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Improved Results of Perturbed Inequalities for Higher-Order Differentiable Functions and their Various Applications

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Abstract. A new integral equality for the function whose higher order derivatives are absolutely continuous are first improved by using the quadratic kernel mapping with five sections. After that, refined inequalities of perturbed Ostrowski type for bounded functions and mappings of bounded variation are developed. What's more, new effective composite quadrature rules are derived to find closer estimates of the integral of a mapping. Some applications for exponential and logarithmic functions are also obtained by using inequalities presented in this study. Finally, new results involving Cumulative Distribution, the reliability function and expectation value of random variable are given.

1. Introduction

For well over a century, inequality theory has attracted great attention by a good many researchers, concerned in pure and applied science, because mathematical inequalities determine upper and lower bounds to mathematical statements values of which are unknown precisely. Specifically, two integral inequalities, which are named as Hermite-Hadamard and Ostrowski, have become a cornerstone in many areas of mathematics such as numerical integration, probability, special means and stochastic. One of a large number of mathematical explorations of A. M. Ostrowski [21] is the following fundamental integral inequality based on the differentiable mappings:

Theorem 1.1. *Suppose that $\varphi : [\alpha, \beta] \to \mathbb{R}$ is a differentiable function on (α, β) whose derivative $\varphi' : (\alpha, \beta) \to \mathbb{R}$ is bounded on (α, β), that is, $\|\varphi'\|_\infty := \sup_{\xi \in (\alpha,\beta)} |\varphi'(\xi)| < \infty$. Then, we possess the inequality*

$$\left| \varphi(xy) - \frac{1}{\beta-\alpha} \int_\alpha^\beta \varphi(\xi) d\xi \right| \le \left[\frac{1}{4} + \frac{\left(y - \frac{\alpha+\beta}{2}\right)^2}{(\beta-\alpha)^2} \right] (\beta-\alpha) \|\varphi'\|_\infty , \tag{1}$$

for all $y \in [\alpha, \beta]$. The constant $\frac{1}{4}$ is the best possible.

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Since 1938, the year of its publication, many research papers and books related to this inequality have been written and their numerous applications have been observed by concerned researchers. Especially in the last three decades, a great many authors have investigated refinements, improvements, counterparts and generalizations of the inequality (1). In general, the generalizations to the Ostrowski type inequalities are examined by adding arbitrary parameters in the Peano kernels involved. Some authors have also worked on new results related to Ostrowski inequality under various assumptions of the functions. To illustrate, a new generalization of the inequality (1) was established by Dragomir et al., and they gave some applications for special means and numerical integration in [7]. In addition, an Ostrowski type inequality for twice differentiable mappings was presented by Cerone et al. in [4]. Afterwards, in [6], a different version of the inequality given in [4] was obtained by Dragomir and Barnett. In [25], via a new Montgomery type identity, Sarikaya and Set provided some inequalities for functions second derivatives of which are elements of different Lebesgue spaces.

On the other side, perturbed method has been used to generalize Ostrowski type integral inequalities in recent years. For example, Perturbed types of Ostrowski inequalities for absolutely continuous functions was derived by Dragomir in [8] and [9]. After that, some mathematicians presented similar results to Dragomir's perturbed inequalities for twice differentiable mappings in [3], [12] and [26]. In addition to all these results, some researchers have focused on companions, refinements, and improvements of these recent inequalities for various classes of function. For instance, researchers deduced new companions and improvement companions of Ostrowski type for functions of bounded variation in [1] and [2]. In [10], [11] and [27], authors worked on companion inequalities based on quadratic kernel mapping with three-section for different function classes. Furthermore, new refined inequalities for functions whose second derivatives are the elemet of L_1 or L_2 and whose third derivatives are the element of L_2 are derived by Liu et al. in [20]. Then, companion inequalities based on five steps quadrature kernel are improved, and some applications for composite quadrature rules and Cumulative distribution functions of random variables are presented by Qayyum et al. in [22].

There are problems involving any-order derivative of a function besides the cases when first or second derivatives are required. So, a large number of mathematicians examined how inequalities come out when using higher-order differentiable functions. For instance, some Ostrowski type inequalities for higher-order differentiable functions were attained by Cerone et al. in [5]. What is more, researchers captured new generalized inequalities for functions whose any-order derivatives are element of L_p or L_∞ in [29] and [28]. In [16], Erden et al. improved weighted versions of inequalities involving higher-order derivatives, and they also gave some applications for the moments of random variable. In [19], some inequalities for n-times differentiable mappings were proved by Kechriniotis and Theodorou, and they gave some applications for probability density function of random variable. With the help of a three steps quadratic kernel, some companion results of Ostrowski type for functions whose *n.th* derivatives are bounded were provided by Kashif et al. in [17]. Also, Qayyum et al. established more comprehensive Ostrowski type results for the cases when $f^{(n+1)} \in L_1$, $f^{(n)} \in L_2$ or $f^{(n+1)} \in L_2$ by using quadratic kernel function with five sections in [23]. In [17], [18] and [23], new effective quadrature rules were developed, and relations between exact value and estimates of the integral of a mapping were examined. In [13]-[15], some perturbed inequalities of Ostrowski type for different function classes such as convex, bounded, bounded variation and Lipschitzian were provided by Erden. Generalized fractional Ostrowski type inequalities for higher order derivatives were established by Qayyum et al. in [24].

The purpose of this work is to establish new Ostrowski type results and efficient quadrature rules. For this, we first mention the importance of this article and some recent works in literature. In the next section, via quadratic kernel mapping with five sections, an integral inequality for function whose derivatives up to the order $(n-1)$ with $n \geq 1$ is continuous and *n.th* derivative exists is developed. Afterwards, improved inequalities of Ostrowski type for mappings whose higher-order derivatives are element of L_∞ or L_1 are obtained in section 3. In part 4, new refined perturbed results of Ostrowski type are also observed. In section 5, new efficient quadrature rules are developed to obtain closer estimations of the integral of a function. In part 6, new inequalities including exponential and logarithmic functions are derived by using the inequalities given in section 3. In final section, some applications of the inequalities presented in this paper for Cumulative Distribution, the reliability function and expectation value of random variable are given. Briefly, we deal with various function classes in order to obtain new inequalities, and we examine their different applications.

2. An Identity for High Degree Differentiable Functions

An identity involving higher order derivatives of a function is given in this section. This equality will be used to derive new results developed throughout this study. For convenience, we first define four notations that will be frequently used in this work.

