, x_2; \bar{\mu})) \\ &+ e(F(x_2, x_1; \bar{\mu}), F(x_2, y_2; \bar{\mu})) + e(F(x_2, y_2; \bar{\mu}), G(x_2, y_2; \bar{\mu})) + e(G(x_2, y_2; \bar{\mu}), G(x_2, x_1; \bar{\mu})) \\ &\le n(d^{\delta}(x_2, y_1) + d^{\delta}(x_1, y_2)). \square \end{aligned}$$

Example 3.2. *Let the set of parameters be $M = [0, \infty[$ and $X = \mathbb{R}^2_+$ be the first quadrant of the complex plane, $Y = \mathbb{R}^2$. Consider the inclusion $B(O, \mu) \subseteq B(A, \frac{1}{4}d(A, B) + \mu)$, where A, B are points in X, $d(A, B)$ is the euclidean distance between A and B and $B(A, r)$ denotes the closed ball centered at A, of radius r. We prove that the mappings $F(A, B; \mu) = B(O, \mu)$ and $G(A, B; \mu) = B(A, \frac{1}{4}d(A, B) + \mu)$ verify conditions (ii')-(iv'), for $\bar{\mu} = 0$.*

Let $A(a_1, a_2), B(b_1, b_2) \in X$. We have

$$e(\{O\}, B(A, \frac{1}{4}d(A, B))) = \begin{cases} 0, \text{ if } d(O, A) \le \frac{1}{4}d(A, B) \\ d(O, A) - \frac{1}{4}d(A, B), \text{ otherwise.} \end{cases} \tag{12}$$

From the triangle inequality, it is impossible that both $e(\{O\}, B(A, \frac{1}{4}d(A, B)))$ and $e(\{O\}, B(B, \frac{1}{4}d(A, B)))$ are null. Further suppose, for instance, that the first excess is nonzero and the second one is zero. Then, from (12),

$$\begin{aligned} & e(\{O\}, B(A, \frac{1}{4}d(A, B))) + e(\{O\}, B(B, \frac{1}{4}d(A, B))) = d(O, A) - \frac{1}{4}d(A, B) \\ & \ge d(O, A) - \frac{1}{4}d(A, B) + d(O, B) - \frac{1}{4}d(A, B) = \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} - \frac{1}{2}d(A, B) \\ & \ge \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} - \frac{1}{2}d(A, B) = \frac{1}{2}d(A, B), \end{aligned}$$

which verifies (ii'). The same can be obtained if both terms of the sum are nonzero.

We have $e(\{O\}, B(O, \mu))) = 0$ and $e(B(A, \frac{1}{4}d(A, B) + \mu), B(A, \frac{1}{4}d(A, B))) = \mu$, which confirms (8) with $a = \varepsilon = 1$ and $\theta = 0$.

Finally, to verify (iv') we notice that

$$e(B(A, \frac{1}{4}d(A, B_1)), B(A, \frac{1}{4}d(A, B_2))) = \begin{cases} 0, \text{ if } d(A, B_1) \le d(A, B_2) \\ \frac{1}{4}d(B_1, B_2) \text{ otherwise.} \end{cases} \tag{13}$$

II. Consider now a more particular case, where F is single-valued and G is a constant mapping. More precisely, let $F(x, y; \mu) = \{f(x, y; \mu)\}$, with $f : X \times X \times M \to \mathbb{R}$ and $G(x, y; \mu) = [0, +\infty[$ for all $x, y \in X$ and all $\mu \in M$. The variational inclusion problem becomes the classical equilibrium problem:

$$(EP)(\lambda, \mu) \quad \text{Find } x(\lambda, \mu) \in K(\lambda) \text{ such that } f(x(\lambda, \mu), y; \mu) \ge 0 \text{ for every } y \in K(\lambda).$$

Directly from Theorem 3.1 we get

Theorem 3.3. *Let $\bar{\mu} \in M$, $\bar{\lambda} \in \Lambda$ and $x(\bar{\lambda}, \bar{\mu}) \in S(\bar{\lambda}, \bar{\mu})$ be fixed. Suppose that there exist some neighborhoods $U(\bar{\lambda})$, $V(\bar{\mu})$ and $\bar{W}$ of $\bar{\lambda}$, $\bar{\mu}$ and $x(\bar{\lambda}, \bar{\mu})$ respectively, such that*

(i'') there exist $l, \alpha > 0$ such that, for all $\lambda \in U(\bar{\lambda})$,

$$H(K(\lambda), K(\bar{\lambda})) \le l d^{\alpha}(\lambda, \bar{\lambda})$$

(ii'') there exist $h, \psi > 0$ such that, for all $x, y \in X$

$$hd^{\psi}(x, y) \le f(x, y; \bar{\mu})_- + f(y, x; \bar{\mu})_-.$$

(iii″) there exist $a, \varepsilon, \theta > 0$ such that, for all $\mu \in V(\bar{\mu})$, $x \in \bar{W}$ and $y \in X$, with $x \neq y$,

$$|f(x, y; \bar{\mu}) - f(x, y; \mu)| \leq a d^{\varepsilon}(\mu, \bar{\mu}) d^{\theta}(x, y)$$

(iv″) there exist $n, \delta > 0$ such that, for any $x, y_1, y_2 \in X$,

$$|f(x, y_1; \bar{\mu}) - f(x, y_2; \bar{\mu})| \leq \frac{n}{2} d^{\delta}(y_1, y_2)$$

(v″) $\theta < \psi$.

Then:

(a) the set $S(\bar{\lambda}, \bar{\mu}) \cap \bar{W}$ has only one element;

(b) for every $\lambda \in U(\bar{\lambda})$ and $\mu \in U(\bar{\mu})$,

$$e(S(\lambda, \mu) \cap \bar{W}, S(\bar{\lambda}, \bar{\mu})) \leq k_1 d^{\gamma_1}(\lambda, \bar{\lambda}) + k_2 d^{\gamma_2}(\mu, \bar{\mu})$$

where $k_1, k_2, \gamma_1, \gamma_2$ are the same as in Theorem 3.1.

Remark 3.4. *1) The function $f(\cdot, \cdot; \bar{\mu})$ is said to be* strongly monotone *if there exist some constants $h, \psi > 0$ such that $f(x, y; \bar{\mu}) + f(y, x; \bar{\mu}) \leq -h d^{\psi}(x, y)$, for any $x, y \in X$ with $x \neq y$. Hypothesis (ii″) is well-known in the literature (see for instance [3], [4]) and is weaker than the strong monotonicity property of the function $f(\cdot, \cdot; \bar{\mu})$. Condition (iii″) is called* uniform Hölder calmness.

2) The above result is very close to Theorem 3.1 in [4]. There, the Hölder continuity of f in the second argument (condition (iv″)) is requested for every parameter μ in a neighbourhood of $\bar{\mu}$. For the evaluation of the distance between the unperturbed and the perturbed solution set is used $\rho(A, B) = \sup_{a \in A, b \in B} d(a, b)$.

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