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The Trascendence of Kac-Moody Algebras

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Abstract. With the main objective that it can be consulted by all researchers, mainly young people, interested in the study of the Kac-Moody algebras and their applications, this paper aims to present the past and present state-of-the-art developments and results on these algebras at various levels of analysis and fields of application in different disciplines. Indeed, lot of references in the literature on these algebras are analyzed and a study of the most relevant topics regarding them is shown.

Introduction

In 1968, Victor Kac and Robert Moody published in an independent way each papers about the later so-called *Kac-Moody algebras* (see [139] and [187], respectively). Therefore, these algebras have already turned fifty years. More than to show new results on them, this paper is written with the intention of summarizing the main results on this topic published in different journals, from that year 1968 to 2018 (up to December of this last year, there are 754 publications referenced in MathSciNet with the words *Kac-Moody algebras* in the title. The first of them is due to J. Morita in 1980 [190]), so that they can be known all together by a wide mathematical audience, particularly by young people who are interested in researching as many these algebras as on their applications to disciplines other than purely Mathematics, such as Physics, for example and in a lesser extent Engineering. In this way, the authors contribute to celebrate the fifty anniversary of the introduction of Kac-Moody algebras (note that a similar survey paper was published by R.V. Moody and A. Pianzola in 1996, with the occasion of the thirsty anniversary of these algebras [189]).

However, it is not understandable the study of these algebras if one does not firstly know the historical evolution of nilpotent Lie algebras. Note that Kac-Mody algebras were firstly introduced as a tool to classify nilpotent Lie algebras of maximal rank. Therefore, the first part of this paper is devoted to show how the evolution of nilpotent Lie algebras has occurred.

Moreover, we would like to comment that at the best of our knowledge, the problem of classifying all the different types of Lie algebras is still unsolved. As it is already well known, there exist three different types of Lie algebras: the semi-simple, the solvable and those which are neither semi-simple nor solvable. So, to know the classification of Lie algebras, in general, is equivalent to know the classification of each of these three types.

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However, by the Levi-Maltsev theorem, which groups together the results formulated in the first place by E.E. Levi [171], in 1905, and later, by A.I. Maltsev [178] (it can be also written Malcev), in 1945, any finite-dimensional Lie algebra over a field of characteristic zero can be expressed, according to the Levi-Maltsev Decomposition, as a semidirect sum of a semi-simple subalgebra (called the Levi factor) and its radical (its maximal solvable ideal). It reduces the task of classifying all Lie algebras to obtain the classification of semi-simple and of solvable Lie algebras.

With respect to the first problem, the classification of semi-simple Lie algebras, it is completely solved by the well-known Cartan theorem: any semi-simple complex or real Lie algebra can be decomposed into a direct sum of ideals which are simple subalgebras being mutually orthogonal with respect to the Cartan-Killing form. So, the problem of classifying semi-simple Lie algebras is then equivalent to that of classifying all non-isomorphic simple Lie algebras and since the classification of simple Lie algebras was already obtained by Killing, Cartan and others in the last decade of the 19th century, it can be admitted that the problem of the classification of semi-simple Lie algebras is, at present, totally solved.

With respect to the classification of solvable Lie algebras, several authors have tried to obtain it, since J. Dozias [83], who was the first researcher in dealing with that problem, to recent ones, such as W. de Graaf, for instance.

However, the previously cited Malcev had already reduced in 1945 the classification of complex solvable Lie algebras to the classification of one of its subsets, the nilpotent Lie algebras (see [178]). To do it, Malcev defined in the first place a particular type of algebra, which he called *splittable algebra*, whose structure is completely determined from its maximal nilpotent ideal and proved that an arbitrary solvable Lie algebra is contained in a uniquely minimal splittable algebra. The relation between an algebra and its splittings leads him to the construction of all solvable Lie algebras with a given splitting. So, in this way, the classification of all solvable Lie algebras is reduced to the classification of the nilpotent Lie algebras. However, at present, this last problem continues unsolved, in spite of the efforts made by many researchers to tackle it.

The structure of this survey article is as follows: Section 1 is devoted to briefly show the evolution of the classification of finite-nilpotent Lie algebras in general. Certain paragraphs dedicated to show the contributions which some authors have got on partial classifications concerning other particular properties of nilpotent Lie algebras are also included.

Section 2 constitutes a comprehensive survey on Kac-Moody algebras, which is the main goal of this paper. To this end, four subsections dealing with classifications, other concept related, physical applications of these algebras and books existing on them, respectively, are considered. Comments on certain open problems and possible steps to solve them in each case are also included. In any case, logical reasons of length and of progress imply that papers published in the current century are preferably commented instead of those appeared earlier.

Particularly, Subsection 2.4 is devoted to show the physical applications of Kac-Moody algebras. Indeed, since their introduction in 1967, in an independent way, by Victor Kac (at present in MIT, although working first in Moscow) [138] and Robert Moody (University of Alberta) [186], Kac-Moody algebras, which enlarge the paradigm of classical Lie algebras, have resulted in many and varied applications in the world of Physics and have been widely used as tools by numerous physicists in the theoretical development of their research. Particularly, the representation theory of a subclass of them, the *affine Kac-Moody algebras*, has also developed into a mature Mathematics.

Indeed, since 1980, these algebras had been used by physicists working in the areas of elementary particle theory, gravity, and two-dimensional phase transitions as an obvious framework from which to consider descriptions of non-perturbative solutions of gauge theory, vertex emission operators in string theory on compactified space, integrability in two-dimensional quantum field theory and conformal field theory, among many other physical theories (see [82] and the references therein). Particularly, at present, Kac-Moody algebras have an important application in Physics in the Sugawara construction [253], the Virasoro algebra and the coset construction, which are fundamental to string theory and the basic constructions of two-dimensional conformal field theory.

Moreover, and apart from their applications in Mathematics, where Kac-Moody algebras are also relevant to number theory and modular forms, and for their links with other different types of algebras (see [203] for instance), they have been recently shown to serve as duality symmetries of non-perturbative strings appearing to relate all superstrings to a single theory. Particularly, the infinite-dimensional Lie algebras and groups have been suggested as candidates for a unified symmetry of superstring theory.

Related to the concept of *symmetry*, this mathematical concept appears for the first time when the Greek discover the five regular solids, which are remarkably symmetrical. In the 19th century, this property was codified in the mathematical concept of a *group* invented by Galois and then that of a *continuous group* by Sophus Lie.

It is in Physics where the symmetries of a theory provide significant information about the general solution of the system. For instance, for a spherical conductor, limited to the case where U is $U(r, \theta)$, the solution can be expressed in terms of specific functions of r and the Legendre polynomials $P_n(\theta)$. In Quantum Field Theories, useful applications of symmetry include also the invariance imposed on the regularization of an approximation scheme, symmetry transformations among different specific solutions, and Noether constants of the motion used to label them. Moreover, *familiar symmetries*, such as rotations or translation or the electromagnetic gauge symmetry, reflect not only spacetime invariance, but also internal gauge invariance [82].

Then, related with this concept, Kac-Moody algebras constitute a certain class of symmetries. In contrast to the symmetries of the Standard Model and gravity, they are infinite and appear in the context of M-theory, an as of yet unknown theory that might both describe the Standard Model and gravity. At present, these algebras unify all the low-energy limits of M-theory, which are known as supergravities. Moreover, they contain information that corresponds exactly to all the known gauge deformations of these supergravities (see [201], for instance).

Therefore, all of these stunning results of mathematical Physics suggested that infinite-dimensional algebras as well as the finite-parameter symmetry algebras might be important for physical theories. And let us not forget that among infinite dimensional algebras, the Kac-Moody algebras play a relevant role.

Finally, an extensive list of references on this topic is included.

1. Nilpotent Lie algebras: classification

Many attempts related to the classification of nilpotent Lie algebra have been made, and lots of lists of algebras have been published with bigger or less fortune. The earliest one was given by an Engel's pupil, K.A. Umlauf (see [261]), who classified in 1891 all nilpotent Lie algebras up to dimension 6 over the field of complex numbers. Long after him, several authors dealt with these algebras trying to get its classification. So, it can be cited, for instance, Vranceanu, Morosov, Dixmier, Vergne, Safiullina, Favre, Magnin, Romdhani and Seeley.

Other contributions to this subject were made by M. Goze and J.M. Ancochea, who classified nilpotent Lie algebras of dimension 7 [8], G. Tsagas [3, 258], who classified in 1999 the real nilpotent Lie algebras of dimension 8, and by that same author, together with A. Kobotis and T. Koukouvinos [260], who dealt with those real nilpotent Lie algebras of dimension 9 which have a maximum abelian ideal of dimension 7. Since then, the classifications of both real or complex nilpotent Lie algebras of dimension greater than or equal to 8 have not been obtained yet, in spite of the frequent attempts made by several authors to get them. The problem is so unsolved at present and it has made that other subjects related with these algebras are being now considered before it. To that respect, it is quite significative the opinion of some experts, like Shalev and Zelmanov (Efim I. Zelmanov, URSS 1955, Fields Medal 1994), who think that it is already no possible to obtain in an explicit way the classification of nilpotent Lie algebras of bigger dimensions (see [232]). Another opinion in this sense is due to Kac, who affirms that *this is a wild problem and the general classification of nilpotent Lie algebras is believed to be impossible, if not very difficult* (see on page 4 of [142]).

In any case, it is really obvious that nowadays it is totally necessary to use the computer and a systematic approach to study the problem of classifying Lie algebras, in general, and that the more the computer is involved, the more systematic the methods have to be. So, in the last few years of the current century, several symbolic computational packages have been being specially constructed with the finality of being used to deal with Lie algebras, like MAGMA, GAP or ACTION, for instance. Apart from that, nilpotent Lie algebras over fields different from $\mathbb{R}$ or $\mathbb{C}$ are now being considered.