$$\begin{aligned}
&\Omega_n(x;\varphi(x)) \qquad (2)\\
:&= \sum_{k=0}^{n-1}\frac{(-1)^{n+1-k}}{(k+1)!}\left\{\frac{1}{2^{k+1}}\left[(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right]\varphi^{(k)}\left(\frac{a+x}{2}\right)\right.\\
&+\left[\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right]\varphi^{(k)}(x)\\
&+\left[\left(\frac{a+b}{2}-x\right)^{k+1}-\left(\frac{3a+b}{4}-x\right)^{k+1}\right]\varphi^{(k)}(a+b-x)\\
&\left.+\frac{1}{2^{k+1}}\left[\left(\frac{a+b}{2}-x\right)^{k+1}-(-1)^{k+1}(x-a)^{k+1}\right]\varphi^{(k)}\left(\frac{a+2b-x}{2}\right)\right\},
\end{aligned}$$

$$\begin{aligned}
\Psi_n(x) \; :&= \frac{(x-a)^{n+1}}{2^{n+1}(n+1)!}\left[\psi_1(x)+(-1)^n\psi_5(x)\right]+\frac{1}{(n+1)!}\left(x-\frac{3a+b}{4}\right)^{n+1}\left[\psi_2(x)+(-1)^n\psi_4(x)\right] \qquad (3)\\
&+\frac{1}{2^{n+1}(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+1}\left[\psi_4(x)+(-1)^n\psi_2(x)\right]+\frac{[1+(-1)^n]}{(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+1}\psi_3(x),
\end{aligned}$$

$$\mathcal{P}(x)=\frac{(x-a)^{n+1}}{2^{n+1}}+\left(x-\frac{3a+b}{4}\right)^{n+1}+\frac{2^{n+1}+1}{2^{n+1}}\left(\frac{a+b}{2}-x\right)^{n+1}, \qquad (4)$$

and

$$\begin{aligned}
&\mathcal{R}_n(\varphi;x) \qquad (5)\\
:&= \frac{(x-a)^{n+1}}{2^{n+1}(n+1)!}\left[\varphi^{(n)}(a)+(-1)^n\varphi^{(n)}(b)\right]+\left[\frac{\varphi^{(n)}(a+b-x)+\varphi^{(n)}(\frac{a+2b-x}{2})}{2}+(-1)^n\frac{\varphi^{(n)}(\frac{a+x}{2})+\varphi^{(n)}(x)}{2}\right]\\
&\times\left[\frac{1}{2^{n+1}(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+1}-\frac{1}{(n+1)!}\left(\frac{3a+b}{4}-x\right)^{n+1}\right]\\
&+\frac{[1+(-1)^n]}{(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+1}\frac{\varphi^{(n)}(x)+\varphi^{(n)}(a+b-x)}{2}.
\end{aligned}$$

Now, we establish the required equality in the following theorem.

Lemma 2.1. *Let $\varphi:[a,b]\to\mathbb{R}$ be an $n-$times differantiable function such that $(n-1)$ .th derivative of φ is absolutely continuous on $[a,b]$, for $n\in\mathbb{N}$. Then, for any $\psi_i(x),\ i=1,2,...,5$ real mappings and all $x\in\left[a,\frac{a+b}{2}\right]$, we have the* *identity*

$$\begin{aligned}
&\Omega_{n,k(0)}(x;\varphi(x)) - \Psi_n(x) + (-1)^n \int_a^b \varphi(t)\,dt \qquad (6)\\
= &\int_a^{\frac{a+x}{2}} \frac{(t-a)^n}{n!}\left[\varphi^{(n)}(t) - \psi_1(x)\right]dt + \int_{\frac{a+x}{2}}^{x} \frac{1}{n!}\left(t - \frac{3a+b}{4}\right)^n \left[\varphi^{(n)}(t) - \psi_2(x)\right]dt\\
&+ \int_x^{a+b-x} \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n \left[\varphi^{(n)}(t) - \psi_3(x)\right]dt\\
&+ \int_{a+b-x}^{\frac{a+2b-x}{2}} \frac{1}{n!}\left(t - \frac{a+3b}{4}\right)^n \left[\varphi^{(n)}(t) - \psi_4(x)\right]dt + \int_{\frac{a+2b-x}{2}}^{b} \frac{(t-b)^n}{n!}\left[\varphi^{(n)}(t) - \psi_5(x)\right]dt
\end{aligned}$$

where $\Omega_n(x)$ and $\Psi_n(x)$ are defined as in (2) and (3), respectively.

Proof. Combining the resulting equalities side by side after applying the integration by parts n times to the integrals in the left of (6), from the definition of $\Omega_n(x)$ and $\Psi_n(x)$, the identity (6) can be captured. □

It should be noted that a refinement equality can be obtained for differentiable functions if $n = 1$ is taken in (6). We also note that if we choose $n = 2$ in the identity (6), then we reach the identity that is proved by Sarikaya et al. in [26].

3. Inequalities for Absolutely Continuous Derivatives

In this section, we first observe how the results will come out in the case when $\varphi^{(n+1)}$ is an element of $L_\infty[a,b]$.

Theorem 3.1. *Let $\varphi : [a,b] \to \mathbb{R}$ be an $n-$times differantiable function such that n.th derivative of φ is absolutely continuous on $[a,b]$, for $n \in \mathbb{N}$. If $\varphi^{(n+1)} : (a,b) \to \mathbb{R}$ is bounded, i.e.,*

$$\left\|\varphi^{(n+1)}\right\|_{[a,b],\infty} = \sup_{t\in(a,b)} \left|\varphi^{(n+1)}(t)\right|,$$

then, for any $x \in \left[a, \frac{a+b}{2}\right]$, we possess the inequality

$$\begin{aligned}
&\left|\Omega_n(x;\varphi(x)) - \frac{[1+(-1)^n]}{(n+1)!}\mathcal{P}(x)\varphi^{(n)}(x) + (-1)^n \int_a^b \varphi(t)\,dt\right| \qquad (7)\\
\le\ & \frac{1}{(n+2)!}\left\|\varphi^{(n+1)}\right\|_{[a,x],\infty} \mathcal{G}_n(x) + \frac{1}{(n+2)!}\left\|\varphi^{(n+1)}\right\|_{[x,b],\infty} \mathcal{H}_n(x)
\end{aligned}$$

where $\Omega_n(x;\varphi(x))$ and $\mathcal{P}(x)$ are as in (2) and (4), respectively. Also, $\mathcal{G}_n(x)$ and $\mathcal{H}_n(x)$ are defined by

$$\begin{aligned}
\mathcal{G}_n(x) \;=\; &\frac{(n+3)(x-a)^{n+2}}{2^{n+2}} + \frac{n+2}{2^{n+1}}\left(x - \frac{3a+b}{4}\right)\left(\frac{a+b}{2} - x\right)^{n+1} \qquad (8)\\
&+ \frac{n+1}{2^{n+2}}\left(\frac{a+b}{2} - x\right)^{n+2} + \left(x - \frac{3a+b}{4}\right)^{n+2}
\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{H}_n(x) \qquad (9)\\
=\ &\left(x-\frac{3a+b}{4}\right)^{n+1}\left[(n+2)\left(\frac{a+3b}{4}-x\right)-(n+1)\left(x-\frac{3a+b}{4}\right)\right]\\
&+\frac{1}{2^{n+1}}\left(\frac{a+b}{2}-x\right)^{n+1}\left[\frac{n+1}{2}\left(\frac{a+b}{2}-x\right)+(n+2)\left(\frac{a+3b}{4}-x\right)\right]\\
&+2(n+2)\left(\frac{a+b}{2}-x\right)^{n+2}+(x-a)^{n+1}\left[(n+2)(b-x)-(n+1)(x-a)\right],
\end{aligned}
$$

respectively.