So, in the first years of the current century, C. Schneider adapted the p-group generation algorithm to classify small-dimensional nilpotent Lie algebras over small fields. Using the implementation of his algorithm, he listed in 2005 the nilpotent Lie algebras of dimension up to 9 over $\mathbb{F}_2$ and those of dimension up to 7 over $\mathbb{F}_3$ and $\mathbb{F}_5$, where $\mathbb{F}$ denotes either the real or the complex number field (see [228]).

By using the new packages, de Graaf has also got significative advances in the classification of nilpotent Lie algebras. Indeed, he got in 2005 the classification of 6-dimensional nilpotent Lie algebras over fields of characteristic different from 2. To do this, de Graaf used a essentially constructive method, based on a previous method by Skjelbred and Sund, along with another one based on Gröbner bases to find isomorphisms (see [114]). Moreover, this author has recently also obtained the classification of small-dimensional solvable Lie algebras by using a simple method which allowed him to classify three- and four-dimensional solvable Lie algebras, over fields of any characteristic (recall that nilpotent Lie algebras are a subset of solvable Lie algebras) [113].

C. Bartolone, A. di Bartolo and G. Falcone (see [16]) published a paper devoted to the classification of nilpotent Lie algebras over an arbitrary field with a 2-dimensional commutator ideal. They extended some results by M.A. Gauger ([104]) for the 2-step nilpotent case to the cases where the commutator is not abelian and where the field is not necessarily algebraically closed but has sufficiently many elements.

Finally, in 2012, S. Cicalò, W. de Graaf and C. Schneider [64] have presented a uniform classification of six-dimensional nilpotent Lie algebras over an arbitrary ground field $\mathbb{F}$ of any characteristic. They distinguished two cases depending on the characteristic was 2 or different from 2. This last case had been solved in a previous paper by de Graaf (see [114]) and the authors improved and simplified in [64] some of the arguments there. In characteristic 2 (see [64]), the authors agreed with the results obtained by M.P. Gong in his Ph.D. Thesis [111], correcting an inaccuracy there.

Apart from all facts just mentioned, one of the authors of this contribution dealt in his Ph.D. Thesis, in 1991, with filiform Lie algebras, that is, with that special class of n-dimensional nilpotent Lie algebras whose *characteristic sequence* (concept introduced by J.M. Ancochea and M. Goze in [7]) is $(n-1, 1)$. Let us recall that these algebras had been introduced by M. Vergne in the late 1960's (in her Ph. D. Thesis, later published in 1970 (see [266])), although before that, N. Blackburn [29] had already studied the analogous class of finite Lie p-groups and used the term *maximal class* to call them, which is also now used for Lie algebras. In fact, both terms *filiform* and *maximal class* are synonymous. In her Ph.D. Thesis ([265], Vergne showed that within the variety of nilpotent Lie multiplications on a fixed vector space, non-filiform ones can be relegated to small-dimensional components; thus, from some intuitive point of view, it is possible to consider that *quite a lot* nilpotent Lie algebras are filiform. Moreover, filiform Lie algebras are the most structured subset of nilpotent Lie algebras, which allows them to be studied and classified easier than the set of nilpotent Lie algebras.

With respect to the classifications of these algebras, several works following the seminal paper by Vergne can be cited. Indeed, Ancochea and Goze [7] already reached in 1988 the classification of complex filiform Lie algebras of dimension 8. However, this list of algebras was incomplete and the authors rectified it themselves in 1992 [9], almost at the same time as C. Seeley [230] did it in an independent way.

In 1991, J.R. Gómez classified those of dimension 9 in his Ph. D. Thesis, later published by F.J. Echarte and himself in [84], and L. Boza, F.J. Echarte and J. Núñez [39] those of dimension 10 in 1994. Later, J.R. Gómez, A. Jiménez and Y.B. Khakimdjanov [110], in 1996, gave the classification of these algebras on dimension 11 as well as some corrections of previous classifications. J.C. Benjumea *et al.* [21] also tried to tackle bigger dimensions by obtaining in 1996 the laws of the filiform Lie algebras of dimension 13 and 14.

Regarding the explicit lists, L. Boza, E.M. Fedriani and J. Núñez [40] got in 1998 the classification of complex filiform Lie algebras of dimension 12, and also showed in 2003 ([42]) an explicit classification of such algebras of dimension 11, in a different way as the one used in [110]. Although [41] supplied a method valid for every dimension over the complex field, the involved computations are too hard to be nowadays tacked successfully.

Moreover, some subsets of filiform Lie algebras have also been studied. In this sense, as a recent illustration, L.R. Bosko computed in [37] all the filiform Lie algebras (called in the paper as of maximal class) for the Schur multiplier $t(\mathfrak{g}) \leq 16$, in both the complex and real fields. This type of algebras does not exist for dimension 14, 15 and 16. Remember that in mathematical group theory, the Schur multiplier (or Schur multiplicator) is the second homology group $H_2(G)$ of a group G. And notice that the Schur multiplier of the Lie algebra has also been used in [199] to classify nilpotent Lie algebras in general.

So, as a summary of this section, it can be affirmed that up to isomorphism the classification of nilpotent Lie algebras of dimension less than or equal to 5 is well known and does not depend on the ground field, in the sense that the tables of structure constants of the algebras can be chosen with integer entries. The classification of six-dimensional nilpotent Lie algebras goes back to the Ph.D. Thesis of K.A. Umlauf and then, Morozov published a classification of such Lie algebras over fields of characteristic 0. Recently several classifications have appeared, over various ground fields, e.g., over algebraically closed fields, over various finite fields, and over fields of characteristic different from 2. However, in spite of there is indeed lot of works devoted to classification of nilpotent Lie algebras of low dimension (those cited above and many more), with numerous mistakes and omissions. Even worse, all they are using different nomenclature and invariants to classify the algebras, and it is a non-trivial task to compare different lists. Luckily, W. de Graaf undertook a painstaking task to make an order out of this somewhat messy situation in [114]. Even better, he provided an algorithm for identifying any given nilpotent Lie algebra with one in his list, and the corresponding code is available as a part of GAP package. He built on earlier work of Skjelbred-Sund cited above and his own method of identification of Lie algebras by means of Groebner bases. In this way, he might have detected the previously mentioned mistake in the Thesis of Gong [111], in which he classified all algebras up to dimension 7 over algebraically closed fields of any characteristics except 2, and also over reals and he checked and revised previous classifications. Indeed, de Graaf claims that Gong lost a one-parameter family of 6-dimensional algebras. If this is true, then his results on 7-dimensional algebras may be also incomplete, unfortunately.

To conclude this section we would also like to cite some authors who have got partial classifications concerning other particular properties of nilpotent Lie algebras, like J. Scheuneman in 1967 [227], M.A. Gauger in 1973 [104], P. Revoy in 1980 [217] and D. Fernández-Ternero and J. Núñez, in 2001, who classified two-step nilpotent Lie algebras or those dealing with characteristically nilpotent Lie algebras [93]. These last algebras were introduced in the first place by J. Dixmier and W.G. Lister in 1957 [78], being G. Leger and S. Togo who first published a paper about them in 1959 (see [168]). Later, F.J. Castro and J. Núñez dealt with these algebras in the particular case of being also filiform (see [53, 54], for instance). A very interesting monograph on these characteristically nilpotent Lie algebras, in which its historical evolution is commented, was published by Ancochea and Campoamor (see [5]).

In that same line, different subclasses of nilpotent Lie algebras have also been classified. For instance, recently, B. Ren and L.S. Zhu gave in 2011 a complete classification of 2-step nilpotent Lie algebras N (over the field of complex numbers $\mathbb{C}$) of dimension 8 which have a 2-dimensional center. They showed a description of five algebras with those characteristic which are mutually non-isomorphic, and showed that they are the only ones that may occur. However, as themselves indicate, the descriptions of those algebras were too complicated to be presented in their paper (see [215]).

In 2014, A.Kh. Khudoyberdyev and Z.Kh. Shermatova classified five-dimensional solvable Leibniz algebras with a four-dimensional non-Lie nilradical [156] and S. Benayadi and A. Elduque, quadratic Lie algebras of low dimension [20].

In 2015, J.M. Ancochea and R. Campoamor-Stursberg [6] classified solvable real rigid Lie algebras with a nilradical of dimension $n = 6$, and a year later, in 2016, several authors published on the classifications of Lie algebras. Indeed, R.M. Navarro classified filiform Lie algebras of order 3, [195], Falcón, Falcón, Núñez, Pacheco and Villar the filiform Lie algebras up to dimension 7 over finite fields [86], and I. Darijani and H. Usefi [74] the 5-dimensional p-nilpotent restricted Lie algebras over perfect fields.

In he same year, 2016, Schneider and Usefi classified p-nilpotent restricted Lie algebras of dimension at most 4 [229] and L. Cagliero and F. Szechtman the linked indecomposable modules of a family of solvable Lie algebras over an arbitrary field of characteristic 0 [48]. Besides, in [73], H. Darabi, F. Saeedi and M. Eshrati dealt with capable n-Lie algebras and the classification of nilpotent n-Lie algebras with $s(A) = 3$ and A.S. Hegazi, H. Abdelwahab and A.J. Calderon Martin classified the n-dimensional non-Lie Malcev algebras with $(n-4)$-dimensional annihilator [121].

In 2017, M. Ceballos, J. Núñez and A.F. Tenorio obtained new results on the classification of filiform Lie algebras [55], Ren and Zhu classified 2-step nilpotent Lie algebras of dimension 9 with 2-dimensional center [216] and K. Furutani and I. Markina, gave a complete classification of pseudo H-type Lie algebras [100].

Finally, in 2018, D.S. Shirokov classified Lie algebras of specific type in complexified Clifford algebras [238] and E. Khamseh and S.A. Niri gave in [153] the classification of pair of nilpotent Lie algebras by their Schur multipliers.

2. Kac-Moody algebras

In this section, an overview of the evolution of the study of different aspects concerning Kac-Moody algebras is considered.

2.1. Kac-Moody algebras: General aspects and classifications

As it was mentioned in the introduction, Victor Kac and Robert Moody published in 1968 in an independent way each papers about the later so-called *Kac-Moody algebras* (see [139] and [187], respectively). It produced a different way of research in the study of the classification of nilpotent Lie algebras due to these Kac-Moody algebras are of infinite dimension, in general, and constitute a generalization of the semi-simple Lie algebras.

D.H. Peterson and V.G. Kac (see [207]) gave in 1983 a great step forward to advance in the completion of the classification of Kac-Moody algebras with a demonstration of the isomorphism theorem which claims that two isomorphic Kac-Moody algebras must have, up to permutations of the indices, the same generalized Cartan matrix, or equivalently, their root systems isomorphic to each other.

However, a year earlier, in 1982, L. Santharoubane, by using the *system of weights* previously introduced by F. Bratzlavsky [43] in 1972 and G. Favre [87] in 1973, had canonically associated a Kac-Moody algebra $\mathfrak{g}(A)$ to each finite nilpotent Lie algebra of maximal rank $\mathfrak{L}$ (it is said then that $\mathfrak{L}$ is of the type A) and used this fact to reduce the problem of classifying these nilpotent Lie algebras to obtain for each type A some ideals of the positive part of the Kac-Moody algebra $\mathfrak{g}(A)$ up to the action of the automorphism group of the Dynkin diagram of A, $\widetilde{\mathfrak{S}}_\ell(A)$ (see [222]).

There exist three families of generalized Cartan matrices: finite, affine and indefinite. Various authors, such as Tsagas, Agrafiotou, Favre, Santharoubane and the authors of this contribution have studied the nilpotent Lie algebras of maximal rank and of type A, when A is of finite or affine type.

Indeed, Santharoubane proved that the problem of finding all nilpotent Lie algebras of maximal rank and of type A is equivalent to the concrete problem of finding some ideals of the positive part of the Kac-Moody algebra $\mathfrak{g}(A)$ up to the action of $\widetilde{\mathfrak{S}}_\ell(A)$.

So, by using this technique, the previously mentioned authors have studied the nilpotent Lie algebras of maximal rank and of type A, when A is of finite or affine type (they published their papers between 1996 and the first decade of the current century).

Regarding the first family of Kac-Moody algebras, i.e. the simple Lie algebras, the nilpotent Lie algebras of maximal rank corresponding to the types A_l, B_l, C_l and D_l, with $l \geq 1$, $l \geq 2$, $l \geq 3$, and $l \geq 4$, respectively, were dealt by G. Favre and L.J. Santharoubane in [88]. Later, other authors have made research on these algebras for other finite types. So, the types E_6, E_7 and E_8 were studied by Agrafiotou and Tsagas in [3] and the type F_4 by G. Favre and G. Tsagas [89]. There also exist several results on this family of algebras, in the case of being defined over algebraically closed fields of non-null characteristic. The main contributions are cited at the end of this subsection.

The second family of Kac-Moody algebras are the affine Lie algebras. Santharoubane studied in 1982 (see [222]) the affine types $A_1^{(1)}$ and $A_2^{(2)}$. He proved that there are exactly (up to isomorphism), 3 infinite series and 10 infinite series, respectively, of nilpotent Lie algebras of maximal rank such that $A_1^{(1)}$ and $A_2^{(2)}$ are associated Cartan matrices.

S. Kanagavel considered in [143] the non-twisted affine type of rank 3 and of types $A_2^{(1)}$, $B_2^{(1)}$ and $G_2^{(1)}$. He proved that there are exactly $n(X)$ infinite series and a continuous family of nilpotent Lie algebras of maximal rank and of type X, where $n(A_2^{(1)}) = 9$, $n(B_2^{(1)}) = 18$ and $n(G_2^{(1)}) = 39$.

Next, the types $D_4^{(1)}$ and $D_5^{(1)}$ were studied by X. Agrafiotou in [1] and [2], respectively. This author found that there are exactly 81 infinite series (up to isomorphism) with discrete parameters and 8 infinite series with continuous parameters of nilpotent Lie algebras of maximal rank and of type $D_4^{(1)}$ and 217 infinite series (up to isomorphism) with discrete parameters and 18 infinite series with continuous parameters of these algebras of type $D_5^{(1)}$.

Later, the authors of this contribution studied nilpotent Lie algebras of maximal rank of non-twisted and twisted affine types, in 2001 and 2002. With respect the non-twisted affine types $F_4^{(1)}$ and $E_6^{(1)}$ (see [94] and [95]) they proved that up to isomorphism there are 1095 infinite series with discrete parameters and 28 infinite series with continuous parameters of nilpotent Lie algebras of maximal rank and of type $F_4^{(1)}$ and 2087 infinite series with discrete parameters and 126 infinite series with continuous parameters of nilpotent Lie algebras of maximal rank and of type $E_6^{(1)}$.

Later, D. Fernández-Ternero studied the nilpotent Lie algebras associated with the twisted affine Kac-Moody algebra. She found that there are 88 infinite series (up to isomorphism) with discrete parameters and 1 infinite series with continuous parameter of nilpotent Lie algebras of Kac-Moody type $D_4^{(3)}$ (see [92]).

In 2006, Santharoubane studied in [223] the nilpotent Lie algebras of maximal rank and general non-twisted affine type $X_\ell^{(1)}$ by the previous technique in addition to the study of grassmanians. For rank 2 he showed the explicit list of the series with discrete and continuous parameters of nilpotent Lie algebras of Kac-Moody types $A_1^{(1)}$, $A_2^{(1)}$, $C_2^{(1)}$ and $G_2^{(1)}$. Fernández-Ternero and Núñez also considered a fixed (and low) nilpotent index instead of a fixed Kac-Moody type. So, with respect to the classification of *metabelian Lie algebras* (nilpotent Lie algebras with nilpotent index 2), the main results obtained can be checked in [93].

Regarding these classifications, authors would like to point out certain reflections with respect to the problems still unsolved. So, we consider that in view of the research on nilpotent Lie algebras of maximal rank previously exposed, the following natural steps should be to complete the study of the twisted affine case and to begin with the indefinite one.

Regarding the twisted affine type, as it has been commented before, there exist some works dealing with certain Cartan matrices: Santharoubane studied in [222] the algebras of Kac-Moody type $A_2^{(2)}$ and Fernández-Ternero studied in [92] those of Kac-Moody type $D_4^{(3)}$. Although in both cases the considered technique is the same as the one used for the non-twisted type, it occurs that in the twisted affine case the complexity is greater when the roots system of the Kac-Moody algebras are dealt. We think that to obtain the ideals of Δ_+ included in Δ_{++} could be useful to consider the digraph associated with $\mathfrak{n}_+$ in a similar way as it was done in [75] for the non-twisted case.

With respect to the indefinite Kac-Moody algebras, they are scarcely known nowadays. It could be a good idea to begin this study by continuing the study of the hyperbolic type since there are recent studies about the roots system and other aspects of the Kac-Moody algebras of this type (see [248], for instance).

Apart from all of this information just mentioned on classifications of Kac-Moody algebras, it is convenient to note that there exist in the literature many other classifications of other concepts related to Kac-Moody algebras. Indeed, in 1987 (later published in 1988), J.B. Zuber gave in [280] the classification of modular invariants made of characters of the Virasoro or Kac-Moody algebras. He also showed their physical implications for the classification of universality classes of 2D critical phenomena.

In 1988, F. Levstein [172] classified the conjugacy classes of involutive automorphisms of affine Kac-Moody Lie algebra. Fix a triangular decomposition of an affine Lie algebra, he considered two types of automorphisms and he gave a complete non-redundant list of involutive automorphisms of the first kind, together with the corresponding fixed point sets. He next used a result of Kac and Peterson to describe the involutions of the second kind. However, he was unable to write down the fixed point sets in this case and so could not check if the list was redundant. At this respect, J. Bausch [17] studied and classified automorphisms of finite order and the first kind of affine Kac-Moody algebras in 1989. A year later, V.K. Dobrev and A.H. Ganchev [79] gave the classification of modular invariant partition functions for the twisted $N = 2$ superconformal algebra, twisted $SU(2)$ Kac-Moody algebra and D_{2K} parafermions.

In 1990, K.C. Pati and K.C. Tripathy [204] dealt with super Kac-Moody algebras and extended the simple root system of a super Lie algebra of rank r by adding the negative of the highest root of this super Lie algebra. Then an $(r + 1) \times (r + 1)$ matrix can be constructed by the extended simple root system. It was called the *extended Cartan matrix*, and the corresponding Dynkin diagram was called the *extended Dynkin diagram*. These extended simple root systems and extended Dynkin diagrams yield the untwisted super Kac-Moody algebras. They obtained twisted super Kac-Moody algebras from the outer automorphisms of super Lie algebras and gave the Chevalley basis and canonical basis. The central extension problem was also discussed.

Bennett in 1993 [26] got the the complete classification of the indecomposable symmetrizable generalized Cartan matrices M of indefinite type, admitting special roots and also showed the determination of the set of special roots in such cases. That same year and in the next one, J. Fröhlich and E. Thiran (see [98]) studied integral quadratic forms, Kac-Moody algebras, and fractional quantum Hall effect and they gave an $ADE-O$ classification (remember that the ADE classification scheme is encountered in many areas of mathematics, most notably in the study of Lie algebras). Also in 1994, D. Casperson gave a complete characterization of those Kac-Moody algebras which have the property that all of their imaginary roots are strictly imaginary (recall that an imaginary root τ is defined to be a strictly imaginary root if for every real root α either $\tau + \alpha$ or $\tau - \alpha$ is a root). After deriving a simple test for whether or not a given imaginary root is strictly imaginary, Casperson gave a complete classification of the strictly imaginary roots of 2×2 Kac-Moody algebras and he proved that a minimal non-strictly imaginary root must necessarily be a minimal imaginary root (see [51]).