Proof. Taking modulus in both sides of the resulting identity after writing $\varphi^{(n)}(x)$ instead of all $\psi_i(x)$ for $i=1,...,5$ in the equality (6), it follows that

$$
\begin{aligned}
&\left|\Omega_n(x;\varphi(x))-\frac{[1+(-1)^n]}{(n+1)!}\mathcal{P}(x)\varphi^{(n)}(x)+(-1)^n\int_a^b\varphi(t)\,dt\right| \qquad (10)\\
\leq\ &\int_a^{\frac{a+x}{2}}\frac{(t-a)^n}{n!}\left|\varphi^{(n)}(t)-\varphi^{(n)}(x)\right|dt+\int_{\frac{a+x}{2}}^{x}\frac{1}{n!}\left|t-\frac{3a+b}{4}\right|^n\left|\varphi^{(n)}(t)-\varphi^{(n)}(x)\right|dt\\
&+\int_x^{a+b-x}\frac{1}{n!}\left|t-\frac{a+b}{2}\right|^n\left|\varphi^{(n)}(t)-\varphi^{(n)}(x)\right|dt+\int_{a+b-x}^{\frac{a+2b-x}{2}}\frac{1}{n!}\left|t-\frac{a+3b}{4}\right|^n\left|\varphi^{(n)}(t)-\varphi^{(n)}(x)\right|dt\\
&+\int_{\frac{a+2b-x}{2}}^{b}\frac{(b-t)^n}{n!}\left|\varphi^{(n)}(t)-\varphi^{(n)}(x)\right|dt.
\end{aligned}
$$

It is easy to see that

$$
\left|\varphi^{(n)}(t)-\varphi^{(n)}(x)\right|=\left|\int_x^t\varphi^{(n+1)}(u)du\right|. \qquad (11)
$$

Then, for the first integral in the right side of (10), we find that

$$
\begin{aligned}
&\int_a^{\frac{a+x}{2}}\frac{(t-a)^n}{n!}\left|\int_x^t\varphi^{(n+1)}(u)du\right|\\
\leq\ &\int_a^{\frac{a+x}{2}}\frac{(t-a)^n}{n!}(x-t)\left\|\varphi^{(n+1)}\right\|_{[t,x],\infty}dt\\
\leq\ &\left\|\varphi^{(n+1)}\right\|_{[a,x],\infty}\int_a^{\frac{a+x}{2}}\frac{(t-a)^n}{n!}(x-t)\,dt=\frac{(n+3)(x-a)^{n+2}}{2^{n+2}(n+2)!}\left\|\varphi^{(n+1)}\right\|_{[a,x],\infty}.
\end{aligned}
$$

If we similarly examine the other four integrals in the right side of (10) and later we substitute all the resulting inequalities in (10), then we attain desired inequality (7) which finishes the proof. □

Corollary 3.2. *Substitution of $n = 1$ in (7) gives the result*

$$\left|\frac{b-a}{4}\left[\varphi\left(\frac{a+x}{2}\right)+\varphi(x)+\varphi(a+b-x)\right]+\frac{b-a}{4}\varphi\left(\frac{a+2b-x}{2}\right)-\int_a^b\varphi(t)\,dt\right| \tag{12}$$
$$\leq \frac{1}{6}\left\{\left\|\varphi''\right\|_{[a,x],\infty}\mathcal{G}_1(x)+\left\|\varphi''\right\|_{[x,b],\infty}\mathcal{H}_1(x)\right\}$$

where $\mathcal{G}_1(x)$ and $\mathcal{H}_1(x)$ are defined as in (8) and (9), respectively.

In the following theorem, we investigate how the inequality will come out the case when $\varphi^{(n+1)} \in L_1[a,b]$.

Theorem 3.3. *Let $\varphi : [a,b] \to \mathbb{R}$ be an $n-$times differantiable function such that n.th derivative of φ is absolutely continuous on $[a,b]$, for $n \in \mathbb{N}$. If $\varphi^{(n+1)} \in L_1[a,b]$, i.e.,*

$$\left\|\varphi^{(n+1)}\right\|_{[a,b],1}=\int_a^b\left|\varphi^{(n+1)}(t)\right|dt<\infty,$$

then, for any $x \in \left[a, \frac{a+b}{2}\right]$, one has

$$\left|\Omega_n(x;\varphi(x))-\frac{[1+(-1)^n]}{(n+1)!}\mathcal{P}(x)\varphi^{(n)}(x)+(-1)^n\int_a^b\varphi(t)\,dt\right| \tag{13}$$
$$\leq \left[\frac{(x-a)^{n+1}}{2^{n+1}(n+1)!}+\frac{\mathcal{K}_n(x)}{(n+1)!}\right]\left[\left\|\varphi^{(n+1)}\right\|_{[a,x],1}+\left\|\varphi^{(n+1)}\right\|_{[x,b],1}\right]+\frac{2}{(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+1}\left\|\varphi^{(n+1)}\right\|_{[x,b],1}$$

where $\Omega_n(x;\varphi(x))$ and $\mathcal{P}(x)$ are as in (2) and (4), respectively. Also, $\mathcal{K}_n(x)$ is defined by

$$\mathcal{K}_n(x)=\frac{1}{2^{n+1}}\left(\frac{a+b}{2}-x\right)^{n+1}+\left(x-\frac{3a+b}{4}\right)^{n+1}. \tag{14}$$

Proof. Calculating the first integral in the right side of (10) by considering that $\varphi^{(n+1)}$ is the element of $L_1[a,b]$, from the equality (11), we find that

$$\int_a^{\frac{a+x}{2}}\frac{(t-a)^n}{n!}\left|\int_t^x\varphi^{(n+1)}(s)ds\right|dt$$
$$\leq \left\|\varphi^{(n+1)}\right\|_{[a,x],1}\int_a^{\frac{a+x}{2}}\frac{(t-a)^n}{n!}dt=\frac{(x-a)^{n+1}}{2^{n+1}(n+1)!}\left\|\varphi^{(n+1)}\right\|_{[a,x],1}.$$

Substituting all the resulting inequalities in (10) after calculating the other four integrals in the right side of (10), the required inequality (13) can be easily obtained. □

Corollary 3.4. *Under the assumptions of Theorem 3.3 with $n = 1$, the following inequality holds*

$$\left|\frac{b-a}{4}\left[\varphi\left(\frac{a+x}{2}\right)+\varphi(x)+\varphi(a+b-x)\right]+\frac{b-a}{4}\varphi\left(\frac{a+2b-x}{2}\right)-\int_a^b\varphi(t)\,dt\right| \tag{15}$$
$$\leq \frac{1}{2}\left[\frac{(x-a)^2}{4}+\mathcal{K}_1(x)\right]\left[\left\|\varphi''\right\|_{[a,x],1}+\left\|\varphi''\right\|_{[x,b],1}\right]+\left(\frac{a+b}{2}-x\right)^2\left\|\varphi''\right\|_{[x,b],1}$$

where $\mathcal{K}_1(x)$ is defined as in (14).

It should be noted that if we take $x = a,\ x = \frac{a+b}{2}$ or $x = \frac{3a+b}{4}$ in the inequalities (7) and (13), then we can capture new perturbed type inequalities. Also, it can be observed how results will be obtained, when $n = 2$ is taken in the inequalities provided in this part.

4. Some Estimations for Bounded Variation Functions

We need to give the definition of bounded variation function and concept of total variation which are necessary to express and prove the results presented in this section.

Definition 4.1. *Suppose that $D : a = \xi_0 < \xi_1 < ... < \xi_m = b$ is any division of $[a,b]$, and let $\Delta\phi(\xi_j) = \phi(\xi_{j+1}) - \phi(\xi_j)$. If the sum*

$$\sum_{j=1}^{m} \left|\Delta\phi(\xi_j)\right|$$

is bounded for all such divisions, then ϕ is said to be of bounded variation.

Definition 4.2. *Assume that ϕ is of bounded variation on $[a,b]$, and $\sum \Delta\phi\,(D)$ denotes the sum $\sum\limits_{j=1}^{n} \left|\Delta\phi(\xi_i)\right|$ corresponding to the division D of $[a,b]$. The number*

$$\bigvee_a^b \left(\phi\right) := \sup\left\{\sum \Delta\phi\,(D) : D \in D([a,b])\right\}$$

is called the total variation of f on $[a,b]$. Here, $D([a,b])$ indicates the family of partitions of $[a,b]$.