In 1995, V. Back-Valente, N. Bardy-Panse, H. Ben Messaoud and G. Rousseau [12] classified almost split forms of Kac-Moody algebras, thus generalizing results by J. Tits about the semi-simple finite-dimensional case. In particular, their paper contained a complete list of all the real split forms of affine Kac-Moody algebras, and of their relative root systems.

As we have previously mentioned, the completion of the classification of Kac-Moody algebras was done by Peterson and Kac in 1983 [207]. However, the corresponding classification for Kac-Moody groups remains unsolved before 2000. That year, Y. Chen [59], used one of his previous works on isomorphisms of adjoint Chevalley groups over integral domains (see [58]) to carry it out for non-twisted affine Kac-Moody groups over algebraically closed fields of characteristic 0 and for the corresponding loop groups (which are closely related): two groups of this kind are (abstractly) isomorphic to each other if and only if the root systems and the fields are isomorphic to each other.

V.A. Gritsenko and V.V. Nikulin dealt with the classification of Lorentzian Kac-Moody algebras in 2002 [116]. These algebras are the next important class of (generalized) Kac-Moody algebras after semi-simple and affine algebras and play the decisive role in the solution of the *Moonshine Conjecture* on representations of the Monster, due to R. E. Borcherds. Starting from Borcherds' example, they defined the class of Lorentzian algebras and finally proved that there are 29 such algebras.

Ben Messaoud and Rousseau gave in 2003 (see [18, 19]) the classification of almost compact real forms of affine Kac-Moody algebras, and this same year, V. Bouchard and H. Skarke dealt with the link between affine Kac-Moody algebras and tops [38]. By a *top* the authors meant a 3-dimensional convex lattice polytope with one facet containing the origin and the other facets at the distance one from the origin. Such polytopes sometimes appear as halves of the reflexive polytopes that have a hyperplane section which is also a reflexive polytope. To every top the authors associated an affine Kac-Moody algebra by relating it via a generic hypersurface in a toric variety kind of construction to an elliptic fibration, as studied by K. Kodaira in 1966. All possible tops were then classified, for each of the 16 possible base polygons. It turns out that there are finitely many series of tops. Besides listing them, the authors calculated the corresponding affine Kac-Moody algebras and finally discussed the string theory motivation and connections.

In 2006, N.R. Scheithauer worked on the classification of *automorphic products* and generalized Kac-Moody algebras [226] and a year later, C. Hoyt and V. Serganova classified finite-growth general Kac-Moody superalgebras [126]. As it was known, a classical result by V. G. Kac established that a contragredient (sic) Lie algebra of finite growth corresponding to an indecomposable Cartan matrix is either simple finite-dimensional or affine (finite growth in this context means that the dimension of the graded components, in the natural grading, depends polynomially on the degree). Then, if A is a Cartan matrix and $g(A)$ the corresponding contragredient Lie superalgebra (it should be noted that, unlike in the case of Kac-Moody Lie algebras, the Cartan matrix of a Lie superalgebra is not unique), both authors classified algebras $g(A)$ of finite growth without imposing any conditions on A. Their main result stated that if A is indecomposable and has no zero rows then A either is symmetrizable or is of one of the following types: $D(2,1,0), \hat{D}(2,1,0), S(1,2,0), S(1,2,a)$, $a \neq \mathbb{Z}$ or $q(n)^{(2)}$. The authors also considered the case when A has a row of zeroes. In this case the algebra $g(A)$ is not simple and is obtained by extending a finite-dimensional algebra by a Heisenberg algebra. They emphasized that a crucial role in the classification is played by the odd reflections.

Finally, it is also convenient to note that as it was well-known, the theory of a generalized Kac-Moody algebra associated to a generalized Cartan matrix was initiated by R. E. Borcherds in 1988, as part of his proof of the Conway-Norton conjectures on the representation theory of the Monster simple group (see [32]). In 2007, N. Sthanumoorthy and P.L. Lilly [246] completed the classifications of this type of algebras possessing special imaginary roots and strictly imaginary property. To this end, he used the classifications of Kac-Moody algebras which possess special imaginary roots and satisfy the (so-called) *SIM property*, given by C. Bennett [26] and D. Casperson [51] to give a complete classifications in the generalized Kac-Moody case by discussing whether or not the matrix A is an extension of a generalized Cartan matrix.

In the years 2013, 14 and 15, there are lots of references dealing with generalized Kac-Moody algebras in the main data base. Some of the most significative are the following:

In 2013, S.J. Kang, M. Kashiwara and S.J. Oh studied categorification of highest weight modules over quantum generalized Kac-Moody algebras [144] and M. Ishii dealt with a path model for representations of generalized Kac-Moody algebras [128, 129, 144].

A year later, G. Höhn and N. R. Scheithauer [123] introduced a generalized Kac-Moody algebra of rank 14 and finally, in 2015, H. Gan and Y. Tan dealt with a class of generalized Kac-Moody algebras arising from 2-fold affinizations [103], Ishii [129] studied Demazure character formulas for generalized Kac-Moody algebras and J. Chen and Z. Zhao proposed a fundamental representation of quantum generalized Kac-Moody algebras with one imaginary simple root [57].

Continuing with this subsection and as we previously indicated, we show at this point some results obtained on algebras defined over algebraically closed fields of non-null characteristic. Indeed, the research on finite-dimensional simple Lie algebras over algebraically closed fields of characteristic $p > 0$ was begun by N. Jacobson [131, 132] and H. Zassenhaus [278] in the 30s. In these papers, Jacobson noted certain identities which connected addition, scalar multiplication, commutation and pth powers in an arbitrary associative algebra of characteristic $p \neq 0$. These lead naturally to the definition of a class of abstract algebras called *restricted* Lie algebras which in many respects bear a closer relation to Lie algebras of characteristic 0 than ordinary Lie algebras of characteristic p.

Later, in 1966, A.I. Kostrikin and I.R. Safarevic [159] conjectured that each finite-dimensional restricted simple Lie algebra over an algebraically closed field of characteristic $p > 5$ is either of classical or Cartan type and they made substantial progress towards establishing it. The conjecture was proved for characteristic $p > 7$ by R.E. Block and R.L. Wilson in 1988 [31]. But it was necessary to reformulate the statement due to the fact that G.M. Melikyan [183] discovered in $p = 5$ a new class of simple Lie algebras neither of classical nor of Cartan type. During the last half-century, many authors contributed to different aspects of this classification. Indeed, different recognition results were obtained first by Wilson [270] for filtered Lie algebras and by Kostrikin and Safarevic [160] and later by V. G. Kac [140] for graded Lie algebras.

Other important classification results about subfamilies of finite simple Lie algebras were obtained by Kostrikin [158] when the dimension of the algebra is less than or equal to the characteristic of the field and by B.Y. Veisfeiler and V.G. Kac [263] for $p > 7$ and contragredient Lie algebras. These two last authors gave an uncomplete classification result. Other authors, like I. Kaplansky [152], R.L. Wilson [271], G. Benkart and J.M. Osborn [25], and A. Premet [210] studied finite simple Lie algebras containing a subalgebra of Cartan type with additional properties. Concretely, Premet proved that if L is a finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$ of absolute toral rank 2 and T is a 2-dimensional torus in the semisimple p-envelope of L, then L is either classical or a Block algebra or contains sandwich elements which are homogeneous with respect to T or a conjugate to T. He also gave several dimension estimates.

In the last twenty eight years this last author and H. Strade obtained great advances in the resolution of the problem and finally proved the Classification Theorem for $p > 3$, that is, they showed that any finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$ is of classical, Cartan type or Melikyan type. Using heavily an important intermediate result of the paper of R.E. Block and R.L. Wilson and many ideas for solutions to problems arising in particular cases, Strade completed the proof in the case $p > 7$ in 1998 [249]. The proof of the Classification Theorem in the case $p = 5$ was given by Premet and Strade [211]. Some papers about characteristic two or three, for mentioning some, are those by S.M. Skryabin [242], who showed an example not found in the classification by B.Y. Veisfeiler and V.G. Kac, G. Brown [45], who dealt with families of simple Lie algebras of characteristic two, and G. Benkart, A.I. Kostrikin and M.I. Kuznetsov [24]. In this last paper, the authors determined all finite-dimensional simple graded Lie algebras L over an algebraically closed field of characteristic 3 with a classical reductive component L_0 and a nonrestricted representation on L_{-1}.

At this respect, Strade has been collecting the most important results published on this topic [250–252].

2.2. *Kac-Moody algebras: other related topics*

There are more than 175 references in the literature which have the words *Kac-Moody algebras* during the period between 1980 and 1990. The first of them is a paper by L. Dolan, entitled *Kac-Moody algebra is hidden symmetry of chiral models*, published in Physical Review Letters in 1981 (see [80]). In the paper, according to the recension by D.B. Creamer, the author studies the relationship between the infinite set of non-local currents in two-dimensional chiral models and Kac-Moody algebras (affine Lie algebras).

In order to construct groups naturally associated with a Kac-Moody Lie algebra L, J. Tits had emphasized the use of amalgamated products of groups coming from certain subalgebras of L (see [257]). For example, the sum L^+ of positive root spaces should yield a prounipotent group; roughly speaking, one exponentiates and then passes to the closure in the completion of the universal enveloping algebra relative to the natural grading. But when L is permitted to have a radical (maximal graded ideal meeting the Cartan subalgebra trivially), the procedure is more subtle. O. Mathieu, at this respect, takes advantage of this fact to characterize in 1984 the unique largest prounipotent group which meets the requirements of Tits's construction (see [180]).

Next, some of the most relevant references which one can find in the literature over Kac-Moody algebras among the more than 266 which exist between 1990 and 2000 are the following. In 1990, S. Kumar [162] dealt with the Bernstein-Gelfand-Gelfand resolution for arbitrary Kac-Moody algebras and E. Neher and D. Nguyen studied finite gradings of these algebras (see [197]). In 1991, N. Andruskiewitsch introduced the representation theory of inner almost compact forms of Kac-Moody algebras [10] and Borcherds considered central extensions of generalized Kac-Moody algebras [33]. S. Naito dealt in 1992 and 1993 with the Kostant's formula and homology vanishing theorems for generalized Kac-Moody algebras (see [192, 193], respectively).

In 1994, G. Benkart, S.K. Kang and K.C. Misra studied indefinite Kac-Moody algebras of special linear type [22] and the same authors published two years later a paper on weight multiplicity polynomials for affine Kac-Moody algebras of type $A_r^{(1)}$ (see [23]). Borcherds gave a characterization of generalized Kac-Moody algebras in 1995 [35] and Mathieu, in 1996 [182] dealt with some modular representations of affine Kac-Moody algebras at the critical level. Shalev studied one year later the relation among the subgroup structure, the fractal dimension and Kac-Moody algebras, in a paper later published in 1998 [231] and M.V. Zaitsev in 1997 dealt with varieties of affine Kac-Moody algebras [277]. Y. Barnea, A. Shalev and E. Zelmanov considered graded subalgebras of affine Kac-Moody algebras in 1998 [14] and to finish the century, E. Poroshenko [209] studied Gröbner-Shirshov bases for Kac-Moody algebras $A_n^{(1)}$ and $B_n^{(1)}$ and two papers on hyperbolic Kac-Moody algebras appeared in 1999, by D.V. Lagno and by Tsagas and Agrafiotou (see [165, 259], respectively).

With the initiation of the current century, papers on Kac-Moody algebras deeply increased. Indeed, starting from 2000, the studies on these algebras were very extended. That year, Q. Zhang and Z.X. Xia studied (see [279]) special imaginary roots of a class of generalized Kac-Moody algebras and Poroshenko (see [209]) considered the Gröbner-Shirshov bases for Kac-Moody algebras $A_n^{(1)}$ and $B_n^{(1)}$. A year later, D.Y. Wang [268] made progress in the study of the overalgebras of Borel subalgebras in Kac-Moody algebras, and Nikulin, who is an author well known for his study of Lorentzian Kac-Moody algebras [198], presented a summary of the current theory. His starting point was a list of five abstract data that served to define a class of Lorentzian generalized Kac-Moody superalgebras. There, generalized meant in the Borcherds sense, and superalgebras were $\mathbb{Z}_2$-graded. His main concern was when the abstract data define a finite or essentially finite (infinite but well-controlled) collection of algebras. He was particularly interested in determining automorphic forms for the data and constructing their product formulas. In this same year, U. Ray [214] reviewed the theory of generalized Kac-Moody algebras focusing on Borcherds proof of the celebrated *Moonshine conjecture* relating representations of the Monster group and elliptic modular functions.

In 2002, the previously cited Nikulin, joint Gritsenko (see [116]), published a paper devoted to the classification of Lorentzian Kac-Moody algebras, which are, as it is known, the next important class of (generalized) Kac-Moody algebras after semi-simple and affine algebras. These algebras play the decisive role in the solution of the *Moonshine Conjecture* on representations of certain type of algebras (a further study on these topics is by J.H. Yang [276] in 2006, where the author dealt with Kac-Moody algebras, the monstrous moonshine, Jacobi forms and infinite product). M. Rausch de Traubenberg and M.K. Slupinski [213] studied the problem of representing any Kac-Moody Lie algebra $g(A)$ associated with an $r \times r$ indecomposable generalized Cartan matrix A as vector fields on the torus $\mathbb{C}^{*r}$. They constructed certain discrete families of representations for these algebras, generalizing the well-known discrete families of representations of $sl(2, \mathbb{C})$ on $\mathbb{C}^*$.

Later, many other papers on Kac-Moody algebras appeared. In one of them, in 2003, Scheithauer [225] attempted to generalize some of this machinery used by R. E. Borcherds in the proof of his celebrated *Moonshine conjecture* (see [34]), which involved constructing the Monster algebra upon which the Monster group acts, calculating its twisted denominator identities, and using these identities to verify that the Thompson series are in fact haupt-modules. As a consequence, these twisted denominator identities could also be thought of as modular forms in two variables. To this end, Scheithauer considered the natural action of certain automorphism groups of the Leech lattice, on the fake Monster algebra and he conjectured that the corresponding twisted denominator identities are in fact automorphic forms of singular weight, and he was able to prove this conjecture for those elements with square-free level and with non-trivial fixed-point lattices. Among many computations and observations, the author noted that those elements of square-free order corresponding to the Mathieu group corresponded to a family of nice generalized Kac-Moody algebras. This same year, C. Lu and H. Xu [173] got a sufficient condition for pairing problem of generators in symmetrizable Kac-Moody algebras.

In 2004, E.M. Souidi and M. Zaoui [244] dealt with the classical problem of the classification of maximal Lie subalgebras of a finite-dimensional Lie algebra, which had been previously treated by many authors. They investigated when a subalgebra is maximal, or when it is maximal among the reductive Lie subalgebras, recovering some classical results in an alternate way. In this same year, A.J. Feingold and H. Nicolai [90] studied (isomorphism classes of) subalgebras of Lorentzian Kac-Moody algebras. In the case of the unique hyperbolic Kac-Moody algebra of rank three that contains both finite and affine Kac-Moody subalgebras, they found all simply laced rank two hyperbolic Lie algebras and an infinite series of indefinite Kac-Moody subalgebras of rank 3. Later, in 2005, T. Damour studied the structure of the general, inhomogeneous solution of (bosonic) Einstein-matter systems in the vicinity of a cosmological singularity. He reviewed the proof (based on ideas of Belinskifii-Khalatnikov-Lifshits and technically simplified by the use of the ArnowittDeser-Misner Hamiltonian formalism) that the asymptotic behavior, as one approaches the singularity, of the general solution is describable, at each (generic) spatial point, as a billiard motion in an auxiliary Lorentzian space (see [70]).

In 2006, Z. Wu [273] associated to a generalized Kac-Moody algebra a collection of associative algebras indexed by the positive integers. These algebras were modeled on the quantized enveloping algebra of a generalized Kac-Moody algebra. They had a Hopf subalgebra isomorphic to the quantized enveloping algebra, and a particular case had already been studied by J.H. Yang for semi-simple Lie algebras a year earlier. P. Fiebig [96] generalized an important result about the structure of category O for a finite-dimensional semi-simple Lie algebra, which was Soergels combinatorial description of the projective modules in O. He studied the case of a symmetrizable Kac-Moody Lie algebra outside the critical hyperplanes. In particular, he showed that the structure of a block of category O in that case depended only on the corresponding integral Weyl group and its action on the parameters of Verma modules. That same year, K. Jeong, S.J. Kang and M. Kashiwara (see [134]) developed a crystal basis theory for quantum generalized Kac-Moody algebras.

In 2008, S. Gaussent and G. Rousseau gave in [105] the definition of a kind of building for a symmetrizable Kac-Moody group over a field endowed with a discrete valuation and with a residue field containing the complex numbers, which they call *a hovel*, due to its pathological behaviour. This concept allowed them to generalize some previous results by themselves, separately, and joint Littelmann, and by using that concept they got good generalizations of Gaussent-Littelmann's results and obtained a new result which is analogue of some previous ones in the semi-simple case.

In 2009, Z.X. Chen and Y.N. Lin [60] dealt with tubular algebras by constructing quotient algebras of complex degenerate composition Lie algebras by some ideals, starting from Hall algebras of tubular algebras, and they proved that the quotient algebras are isomorphic to the corresponding affine Kac-Moody algebras. Moreover, they showed that the Lie algebra generated by A-modules with a real root coincides with the degenerate composition Lie algebra generated by simple A-modules.

Taking into consideration that Borcherds had introduced in 1988 the notion of *generalized Kac-Moody algebras*, Sthanumoorthy and Lilly proved in different contributions some properties of these algebras. Indeed, they extended the notion of purely imaginary roots of Kac-Moody algebras to them and classified all of generalized Kac-Moody algebras having purely imaginary property. They also found among these last algebras those whose purely imaginary roots are strictly imaginary, giving also complete classifications of generalized Kac-Moody algebras which have special imaginary roots and strictly imaginary property. Later, in 2007 (see [247]), both authors continued with this research by determining explicitly the root multiplicities of all roots of generalized Kac-Moody algebras associated with Borcherds-Cartan matrices (of order 3) which are extensions of finite A_3, affine $A_2^{(1)}$ and hyperbolic HA_1, and they also computed the root multiplicities of some generalized Kac-Moody algebras with more than one imaginary simple root.

Starting from 2010 lots of papers on these algebras have been published. Among them, the most relevant are now indicated.

A. Wassermann published in 2010 the notes which he used for a Part III course given in the University of Cambridge in autumn 1998 (see [269]). They contain an exposition of the representation theory of the Lie algebras of compact matrix groups, affine Kac-Moody algebras and the Virasoro algebra from a unitary point of view. The treatment uses many of the methods of conformal field theory, in particular the Goddard-Kent-Olive construction and the Kazami-Suzuki supercharge operator, a generalization of the Dirac operator. The proof of the Weyl character formula is taken from unpublished notes of P. Goddard.

The supersymmetric proof of the Kac character formula for affine Kac-Moody algebras was also found independently at roughly the same time by Greg Landweber in his Harvard Ph.D. dissertation. One of the main novelties of this approach is a very rapid proof of the Feigin-Fuchs character formula for the discrete series representations of the Virasoro algebra.

The same year, T. Bliem [30] introduced the notion of a chopped and sliced cone in combinatorial geometry and proved two structure theorems for the number of integral points in the individual slices of such a cone. He observed that this notion applied to weight multiplicities of Kac-Moody algebras and Littlewood-Richardson coefficients of semi-simple Lie algebras, where he obtained the corresponding results.