In the following theorem, a perturbed type inequality based on the quadratic kernel mapping with five steps kernel for functions whose n.th derivative are of bounded variation are derived.

Theorem 4.3. *Let $\varphi : [a,b] \to \mathbb{R}$ be an $n-$times differantiable function such that $(n-1)$.th derivative of φ is absolutely continuous on $[a,b]$, for $n \in \mathbb{N}$. If n.th derivative of φ is of bounded variation on $[a,b]$, then, for any $x \in \left[a, \frac{a+b}{2}\right]$, we have*

$$\begin{aligned} &\left|\Omega_n(x;\varphi(x)) - \mathcal{R}_n(\varphi;x) + (-1)^n \int_a^b \varphi\,(t)\,dt\right| \qquad (16)\\ \leq\ & \frac{(x-a)^{n+1}}{2^{n+1}\,(n+1)!}\left[\bigvee_a^{\frac{ax}{2}}\left(\varphi^{(n)}\right) + \bigvee_{\frac{a+2b-x}{2}}^{b}\left(\varphi^{(n)}\right)\right] + \frac{\mathcal{K}_n(x)}{2(n+1)!}\left[\bigvee_{\frac{a+x}{2}}^{x}\left(\varphi^{(n)}\right) + \bigvee_{a+b-x}^{\frac{a+2b-x}{2}}\left(\varphi^{(n)}\right)\right]\\ &+\frac{1}{(n+1)!}\left(\frac{a+b}{2} - x\right)^{n+1}\bigvee_x^{a+b-x}\left(\varphi^{(n)}\right) \end{aligned}$$

where $\Omega_n(x;\varphi(x))$, $\mathcal{R}_n(\varphi;x)$ and $\mathcal{K}_n(x)$ are defined as in (2), (5) and (14), respectively.

Proof. We first write $\varphi^{(n)}(a)$, $\frac{\varphi^{(n)}\left(\frac{a+x}{2}\right)+\varphi^{(n)}(x)}{2}$, $\frac{\varphi^{(n)}(x)+\varphi^{(n)}(a+b-x)}{2}$, $\frac{\varphi^{(n)}(a+b-x)+\varphi^{(n)}\left(\frac{a+2b-x}{2}\right)}{2}$ and $\varphi^{(n)}(b)$ instead of $\psi_i(x)$ for $i = 1, ..., 5$ in the expression (6), respectively. Afterwards, if we take absolute value of the resulting identity, then we get

$$\left|\Omega_n(x;\varphi(x)) - \mathcal{R}_n(\varphi;x) + (-1)^n \int_a^b \varphi(t)\,dt\right| \tag{17}$$

$$\begin{aligned}
&\leq \int_a^{\frac{a+x}{2}} \frac{(t-a)^n}{n!}\left|\varphi^{(n)}(t) - \varphi^{(n)}(a)\right| dt + \int_{\frac{a+x}{2}}^{x} \frac{1}{n!}\left|t - \frac{3a+b}{4}\right|^n \left|\varphi^{(n)}(t) - \frac{\varphi^{(n)}(\frac{a+x}{2}) + \varphi^{(n)}(x)}{2}\right| dt \\
&\quad + \int_x^{a+b-x} \frac{1}{n!}\left|t - \frac{a+b}{2}\right|^n \left|\varphi^{(n)}(t) - \frac{\varphi^{(n)}(x) + \varphi^{(n)}(a+b-x)}{2}\right| dt \\
&\quad + \int_{a+b-x}^{\frac{a+2b-x}{2}} \frac{1}{n!}\left|t - \frac{a+3b}{4}\right|^n \left|\varphi^{(n)}(t) - \frac{\varphi^{(n)}(a+b-x) + \varphi^{(n)}(\frac{a+2b-x}{2})}{2}\right| dt + \int_{\frac{a+2b-x}{2}}^{b} \frac{(b-t)^n}{n!}\left|\varphi^{(n)}(t) - \varphi^{(n)}(b)\right| dt.
\end{aligned}$$

In as much as $\varphi^{(n)}$ is of bounded variation on $[a,b]$, one has

$$\left|\varphi^{(n)}(t) - \varphi^{(n)}(a)\right| \leq \bigvee_a^t \left(\varphi^{(n)}\right)$$

for $t \in \left[a, \frac{a+x}{2}\right]$. Then, for the first integral in the right side of (17), we find that

$$\begin{aligned}
&\int_a^{\frac{a+x}{2}} \frac{(t-a)^n}{n!}\left|\varphi^{(n)}(t) - \varphi^{(n)}(a)\right| dt \\
&\leq \bigvee_a^x \left(\varphi^{(n)}\right) \int_a^{\frac{a+x}{2}} \frac{(t-a)^n}{n!} dt = \frac{(x-a)^{n+1}}{2^{n+1}(n+1)!} \bigvee_a^x \left(\varphi^{(n)}\right).
\end{aligned}$$

Also, because $\varphi^{(n)}$ is of bounded variation on $[a,b]$, it is find that

$$\begin{aligned}
&\left|\varphi^{(n)}(t) - \frac{\varphi^{(n)}(\frac{a+x}{2}) + \varphi^{(n)}(x)}{2}\right| \\
&\leq \frac{1}{2}\left[\left|\varphi^{(n)}(x) - \varphi^{(n)}(t)\right| + \left|\varphi^{(n)}(t) - \varphi^{(n)}\left(\frac{a+x}{2}\right)\right|\right] \leq \frac{1}{2} \bigvee_{\frac{a+x}{2}}^{x} \left(\varphi^{(n)}\right)
\end{aligned}$$

for $t \in \left[\frac{a+x}{2}, x\right]$. In this case, we have

$$\begin{aligned}
&\int_{\frac{a+x}{2}}^{x} \frac{1}{n!}\left|t - \frac{3a+b}{4}\right|^n \left|\varphi^{(n)}(t) - \frac{\varphi^{(n)}(\frac{a+x}{2}) + \varphi^{(n)}(x)}{2}\right| dt \\
&\leq \frac{1}{2} \bigvee_{\frac{a+x}{2}}^{x} \left(\varphi^{(n)}\right) \frac{1}{n!} \int_{\frac{a+x}{2}}^{x} \left|t - \frac{3a+b}{4}\right|^n dt = \frac{1}{2(n+1)!}\left[\frac{1}{2^{n+1}}\left(\frac{a+b}{2} - x\right)^{n+1} + \left(x - \frac{3a+b}{4}\right)^{n+1}\right] \bigvee_{\frac{a+x}{2}}^{x} \left(\varphi^{(n)}\right).
\end{aligned}$$

If we similarly analyze the other three integrals in the right side of (17) by considering that $\varphi^{(n)}$ is of bounded variation on $[a,b]$, and later we substitute all the resulting inequalities in (17), then we capture required inequality. The proof is thus completed. □

Corollary 4.4. *If we write* 1 *instead of* n *in (16), we get the inequality*

$$\left|\frac{b-a}{4}\left[\varphi\left(\frac{a+x}{2}\right)+\varphi(x)+\varphi(a+b-x)+\varphi\left(\frac{a+2b-x}{2}\right)\right]\right. \tag{18}$$
$$-\left[\frac{\varphi'(a+b-x)+\varphi'(\frac{a+2b-x}{2})}{2}-\frac{\varphi'(\frac{a+x}{2})+\varphi'(x)}{2}\right]\left[\frac{1}{8}\left(\frac{a+b}{2}-x\right)^2-\frac{1}{2}\left(\frac{3a+b}{4}-x\right)^2\right]$$
$$\left.-\frac{(x-a)^2}{8}\left[\varphi'(a)-\varphi'(b)\right]-\int_a^b\varphi(t)\,dt\right|$$
$$\leq \frac{(x-a)^2}{8}\left[\bigvee_a^{\frac{ax}{2}}(\varphi')+\bigvee_{\frac{a+2b-x}{2}}^{b}(\varphi')\right]+\frac{\mathcal{K}_1(x)}{4}\left[\bigvee_{\frac{a+x}{2}}^{x}(\varphi')+\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(\varphi')\right]+\frac{1}{2}\left(\frac{a+b}{2}-x\right)^2\bigvee_x^{a+b-x}(\varphi')$$

where $\mathcal{K}_1(x)$ *is defined as in (14).*

Remark 4.5. *If we choose* $n=2$ *in (16), then (16) reduce to the inequality (3.2) given in [26].*

We can also derive new inequalities or results provided in the earlier works for special cases of (16).