H. Glöckner, R. Gramlich and T. Hartnick [108] studied final group topologies and their relations to compactness properties. In particular, they were interested in situations where a colimit or direct limit is locally compact, a k_ω-space, or locally k_ω. They also showed, as a first application, that unitary forms of complex Kac-Moody groups can be described as the colimit of an amalgam of subgroups (in the category of Hausdorff topological groups, and the category of k_ω-groups). Their second application concerned Pontryagin duality theory for the classes of almost metrizable topological abelian groups, resp., locally k_ω topological abelian groups, which are dual to each other. In particular, they explored the relations between countable projective limits of almost metrizable abelian groups and countable direct limits of locally k_ω abelian groups.

That same year 2010, Laamara, Belhaj, Boya, Medari and Segui [164] discussed quiver gauge models with bi-fundamental and fundamental matter obtained from F-theory compactified on ALE spaces over a four-dimensional base space. They focussed on the base geometry which consists of intersecting $F_0 = CP^1 \times CP^1$ Hirzebruch complex surfaces arranged as Dynkin graphs classified by three kinds of Kac-Moody algebras: ordinary, i.e finite dimensional, affine and indefinite, in particular hyperbolic. They interpreted the equations defining these three classes of generalized Lie algebras as the anomaly cancelation condition of the corresponding $N = 1$ F-theory quivers in four dimensions.

In 2011 S.J. Kang, S.J. Oh and E. Park [150] introduced the notion of *perfect bases* for integrable modules over generalized Kac-Moody algebras and showed that the colored oriented graphs arising from perfect bases were isomorphic to the highest weight crystals $\mathcal{B}_\lambda$ over quantum generalized Kac-Moody algebras. This same year, A.K. Singh dealt with Dynkin diagrams associated with Kac-Moody algebras in his Thesis (see [241]). Indeed, he introduced the definition of the extended Dynkin diagrams for affinization of Kac-Moody algebras and the Dynkin Diagrams associated with the affine Kac-Moody algebras.

S.J. Kang, D.I. Lee, E. Park and H. Park [148] also introduced and generalized in 2011 the notion of Castelnuovo-Mumford regularity for representations of non-commutative algebras, effectively establishing a measure of complexity for such objects. The Gröbner-Shirshov basis theory for modules over non-commutative algebras is developed, by which a non-commutative analogue of Schreyer's Theorem is proved for computing syzygies. By a repeated application of this theorem, they constructed free resolutions for representations of non-commutative algebras and include some interesting examples. In particular, using the Bernstein-Gelfand-Gelfand resolutions for integrable highest weight modules over Kac-Moody algebras, they computed the projective dimensions and regularities explicitly for the cases of finite type and affine type.

Also, M. Gorelik and V. Kac [112] have studied the complete reducibility of representations of infinite-dimensional Lie algebras from the perspective of representation theory of vertex algebras. Indeed, they proved, in the first place a general result on complete reducibility of a category of modules over an associative algebra U, satisfying certain conditions, and introduced the notions of highest weight modules, Jantzen-type filtrations and Shapovalov-type determinants in this general setup, with the objective of using them to find the conditions under which any extension between two isomorphic (resp. non-isomorphic) highest weight modules over U splits. They also classified admissible modules for the Virasoro and the Neveu-Schwarz algebra, and after recalling some background material on vertex algebras, they have studied the admissible modules over minimal W-algebras. In particular, they classified the admissible irreducible vacuum modules over those W-algebras and described all admissible weights vs KW-admissible highest weights via the quantum Hamiltonian reduction. As a conclusion, they finally have stated a conjecture, which is supposed to imply that all simple W-algebras $W_k(g, f)$ satisfying a certain condition (the C_2-condition), are regular.

A. Felikson and P. Tumarkin [91] classified in 2012 regular subalgebras of Kac-Moody algebras in terms of their root systems. In the process, they established that a root system of a subalgebra is always an intersection of the root system of the algebra with a sublattice of its root lattice. They also discussed applications to investigations of regular subalgebras of hyperbolic Kac-Moody algebras and conformally invariant subalgebras of affine Kac-Moody algebras. In particular, they provided explicit formulae for determining all Virasoro charges in coset constructions that involve regular subalgebras.

Also in 2012, A. Braverman and M. Finkelberg [44] have recently proposed a conjectural analogue of the geometric Satake isomorphism for untwisted affine Kac-Moody groups. As part of their model, they conjecture that (at dominant weights) Lusztig's q-analog of weight multiplicity is equal to the Poincare series of the principal nilpotent filtration of the weight space, as occurs in the finite-dimensional case. At this respect, W. Slofstra shows in [243] that the conjectured equality holds for all affine Kac-Moody algebras if the principal nilpotent filtration is replaced by the principal Heisenberg filtration. The main body of his proof is a Lie algebra cohomology vanishing result. He also gives an example to show that the Poincare series of the principal nilpotent filtration is not always equal to the q-analog of weight multiplicity, and finally, he we gives some partial results for indefinite Kac-Moody algebras.

S.S. Sharma and S. Viswanath have studied in [234, 235] Lusztig's t-analog of weight multiplicities associated to level one representations of twisted affine Kac-Moody algebras. They obtained an explicit closed form expression for the corresponding t-string function using constant term identities of Macdonald and Cherednik. This extends previous work on level 1 t-string functions for the untwisted simply-laced affine Kac-Moody algebras.

Continuing in 2012, A.U. Maheswari has defined in [177] fuzzy sets on the Cartesian product of the simple roots of some of the affine type of Kac-Moody algebras. The fundamental fuzzy properties like normality, convexity and cardinality have been studied for these affine Kac-Moody algebras. The α-level, strong α-level sets and α-cut decomposition for these fuzzy sets, associated with the affine Kac-Moody algebras have also been computed.

In the last years, there are more than seventy references on Kac-Moody algebras in the main data base. Some of the most significative are the following:

In 2013, J, Xiao and F. Xu gave a a refinement (meanwhile, a new proof) of one result by L. Peng and J. Xiao [205], who had proved in 2000 a theorem for constructing Hall type Lie algebras associated to 2-periodic triangulated categories. This theorem gave a global realization of symmetrizable Kac-Moody algebras. In their new proof, Xiao and Xu applied the approach of derived Hall algebras arising in homologically finite triangulated categories, previously dealt with by themselves in 2008 [274].

In that same year, C. Lu and H. Zhang [174] gave the isomorphic realization of nondegenerate solvable Lie algebras of maximal rank, which in turn showed the closed connections between nondegenerate solvable Lie algebras and Kac-Moody algebras, resulting in some new worthy topics in this area.

In 2014, L. Carbone, W. Freyn and K.H. Lee introduced root multiplicities of Kac-Moody algebras and gave various formulas for these multiplicities. They mainly summarized the known results and methods for determining root multiplicities for hyperbolic Kac-Moody algebras and posed some open problems for further study [49]. J. Kübel dealt with a Jantzen sum formula for restricted Verma modules over affine Kac-Moody algebras at the critical level, [161] and P. Shan, M. Varagnolo and E. Vasserot set a Koszul duality of affine Kac-Moody algebras and cyclotomic rational double affine Hecke algebras [233].

In 2015, lots of references have gone appearing regarding this topic. Among them, B. Hou and S. Yang dealt with generalized McKay quivers, root system and Kac-Moody algebras [124], R. Lv and Y. Tan studied the link between some gim algebras and the associated indefinite Kac-Moody algebras [176] and D. Allcock [4] found root systems for Lorentzian Kac-Moody algebras in rank 3.

In 2016, S. Eswara Rao, V. Futorny and S.S. Sharma dealt with global and local Weyl modules, already introduced via generators and relations in the context of affine Lie algebras in [56]. They continued that research to study the representations of an affine Lie algebra over a finitely generated commutative associative unital algebra [85]. M.K. Chuah and R. Fioresi studied admissible positive systems of affine nontwisted Kac-Moody Lie algebras and their Hermitian real forms and proved that for one of those algebras, the existence of Hermitian real forms is equivalent to having admissible positive systems. They also classified Hermitian real forms for affine non-twisted Kac-Moody Lie algebras using Vogan diagrams [62]. Besides, J. Brundan [46] proved that the 2-categories defined by R. Rouquier [219] and by M. Khovanov and A. Lauda [154] are the same.

In 2017, M.I. Dillon [77] constructed graded affine Lie algebras of rank $n+1$ from representations of semi-simple algebras of rank n. Indeed, he set the connection between the grading determined by a simple root of an existing affine Lie algebra and certain modules over an underlying semi-simple Lie algebra. He also constructed an algorithm that either generates the homogeneous components of a graded affine Lie algebra or determines that the module is inadmissible. That algorithm produced both twisted and non-twisted affine Lie algebras, identifying no distinction between both types, which was illustrated in the paper by him showing an example of each of them. Let us recall that after its introduction more than four decades ago, Kac-Moody theory has become a standard generalization of classical Lie theory. However, very little is known beyond the affine case. The first difficulty in the hyperbolic case stems from the wild behavior of root multiplicities. At this respect, S.J. Kang, K.H.Lee and K. Lee took a new approach to the study of root multiplicities for hyperbolic Kac-Moody algebras [149]. And G. Pezzini, under the conditions of being G a complex connected reductive algebraic group, B a Borel subgroup and X a normal irreducible variety with an algebraic action of G, constructed an homogenous spherical datum for spherical subgroups H of G and showed that these objects satisfy the critical combinatorial axioms of the homogenous spherical datum for finite-dimensional spherical homogenous spaces G/H. He also proved that the (abstract) homogenous spherical datum for H is invariant under conjugation [208].

Finally, other papers on this topic have been published recently, in 2018. Among them, Gritsenko and Nikulin studied Lorentzian Kac-Moody algebras corresponding to 2-reflexive hyperbolic lattices with a lattice Weyl vector. Particularly, they constructed and classified some Lorentzian Kac-Moody algebras for rank at least 3, mainly algebras with Weyl groups of 2-reflections [117]. V. Futorny and I. Kashuba [102] dealt with the structure of parabolically induced modules for affine Kac-Moody algebras and B. Mühlherr and K. Struyve prove that the so-called *extremal elements* in a Coxeter system (W, S) only exist in Coxeter systems which are spherical and affine. They showed that for a pair of buildings of irreducible type (W, S), the existence of two distinct twinnings at finite distance between them implies the existence of an extremal element in (W, S) (two twinnings are said to be at finite distance if their sets of opposite chamber pairs are at finite Hausdorff distance from each other). They applied their results to the study of metric commensurators [191].