5. Efficient Quadrature Rules

In this section, novel composite quadrature rules including higher-order derivatives are proposed to calculate approximate values of any given mapping. The main focus is to find closer estimates of the integral of a mapping by dividing the intervals into thinner pieces. These perturbed quadrature rules are applied to several mappings to demonstrate their achievements at estimating the exact value of each mapping. Therefore, the before-mentioned identity (6) are utilized in order to evaluate the composite quadrature rules.

We first derive three quadrature rules as follows:

$$\int_a^b\varphi(t)\,dt\approx Q_{n,1}(\varphi)$$
$$=\left[2^{n+1}+1+(-1)^{n+1}\right]\left(\tfrac{b-a}{4}\right)^{n+1}\tfrac{[1+(-1)^n]}{(n+1)!}\varphi^{(n)}(a)+\sum_{k=0}^{n-1}\tfrac{(b-a)^{k+1}}{2^{k+1}(k+1)!}\left\{\varphi^{(k)}(a)+(-1)^k\varphi^{(k)}(b)\right\},$$

$$\int_a^b\varphi(t)dt\approx Q_{n,2}(\varphi)$$
$$=\tfrac{(b-a)^{n+1}}{2^{2n+1}}\tfrac{[1+(-1)^n]}{(n+1)!}\varphi^{(n)}\left(\tfrac{a+b}{2}\right)+\sum_{k=0}^{n-1}\tfrac{(b-a)^{k+1}}{2^{2k+2}(k+1)!}\left[\varphi^{(k)}\left(\tfrac{a+3b}{4}\right)+(-1)^k\varphi^{(k)}\left(\tfrac{3a+b}{4}\right)\right]$$
$$+\sum_{k=0}^{n-1}\tfrac{(b-a)^{k+1}}{2^{2k+2}(k+1)!}\left[1+(-1)^k\right]\varphi^{(k)}\left(\tfrac{a+b}{2}\right)$$

and

$$\int_a^b\varphi(t)dt\approx Q_{n,3}(\varphi)$$
$$=\tfrac{[1+(-1)^n]}{(n+1)!}\tfrac{2^{n+1}+2}{8^{n+1}}(b-a)^{n+1}\varphi^{(n)}\left(\tfrac{3a+b}{4}\right)+\sum_{k=0}^{n-1}\tfrac{[1+(-1)^k]}{8^{k+1}(k+1)!}(b-a)^{k+1}\left[\varphi^{(k)}\left(\tfrac{7a+b}{8}\right)+\varphi^{(k)}\left(\tfrac{a+7b}{8}\right)\right]$$
$$+\sum_{k=0}^{n-1}\tfrac{1}{(k+1)!}\left(\tfrac{b-a}{4}\right)^{k+1}\left[\varphi^{(k)}\left(\tfrac{3a+b}{4}\right)+(-1)^k\varphi^{(k)}\left(\tfrac{a+3b}{4}\right)\right].$$

For a different choosing of ψ in (6), we also have

$$\int_a^b \varphi(t)\,dt \approx Q_{n,4}(\varphi)$$
$$= \left[\frac{1+(-1)^n}{2^{n+1}(n+1)!}\frac{\varphi^{(n)}(a)+\varphi^{(n)}(b)}{2}(b-a)^{n+1}\right] + \sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{2^{k+1}(k+1)!}\left\{\varphi^{(k)}(a)+(-1)^k\varphi^{(k)}(b)\right\},$$

$$\int_a^b \varphi(t)dt \approx Q_{n,5}(\varphi)$$
$$= \frac{(b-a)^{n+1}}{2^{2n+2}(n+1)!}\left[\varphi^{(n)}(a)+(-1)^n\varphi^{(n)}(b)\right] + \frac{(-1)^n(b-a)^{n+1}}{2^{2n+3}(n+1)!}\left[[1+(-1)^n]\varphi^{(n)}\left(\frac{a+b}{2}\right)+\varphi^{(n)}\left(\frac{a+3b}{4}\right)+(-1)^n\varphi^{(n)}\left(\frac{3a+b}{4}\right)\right]$$
$$+\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{2^{2k+2}(k+1)!}\left[\varphi^{(k)}\left(\frac{a+3b}{4}\right)+(-1)^k\varphi^{(k)}\left(\frac{3a+b}{4}\right)\right]+\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{2^{2k+2}(k+1)!}\left[1+(-1)^k\right]\varphi^{(k)}\left(\frac{a+b}{2}\right)$$

and

$$\int_a^b \varphi(t)dt \approx Q_{n,6}(\varphi)$$
$$= \frac{(b-a)^{n+1}}{8^{n+1}(n+1)!}\left[\varphi^{(n)}(a)+(-1)^n\varphi^{(n)}(b)\right]+\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{4}\right)^{k+1}\left[\varphi^{(k)}\left(\frac{3a+b}{4}\right)+(-1)^k\varphi^{(k)}\left(\frac{a+3b}{4}\right)\right]$$
$$+\frac{1}{2}\frac{(b-a)^{n+1}}{8^{n+1}(n+1)!}\left\{\varphi^{(n)}\left(\frac{a+3b}{4}\right)+\varphi^{(n)}\left(\frac{a+7b}{8}\right)+(-1)^n\left[\varphi^{(n)}\left(\frac{3a+b}{4}\right)+\varphi^{(n)}\left(\frac{7a+b}{8}\right)\right]\right\}$$
$$+\frac{[1+(-1)^n]}{(n+1)!}\frac{(b-a)^{n+1}}{2^{2n+3}}\left[\varphi^{(n)}\left(\frac{3a+b}{4}\right)+\varphi^{(n)}\left(\frac{a+3b}{4}\right)\right]+\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right]}{8^{k+1}(k+1)!}(b-a)^{k+1}\left[\varphi^{(k)}\left(\frac{7a+b}{8}\right)+\varphi^{(k)}\left(\frac{a+7b}{8}\right)\right]$$

The relations of the quadrature rules and exact values are obtained in what follows. The successive rules, which are $Q_{n,1}-Q_{n,4}$, $Q_{n,2}-Q_{n,5}$ and $Q_{n,3}-Q_{n,6}$, are in the same ballpark.

The achievements of the proposed rules $Q_{n,1}-Q_{n,6}$ are tracked for eight mappings, $f_1(x)=x^4+2x^2+1$, $f_2(x)=cosx-x$, $f_3(x)=e^x\sin x$, $f_4(x)=e^{x^2}$, $f_5(x)=\sin x-x^3$, $f_6(x)=\log(x^2+1)\sin(x^2+1)$, $f_7(x)=e^x\cos(e^x-2x)$, $f_8(x)=\log(x^2+2)\sin(\log(x^2+2))$. The approximate and exact values of the obtained functions for this study are given in Table 1. Besides, the approximate and exact results of the widely-used functions in the literature are given in Table 2. All calculations and illustrations are realized by an interface created in MATLAB R2019a.

According to Table 1, the quadrature rules $Q_{n,1}-Q_{n,6}$ give the precise value of the integral of f_1 for $n=4$. It shows that the proposed rules will provide the exact value of integral of any given polynomial function. Similarly, the exact value of integral of f_3 are calculated through $Q_{n,1}-Q_{n,6}$ where $n=7$. Most of the similar studies emphasized that the error calculated for f_2 changes between 5.1×10^{-4} and 6.5×10^{-5}. In this study, the exact value for the function f_2 can be calculated through all rules with near-zero errors when n equals 7. The best results of the integral of f_3 are achieved through the overall quadrate rules with near-zero errors for $n=6$. The rules $Q_{n,2}$, $Q_{n,3}$, $Q_{n,5}$, $Q_{n,6}$ succeed at calculating the exact value of the integral of f_4 when n equals 10. It can be seen from Table 1 that the errors are near to zero for n=10. The approximate results of $Q_{n,1}$ and $Q_{n,4}$ are near to exact value with the errors $2.44E-05$ and $5.88E-06$, respectively.

In this part, the performances of the proposed rules are tested on well-known mappings in literature. According to Table 2, the exact result of integral of f_5 can be calculated through the whole proposed rules with near-zero errors when n equals 7. The approximate values calculated using $Q_{n,2}$, $Q_{n,3}$, $Q_{n,5}$, $Q_{n,6}$ are found as near to the exact value of the integral of f_6 for $n=12$. The errors computed for these quadrature rules are near to zero when n equals 12. $Q_{n,1}$ and $Q_{n,4}$ give worse-results than the remained rules. Similarly, the exact value of the integral of f_7 is achieved through $Q_{n,2}$, $Q_{n,3}$, $Q_{n,5}$, $Q_{n,6}$ with near-zero errors for $n=11$. The results of these quadrature rules are better than the results of $Q_{n,1}$ and $Q_{n,4}$ for the function f_7. The exact value of the integral of f_8 can be achieved through the whole rules with near-zero errors for both $n=5$ and $n=9$. The best results for this function are found when n equals 9.

We observed the relations between exact value of the integrals and estimations of our quadrature rules for a certain closed interval. We now present some figures to see how approachs will come out for the

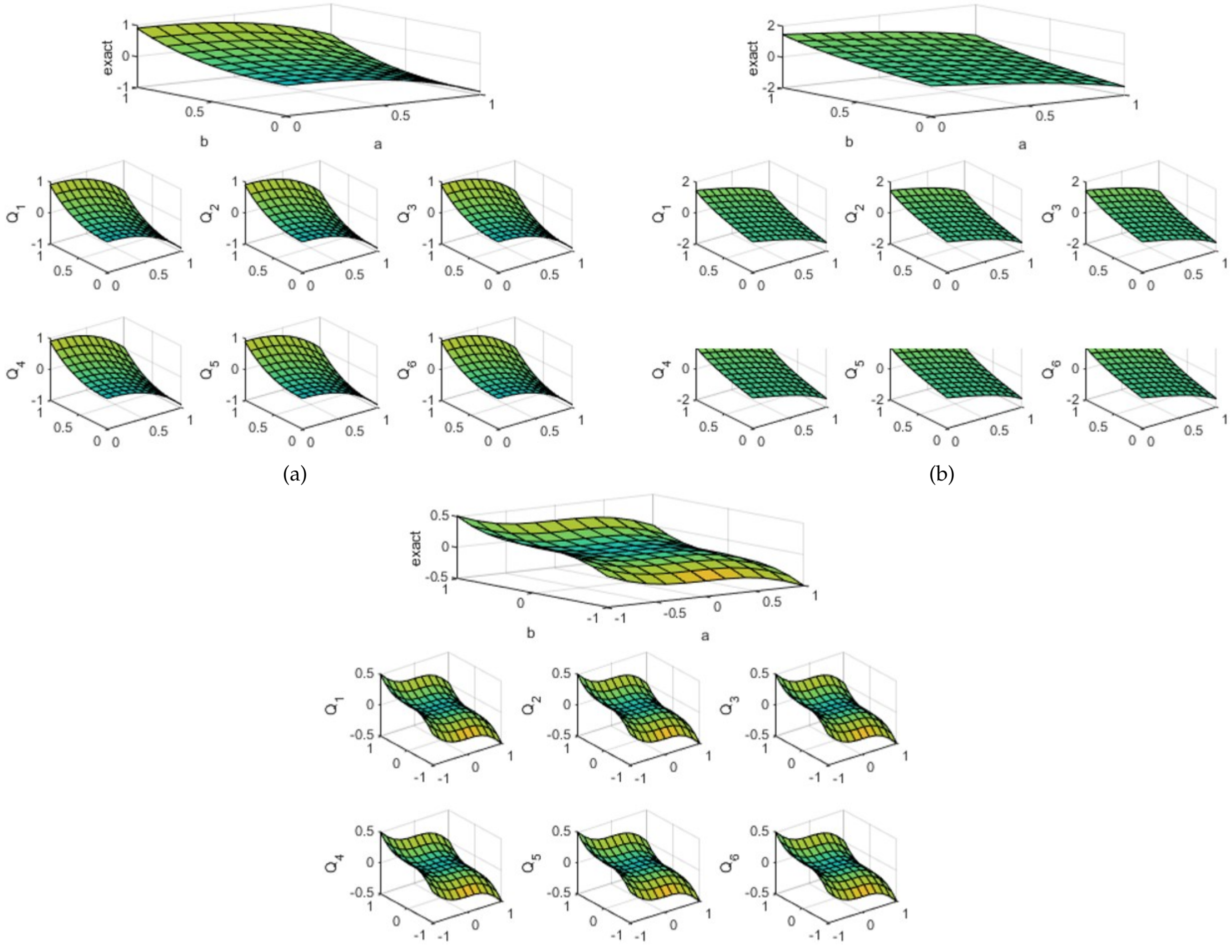
Table 1: The results for f_1, f_2, f_3, f_4.

Function	$[a, b]$	n		Q_1	Q_4	Q_2	Q_5	Q_3	Q_6	Exact
f_1	[0,1]	4	Result	1.866667	1.866667	1.866667	1.866667	1.866667	1.866667	1.866667
			Error	0	0	3.67E-40	1.08E-20	1.08E-20	5.76E-21	---
f_2	$[0,\frac{\pi}{2}]$	3	Result	-0.22062	-0.22062	-0.23445	-0.2357	-0.23327	-0.23329	-0.2337
			Error	0.013078	0.013078	0.000744	0.002004	0.000426	0.000406	---
		7	Result	-0.2337	-0.2337	-0.2337	-0.2337	-0.2337	-0.2337	-0.2337
			Error	3.25E-06	3.25E-06	9.24E-09	2.71E-08	6.78E-09	6.76E-09	---
f_3	[0,1]	4	Result	0.911506	0.909123	0.909339	0.909313	0.909368	0.909329	0.909331
			Error	0.002175	0.000207	8.34E-06	1.76E-05	3.77E-05	1.86E-06	---
		6	Result	0.909338	0.909332	0.909331	0.909331	0.909331	0.909331	0.909331
			Error	7.04E-06	1.22E-06	1.28E-08	3.33E-08	3.20E-08	2.29E-09	---
f_4	[0,1]	5	Result	1.47637	1.47637	1.462536	1.462205	1.462723	1.46272	1.462652
			Error	1.37E-02	0.013718	1.16E-04	0.000447	7.08E-05	6.83E-05	---
		7	Result	1.463982	1.463982	1.462649	1.462642	1.462653	1.462653	1.462652
			Error	1.33E-03	0.00133	2.30E-06	9.90E-06	1.54E-06	1.53E-06	---
		10	Result	1.462627	1.462658	1.462652	1.462652	1.462652	1.462652	1.462652
			Error	2.44E-05	5.88E-06	2.51E-09	1.16E-08	3.10E-09	3.52E-10	---

Table 2: The results for f_5, f_6, f_7, f_8.