To finish this subsection, the authors think that brief comments on *Kac-Moody superalgebras* and *2-Kac-Moody algebras* should not be omitted in this survey, due to their natural importance, although a long space cannot be used for it.

Indeed, the study of Kac-Moody superalgebras is, at present, very extend. At this respect, Hoyt provided in [125] a classification of Kac-Moody superalgebras using integrable modules. The main interest of this research is to be the major part of the proof for the classification of contragredient Lie superalgebras, and, in particular, for the classification of finite-growth Kac-Moody superalgebras. In the paper, the author introduced the concept of regular Kac-Moody superalgebras and classified them using integrable modules, giving also conditions for irreducible highest weight modules of regular Kac-Moody superalgebras to be integrable. To do this, the author classified connected subfinite regular Kac-Moody diagrams and also dealt with the Lie superalgebra $Q^{\pm}(m, n, t)$ and with integrable modules and finally, extends regular Kac-Moody diagrams that are not of finite type.

Later, in 2010, C. Daboul, J. Daboul and Montigny defined the concepts of open Lie superalgebras and their closure and studied their embedding into affine Kac-Moody superalgebras. They distinguished between two types of open algebras, whose closures typically yield twisted or untwisted Kac-Moody superalgebras. They showed that the open dynamical symmetry superalgebra δ_0 of the Dirac theory of the Taub–NUT model, studied by Cotaescu and Visinescu in 2007, could not be embedded into a twisted Kac-Moody superalgebra, in contrast to their claim. Their analysis of the above relativistic model revealed the deeper reason of why the hydrogen algebras N, studied by J. Daboul, P. Slodowy and C. Daboul, must be twisted-like (genuinely and not-genuinely twisted) Kac-Moody subalgebras (see [67]). A recent paper on superalgebras, in 2012, is by K. Iohara and Y. Koga, who dealt with *Enright functors* for these algebras (see [127]).

In 2015, B. Ransingh dealt with Vogan diagrams of untwisted affine Kac-Moody superalgebras [212], S. Nayak and K.C. Pati studied the embedding of simply laced hyperbolic Kac-Moody superalgebras [196], D. Hill and W. Wang dealt with the categorification of quantum Kac-Moody superalgebras [122] and S.J. Cheng, J.H. Kwon and W. Wang researched on irreducible characters of Kac-Moody Lie superalgebras [61]

Finally, in 2017, S. Azam [11] gave a new characterization of Kac-Moody-Malcev superalgebras.

With respect to the topic of 2-Kac-Moody algebras, Rouquier wrote in 2008 (see [219]) that *over the past ten years, authors have advocated the idea that there should exist monoidal categories (or* 2*-categories) with interesting "representation theory"* and he proposed to call 2-*representation theory* this higher version of representation theory and to call 2-*algebras* those interesting monoidal additive categories. The difficulty in pinning down what is a 2-algebra (or a Hopf version) should be compared with the difficulty in defining precisely the meaning of quantum groups (or quantum algebras). The analogy is actually expected to be meaningful: while quantization turns certain algebras into quantum algebras, *categorification* should turn those algebras into 2-algebras. *Dequantization* is specialization $q \mapsto 1$, while *decategorification* is the Grothendieck group construction in the presence of gradings. It leaded to a quantum object.

The starting point of the study of 2-representation theory of Lie algebras was the construction in 2003 of the theory of $\mathfrak{sl}_2$-categorifications by J. Chuang and R. Rouquier [63]. A large part of geometric representation theory should, and can, be viewed as a construction of irreducible 2-representations as categories of sheaves.

The main results of the paper, in which Rouquier defined a 2-category $A_{\mathfrak{g}}$ associated with a Kac-Moody algebra $\mathfrak{g}$, were announced at seminars in Orsay, Paris and Kyoto in the Spring 2007. Certain specializations of the Hecke algebras associated with quivers and the resulting monoidal categories associated with half Kac-Moody algebras were introduced independently by Khovanov and Lauda [155].

2.3. *Kac-Moody algebras: Graphs associated*

Regarding Graph Theory, the study of its properties and applications is currently running in a high level due to its very widespread use as a tool to solve many important problems in other disciplines. Indeed, graphs (and more particularly, trees) have been essential previously to study several properties on different types of algebras (the first ones were semisimple Lie algebras) due to its role in determining the Dynkin diagrams associated to such algebras.

There exist in the literature several and interesting papers dealing with the association between Kac-Moody algebras and graphs. A sample of some of them are the following. One of the first papers dealing with this link is by M.B. Halpern and N.A. Obers in 1991 [118]. In that paper, the announced an isomorphism between a set of generically irrational affine-Virasoro constructions on $SO(n)$ and the unlabelled graphs of order n. On the one hand, they classified the conformal constructions by the graphs, while, conversely, obtained a group-theoretic and conformal field-theoretic identification for every graph of graph theory.

Two years later, P.E. Singer gave a talk entitled "A digraph and a product of subspaces of a Kac-Moody algebra" in a Conference held on the occasion of Albert John Coleman's 75th birthday, in Kingston (Ontario, Canada) [239]. In it he presented several results on products of weight subspaces of Kac-Moody algebras, by himself, which had appeared previously in- [240].

In 2007 and motivated by affine Schubert calculus, T.F. Lam and M. Shimozono [166] constructed a family of dual graded graphs (Γ_s, Γ_w) for an arbitrary Kac-Moody algebra $\mathfrak{g}$. These graded graphs have the Weyl group W of $\mathfrak{g}$ as vertex set and are labeled versions of the strong and weak orders of (Γ_s, Γ_w), respectively. By using a construction of G. Lusztig for quivers with an admissible automorphism, they defined folded insertion for a Kac-Moody algebra and obtained Sagan-Worley shifted insertion from Robinson-Schensted insertion as a special case. Moreover, drawing on work of Proctor and Stembridge, they analyzed the induced subgraphs of (Γ_s, Γ_w) which are distributive posets.

In 2008, C. Lenart and A. Postnikov presented a simple combinatorial model for the characters of the irreducible integrable highest modules for complex symmetrizable Kac-moody algebras. The model can viewed as a discrete counterpart to the Littlemann path model. We describe crystal graphs and give a Littlewood-Richardson rule for decomposing tensor products of irreducible representations. The new model is based on the notion of *λ-chain*, which is a chain of positive roots defined by certain interlacing conditions [169].

In 2012, T.A. Terlep and J. Williford constructed new families of graphs whose automorphism groups are transitive on 3-paths. These graphs are constructed from certain Lie algebras related to generalized Kac-Moody algebras of rank two. They showed that one particular subfamily gives new lower bounds on the number of edges in extremal graphs with no cycles of length fourteen [255].

In 2015, R. Venkatesh and S. Viswanath dealt in [264] with the connections between properties of root systems of Kac-Moody algebras and the chromatic polynomials of their associated Dynkin diagrams. They gave two interesting results. The first of them relates the chromatic polynomial of the graph G associated to the Dynkin diagram of a Kac-Moody algebra $\mathfrak{g}$ to root multiplicities of $\mathfrak{g}$ in a very nice way. They also showed two relevant examples for a tree and a cycle graph. The second main result gives another realization of the chromatic polynomial of G, this time as a summation over the paths in G ending at b_0, the unique minimal element of the bond lattice corresponding to the singleton partition. Note that this paper gives an interesting application of root systems to graph theory (namely the chromatic polynomial of a graph) for readers familiar with the theory of Kac-Moody algebras.

Finally, also in 2015, M. Lanini gave a categorical version of a previous result by Lusztig in 1980 (see [175]), which proved a stabilisation property of the affine Kazhdan-Lusztig polynomials by using the theory of sheaves on moment graphs. In this way she associated any Kac-Moody algebra with its stable moment graph [167]. Previously, Fiebig [97] had dealt with a conjecture by the own Lusztig as a moment graph problem. After this paper and until the end of 2018 there are no more publications linking Kac-Moody algebras and graphs in the literature.

2.4. *Kac-Moody algebras: Physical applications*

This subsection is devoted to show references regarding the applications of Kac-Moody algebras to Physics. These applications have contributed in decisive manner to the development of several physical concepts. Although we cannot extend a lot, due to reasons of length, we consider that showing some of them is strictly necessary to understand the great importance of these algebras. Indeed, the Norwegian Academy of Science and Letters (Det Norske Videnskaps-Akademi), founded in Christiania in 1857, which is responsible for the administration of the Abel Prize and the Kavli Prize, decided to award the Abel Prize for 2008 to John Griggs Thompson, University of Florida and Jacques Tits, Collège de France. Tits was honored due to his research on which is at present called *Tits's construction* which allows an amazing variety of physical application, widely used by theoretical physics, in which these algebras play an important role.

It is a fact that since the 1980s, these algebras had been taken up by physicists which work in the areas of elementary particle theory, gravity, and two-dimensional phase transitions as an obvious framework from which to consider descriptions of non-perturbative solutions of gauge theory, vertex emission operators in string theory on compactified space, integrability in two-dimensional quantum field theory, and conformal field theory. Kac-Moody algebras have also been shown to serve as duality symmetries of non-perturbative strings appearing to relate all superstrings to a single theory. The infinite-dimensional Lie algebras and groups have been suggested as candidates for a unified symmetry of superstring theory. In addition to this wide application to physical theories, the Kac-Moody algebras are relevant to number theory and modular forms. It seems feasible that the infinite discrete set of duality symmetries of String Theory may be related to both the affine E_8 algebra, and to a hyperbolic Kac-Moody Lie algebra E_{10}. The idea of using Kac-Moody symmetry to get at non-perturbative information in theories of particle physics remains viable (see further information on this subject in [82]).