Function	$[a, b]$	n		Q_1	Q_4	Q_2	Q_5	Q_3	Q_6	Exact
f_5	[2,3]	4	Result	-15.676	-15.6762	-15.6762	-15.6762	-15.6762	-15.6762	-15.6762
			Error	1.89E-04	1.08E-05	4.32E-07	9.12E-07	3.36E-06	9.63E-08	---
		7	Result	-15.6762	-15.6762	-15.6762	-15.6762	-15.6762	-15.6762	-15.6762
			Error	4.98E-08	4.98E-08	1.37E-10	4.10E-10	9.98E-11	9.95E-11	---
f_6	$[-1,1]$	7	Result	0.635798	0.635798	0.50897	0.508536	0.509651	0.509652	0.509386
			Error	1.26E-01	0.126412	4.16E-04	0.00085	2.65E-04	0.000266	---
		12	Result	0.511884	0.511884	0.509387	0.509387	0.509386	0.509386	0.509386
			Error	0.002498	0.002498	6.73E-07	8.59E-07	5.63E-08	5.64E-08	---
f_7	$[0,\frac{\pi}{2}]$	6	Result	0.573272	2.521273	2.152776	2.16964	2.154924	2.156929	2.156214
			Error	1.582942	0.365059	0.003437	0.013426	0.001289	0.000715	---
		11	Result	2.005	2.005	2.156219	2.156247	2.156211	2.156211	2.156214
			Error	0.151213	0.151213	5.74E-06	3.37E-05	2.61E-06	2.61E-06	---
f_8	[0,1]	4	Result	0.30545	0.305139	0.305058	0.305075	0.305076	0.305068	0.305066
			Error	0.000384	7.29E-05	7.70E-06	9.12E-06	1.03E-05	1.53E-06	---
		5	Result	0.305139	0.305139	0.305064	0.305063	0.305068	0.305068	0.305066
			Error	7.29E-05	7.29E-05	1.93E-06	3.47E-06	1.50E-06	1.51E-06	---
		9	Result	0.305067	0.305067	0.305066	0.305066	0.305066	0.305066	0.305066
			Error	4.09E-07	4.09E-07	1.46E-09	2.15E-09	1.26E-09	1.26E-09	---

integrals of given functions on different intervals. The approximate approaches and exact values for the mappings f_3, f_4 and f_6 are illustrated in Figure 1.a-c, respectively. These illustrations are given as an alternative view to how the results of the proposed rules bear a resemblance to the exact value of the integrals of these mappings.



(a) (b)

(c)

Fig. 1: The approximations for (a) f_3 , (b) f_4, (c) f_6 .

Finally, we should mention the algorithm times for the numerical results attained using the interface, which was created in MATLAB R2019a. The numerical studies are conducted on a computer with Intel Core i5 2.5 GHz CPU, 8.0 GB memory. According to these calculations, it is seen that the algorithm times spent for calculating the approximate values are lower than the times spent for computing exact values of mappings. For example, it took 0.163124, 0.272809 and 0.29948 seconds to calculate the estimations $Q_{7,1}$, $Q_{7,2}$ and $Q_{7,3}$ for the function f_4, respectively, while it took 0.567873 seconds to calculate the exact value of integral of f_4 in MATLAB R2019a.

6. Applications for Some Elementary Functions

To find new inequalities involving exponential and logarithmic functions, we deal with the results presented in the section 3. Initially, we examine how results will come out from the inequalities (7) and (13), if we consider the exponential function. Suppose that $\varphi(t) = e^t$ with $t \in \mathbb{R}$, then one has

$$\left\|\varphi^{(n+1)}\right\|_{[a,x],\infty} = e^x \ , \quad \left\|\varphi^{(n+1)}\right\|_{[x,b],\infty} = e^b$$

and

$$\left\|\varphi^{(n+1)}\right\|_{[a,x],1} = e^x - e^a \ , \ \left\|\varphi^{(n+1)}\right\|_{[x,b],1} = e^b - e^x$$

for any $x \in \left[a, \frac{a+b}{2}\right]$. In the circumstances, from the inequality (7), we possess the following result involving exponential functions

$$\left|\Omega_n(x; e^x) - \frac{[1 + (-1)^n]}{(n+1)!}\mathcal{P}(x)e^x + (-1)^n\left(e^b - e^a\right)\right| \le \frac{1}{(n+2)!}\left[e^x\mathcal{G}_n(x) + e^b\mathcal{H}_n(x)\right]$$

where $\Omega_n(x; e^x)$, $\mathcal{P}(x)$, $\mathcal{G}_n(x)$ and $\mathcal{H}_n(x)$ are defined as in (2), (4), (8) and (9), respectively.

In addition, considering the inequality (13), we have the inequality

$$\begin{aligned} &\left|\Omega_n(x; e^x) - \frac{[1 + (-1)^n]}{(n+1)!}\mathcal{P}(x)e^x + (-1)^n\left(e^b - e^a\right)\right| \\ \le\ & \left[\frac{(x-a)^{n+1}}{2^{n+1}(n+1)!} + \frac{\mathcal{K}_n(x)}{(n+1)!}\right]\left(e^b - e^a\right) + \frac{2}{(n+1)!}\left(\frac{a+b}{2} - x\right)^{n+1}\left(e^b - e^x\right) \end{aligned}$$

where $\Omega_n(x; e^x)$, $\mathcal{P}(x)$ and $\mathcal{K}_n(x)$ are as in (2), (4) and (14), respectively.

Now, we take into account another function that is named as logarithmic function. Supposing that $\varphi(t) = \ln t$ with $t > 0$. It is easy to see that

$$\varphi^{(k)}(t) = \frac{(-1)^{k-1}(k-1)!}{t^k} \quad k \ge 1 \text{ and } t > 0.$$

Also, we have

$$\left\|\varphi^{(n+1)}\right\|_{[a,x],\infty} = \frac{n!}{a^{n+1}} \qquad \left\|\varphi^{(n+1)}\right\|_{[x,b],\infty} = \frac{n!}{x^{n+1}}$$

for any $0 < a \le x \le \frac{a+b}{2}$. In this case, from the inequality (7), one has

$$\left|\Omega_n(x, \ln x) + \frac{[1 + (-1)^n]}{n(n+1)x^n}\mathcal{P}(x) + (-1)^n\left[\ln\frac{b^b}{a^a} - (b-a)\right]\right| \le \frac{x^{n+1}\mathcal{G}_n(x) + a^{n+1}\mathcal{H}_n(x)}{(n+1)(n+2)a^{n+1}x^{n+1}}$$

where $\Omega_n(x, \ln x)$, $\mathcal{P}(x)$, $\mathcal{G}_n(x)$ and $\mathcal{H}_n(x)$ are defined as in (2), (4), (8) and (9), respectively.

We also consider the inequality (13), for the function $\varphi(t) = \ln t$, we find that

$$\left\|\varphi^{(n+1)}\right\|_{[a,x],1} = \frac{(n-1)!(x^n - a^n)}{x^n a^n}$$

and

$$\left\|\varphi^{(n+1)}\right\|_{[x,b],1} = \frac{(n-1)!(b^n - x^n)}{b^n x^n},$$

for any $0 < a \le x \le \frac{a+b}{2} < \infty$. Then, we have

$$\begin{aligned} &\left|\Omega_n(x, \ln x) + \frac{[1 + (-1)^n]}{n(n+1)x^n}\mathcal{P}(x) + (-1)^n\left[\ln\frac{b^b}{a^a} - (b-a)\right]\right| \\ \le\ & \frac{1}{n(n+1)}\frac{b^n - a^n}{a^n b^n}\left[\frac{(x-a)^{n+1}}{2^{n+1}} + \mathcal{K}_n(x)\right] + \frac{2(b^n - x^n)}{n(n+1)x^n b^n}\left(\frac{a+b}{2} - x\right)^{n+1} \end{aligned}$$

where $\Omega_n(x; e^x)$, $\mathcal{P}(x)$ and $\mathcal{K}_n(x)$ are as in (2), (4) and (14), respectively.

As well as all these results, if we take $x = a$, $x = \frac{a+b}{2}$ or $x = \frac{3a+b}{4}$ in the inequalities given in this part, then we can attain new estimations for exponential and logarithmic functions.

7. Applications for PDF

Let (Ω, U, P) be probability space and X be a random variable, defined over the finite interval $[a, b]$, with the probability density function (PDF) $\varphi : [a, b] \to [0, \infty)$, when $\int_a^b \varphi(t)dt = 1$ and with the cumulative distribution function (CDF) $F(x) = P(X \leq x) = \int_a^x \varphi(t)dt$, for $a < x \leq b$, taking value in $[0, 1]$. It is known that F is monotonic nondecreasing function and absolutely continuous on $[a, b]$, $F(a) = 0$ and $F(b) = 1$, $F' = \varphi$ exists almost everywhere on [a,b]. The reliability function (RF) $R(x) = P(X > x)$, as probability of any system operating at a given time, can be determined by CDF and also PDF as follows:

$$R(x) = 1 - F_X(x) = 1 - \int_a^x \varphi(t)dt.$$

The reliability function is an important tool to determine the survival-probability of a component X, which has the CDF F. RF is mostly used in life time analysis, system reliability studies. Now, we provide some inequalities involving CDF, RF and expectation value of random variable X.

Theorem 7.1. *Substitution of $F = \varphi$ in (12) gives the result*

$$\begin{aligned} &\left|\frac{1}{4}\left[F\left(\frac{a+x}{2}\right) + F(x) + F(a+b-x)\right] + \frac{1}{4}F\left(\frac{a+2b-x}{2}\right) - \frac{b-E(x)}{b-a}\right| \\ \leq\ & \frac{\left\|\varphi'\right\|_{[a,x],\infty} \mathcal{G}_1(x) + \left\|\varphi'\right\|_{[x,b],\infty} \mathcal{H}_1(x)}{6(b-a)} \end{aligned}$$

where $\mathcal{G}_1(x)$ and $\mathcal{H}_1(x)$ are defined as in (8) and (9), respectively.

Proof. If we write F instead of φ in (12), then we have the expectation value of random variable X

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt = b - \int_a^b \varphi(t)dt. \tag{19}$$

Hence, the proof is finished. □

Corollary 7.2. *If we take $R = \varphi$ in (12), then we have*

$$\begin{aligned} &\left|\frac{b-a}{4}\left[R\left(\frac{a+x}{2}\right) + R(x) + R(a+b-x)\right] + \frac{b-a}{4}R\left(\frac{a+2b-x}{2}\right) - (E(x)-a)\right| \\ \leq\ & \frac{1}{6}\left\{\left\|\varphi'\right\|_{[a,x],\infty} \mathcal{G}_1(x) + \left\|\varphi'\right\|_{[x,b],\infty} \mathcal{H}_1(x)\right\} \end{aligned}$$

where $\mathcal{G}_1(x)$ and $\mathcal{H}_1(x)$ are the same as the above theorem.

Theorem 7.3. *Substitution of $F = \varphi$ in (15) gives the result*

$$\begin{aligned} &\left|\frac{1}{4}\left[F\left(\frac{a+x}{2}\right) + F(x) + F(a+b-x)\right] + \frac{1}{4}F\left(\frac{a+2b-x}{2}\right) - \frac{b-E(x)}{b-a}\right| \\ \leq\ & \frac{\left\|\varphi'\right\|_{[a,x],1} + \left\|\varphi'\right\|_{[x,b],1}}{2(b-a)}\left[\frac{(x-a)^2}{4} + \mathcal{K}_1(x)\right] + \frac{\left\|\varphi'\right\|_{[x,b],1}}{(b-a)}\left(\frac{a+b}{2} - x\right)^2 \end{aligned}$$

where $\mathcal{K}_1(x)$ is defined as in (14).

Proof. The proof of this theorem follows the same lines as proof of the previous theorem. □

Corollary 7.4. *If we take $R = \varphi$ in (15), then we possess*

$$\left|\frac{b-a}{4}\left[R\left(\frac{a+x}{2}\right)+R(x)+R(a+b-x)\right]+\frac{b-a}{4}R\left(\frac{a+2b-x}{2}\right)-(E(x)-a)\right|$$
$$\leq \frac{1}{2}\left[\frac{(x-a)^2}{4}+\mathcal{K}_1(x)\right]\left[\left\|\varphi'\right\|_{[a,x],1}+\left\|\varphi'\right\|_{[x,b],1}\right]+\left(\frac{a+b}{2}-x\right)^2\left\|\varphi'\right\|_{[x,b],1}$$

where $\mathcal{K}_1(x)$ is the same as the above theorem.

Theorem 7.5. *On taking $F = \varphi$ in (18), because of the identity (19), we get the inequality*

$$\left|\left[F\left(\frac{a+x}{2}\right)+F(x)+F(a+b-x)+F\left(\frac{a+2b-x}{2}\right)\right]-\frac{(x-a)^2}{2(b-a)}[\varphi(a)-\varphi(b)]-4\frac{b-E(x)}{(b-a)}\right|$$
$$-\frac{2}{b-a}\left[\frac{\varphi(a+b-x)+\varphi(\frac{a+2b-x}{2})}{(b-a)}-\frac{\varphi(\frac{a+x}{2})+\varphi(x)}{(b-a)}\right]\left[\frac{1}{2}\left(\frac{a+b}{2}-x\right)^2-2\left(\frac{3a+b}{4}-x\right)^2\right]$$
$$\leq \frac{(x-a)^2}{2(b-a)}\left[\bigvee_a^{\frac{ax}{2}}(\varphi)+\bigvee_{\frac{a+2b-x}{2}}^{b}(\varphi)\right]+\frac{\mathcal{K}_1(x)}{(b-a)}\left[\bigvee_{\frac{a+x}{2}}^{x}(\varphi)+\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(\varphi)\right]+\frac{2}{(b-a)}\left(\frac{a+b}{2}-x\right)^2\bigvee_x^{a+b-x}(\varphi)$$

where $\mathcal{K}_1(x)$ is defined as in (14).

8. Conclusion

In this work, we establish some integral inequalities for higher-order differentiable functions. Some applications of the inequalities developed in this paper are also given. In order to validate that their generalized behavior, we show the relation of our results with previously published ones. In future works, authors can obtain similar inequalities by using the different classes of functions.

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