Among the first authors who have worked in applications of Kac-Moody algebras to the Physics can be mentioned the following: D.I. Olive, in 1985, published *Kac-Moody algebras: an introduction for physicists* [202] and M.C. Díaz, in 1988, published a short survey on the physical applications of these algebras. Concretely, he dealt with the non-linear model of the Einstein equation with two commutative Killing vectors and its link with the infinite dimensional Lie algebras of the dynamic symmetry transformations associated, which resulted to be Kac-Moody algebras (see [76]).

It can also be cited the wide research carried by J. Daboul on mathematical physics, in particular on dynamical symmetries, generalized Inonu-Wigner contractions, and quantum computations, and the most recent on generalized contractions of finite Lie algebras and of infinite-dimensional Kac-Moody algebras (see [66, 68, 69]).

There are many distinguished researchers working on applications of Kac-Moody algebras to Physics. Between others we can cite C. Schweigert, who is dealing with these algebras and Lie algebras since 1992 (see [99]), A. Kleinschmidt (see [47, 157]), who has more than fifteen papers published on this topic, and C. Castro, who used in a very recent paper [52] these algebras as a tool in String Theory.

D. Persson and N. Tabti (see [206]) gave interesting lectures on Kac-Moody Algebras with Applications in Super-Gravity, at the Third Modave Summer School on Mathematical Physics, held in Modave, Belgium, August 2007. These lectures were divided into two main parts. In the first part they gave an introduction to the theory of Kac-Moody algebras directed towards physicists. In particular, they described the subclasses of affine and Lorentzian Kac-Moody algebras in detail. Their treatment focused on the Chevalley-Serre presentation, and emphasized the importance of the Weyl group. In the second part of the lectures they made use of the contents of part one to describe some recent developments devoted to investigations of the underlying symmetry structures of supergravity theories. They began by describing how toroidal compactifications of gravity theories reveals hidden global and local symmetries of the reduced Lagrangian. They also discussed attempts to extend these symmetry structures to infinite-dimensional algebras and showed how a manifestly Kac-Moody invariant action can be constructed, whose solutions correspond to exact BPS solutions in maximal supergravity theories.

On July 4th, 2012, only few months before this paper was finished, the CMS and the ATLAS experimental teams at the Large Hadron Collider independently confirmed the formal discovery of a previously unknown boson of mass between 125 and 127 GeV/c2, whose behavior so far had been consistent with a Standard Model Higgs boson.

This particle, called *Higgs boson* or *Higgs particle* or, even, *God particle* (after the title of Leon Lederman's book on the topic (1993), although this last epithet is strongly disliked by many physicists, who regard it as inappropriate sensationalism) is an elementary particle in the Standard Model of particle physics. It is the last unobserved particle of that model and had been predicted to exist since the 1960s. Its name is in honor of Peter Higgs, who along with R. Brout and F. Englert, and with G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble proposed the mechanism that suggested such a particle in 1964 and was the only one who emphasized the existence of the particle and calculated some of its properties.

In the Standard Model, the Higgs particle is a boson, a type of particle that allows multiple identical particles to exist in the same place in the same quantum state. It has no spin, electric charge, or color charge. It is also very unstable, decaying into other particles almost immediately. Some extensions of the Standard Model predict the existence of more than one kind of Higgs boson.

There exist several papers in the literature linking bosons with Kac-Moody and String Theory. Among them, A. Bilal studied bosonization of $\mathcal{Z}_N$ parafermions and $su(2)_N$ Kac-Moody algebra in 1989 (see [28]) and, in 1990, M. Sakamoto [221] dealt with the bosonization of bosons in vertex operator representations of affine Kac-Moody algebras. Previously, C.P. Sun had begun this research in 1987 studying Boson realization of Virasoro and Kac-Moody algebras and their indecomposable representations [254] and in 1988, a year later, K. Miki dealt with twisted bosons related to the Kac-Moody algebra [185] and S.N.M. Ruijsenaars [220] continued this research by linking Dirac quantization, Virasoro and Kac-Moody algebras and boson-fermion correspondence in $2N$ dimensions.

Let us recall that in Physics, in contrast to bosons, a *fermion* (a name coined by Paul Dirac from the surname of Enrico Fermi) is any particle characterized by Fermi-Dirac statistics and following the Pauli exclusion principle; fermions include all quarks and leptons, as well as any composite particle made of an odd number of these, such as all baryons and many atoms and nuclei. Fermions contrast with bosons which obey Bose-Einstein statistics. A fermion can be an elementary particle, such as the electron; or it can be a composite particle, such as the proton. The spin-statistics theorem holds that, in any reasonable relativistic quantum field theory, particles with integer spin are bosons, while particles with half-integer spin are fermions.

Similarly to the case of bosons, there exist several papers relating these particles with Kac-Moody algebras. Curiously, this study was begun by P. Windey in 1986 (see [272]), although relating fermions with super-Kac-Moody algebras. A year later, A.L. Carey and S.N.M. Ruijsenaars used Kac-Moody algebras to deal with fermion gauge groups [50] and in 1988, Ruijsenaars himself continued this research [220]. Already in the 90's, H.B. Thacker in 1991 [256] and Dolan, in 1994 [81] continued with this research.

Another application of Kac-Moody algebras to Physics is related with solitons. A *soliton* is a self-reinforcing solitary wave that maintains its shape while it travels at constant speed. Solitons are caused by a cancelation of non-linear and dispersive effects in the medium and arise as the solutions of a widespread class of weakly non-linear dispersive partial differential equations describing physical systems. The soliton phenomenon was first described by John Scott Russell (1808-1882) who observed a solitary wave in the Union Canal in Scotland. He reproduced the phenomenon in a wave tank and named it the "Wave of Translation" (see [200] for further information on these objects).

At this respect, N.S. Craigie, W. Nahm and K.S. Narain [65] and H.J. de Vega [262] in 1985 began this study dealing with Kac-Moody algebras in field theory with solitons and Carey and Ruijsenaars, two years later, studied the relation between fermion gauge groups, current algebras and Kac-Moody algebras (see [50]). Later, in 1990, I. Sterling [245] and V.G. Mikhalëv [184] made progress in this research.

V.D. Ivashchuk and V.N. Melnikov, in 2011 (see [130]) considered a multidimensional gravitational model containing scalar fields and antisymmetric forms. The manifold chosen was $M = M_0 \times M_1 \times \ldots \times M_n$, where M_i are Einstein spaces ($i \geq 1$). They considered the sigma-model approach and exact solutions with intersecting composite branes (e.g., solutions with harmonic functions and black brane ones) with intersection rules related to non-singular Kac-Moody algebras (e.g., hyperbolic ones) and they gave some examples of black brane solutions, e.g., those corresponding to hyperbolic Kac-Moody algebras: $H_2(q, q)$, ($q \geq 2$) and $HA_2^{(1)} = A_2^{++}$ and to the Lorentzian Kac-Moody algebra P_{10}.

There also exist lots of paper dealing with the link between *Crystal Theory* and Kac-Moody algebras. Remember that when the metals described on the periodic table of the elements come into contact with each other or are changed by chemical reactions in a lab, their properties can be drastically altered. The crystal theory explains why these property changes occur and helps scientists to predict exactly which changes will happen. These changes were tried to be explained by using Kac-Moody algebras. So several researchers are working in this way. In the last years can be cited: Jeong, Kang and other joint collaborators in 2005, 07, 09 and 14 (see [134–136, 145], respectively), D.U. Shin in 2007 and 2008 ([237] and [236], respectively), N.C. Leung and M. Xu [170], C. Lenart and A. Postnikov [169] in 2008 and A. Savage [224], A. Joseph and P. Lamprou [137] and S.J. Kang, M. Kashiwara and O. Schiffmann, in 2009 [146, 147]. At present, there are 12 references in MathSciNet containing the words *crystal* and *Kac-Moody algebras* in title. The first paper on this topic dates from 1997 and it is by T. Nakashima and A.Zelevinsky [194]. In that paper, authors studied parametrizations of the crystal basis of the negative part of the quantized universal enveloping algebra of a symmetrizable Kac-Moody algebra. Equivalently, the basis vectors should correspond to the lattice points in some polyhedral convex cone. The authors gave such a description, generalizing a result of Kashiwara in 1993 in the rank 2 case and their proof was given under certain technical assumptions that are checked to hold when the Kac-Moody algebra is of finite or affine type.

More recently, Kang, Oh and Park [151]) constructed and investigated the structure of the Khovanov-Lauda-Rouquier algebras R and their cyclotomic quotients R^λ which give a categorification of quantum generalized Kac-Moody algebras [151]. One of the key ingredients of their approach is the perfect basis theory for generalized Kac-Moody algebras.

Moreover, the same authors had extended in [150] the work of A. Berenstein and D. Kazhdan [27] published in 2007, to generalized Kac-Moody algebras. They developed the theory of perfect bases for such algebras, hence giving a new construction of the highest weight crystals associated to quantum generalized Kac-Moody algebras. Indeed, the authors showed that for all integrable highest weight modules over affine Kac-Moody algebras the exponent of growth equals 2. In the main result of the paper they calculated explicitly also the rate of growth in the mentioned case, showing that it depends only on the type and rank of algebra involved, and the level of representation.

Quantum generalized Kac-Moody algebras have been profusely dealt with in the literature. Indeed, in the paper [144], also cited in another section of this paper, the authors, Seok-Jin Kang and Masaki Kashiwara, used supercategories of modules over quiver Hecke superalgebras to establish supercategorification (i.e., categorification through supercategories) of quantum Kac-Moody algebras and their integrable highest weight modules. In fact, they used supercategories of supermodules of quiver Hecke superalgebras to obtain supercategorification of a general family of quantum Kac-Moody (super)algebras and their integrable highest weight modules. This family included the usual quantum Kac-Moody superalgebra as a special case. At this respect, it is convenient to note that supercategorification of quantum Kac-Moody superalgebras was also established by Hill and Wang in [122], under an extra condition, although the main results by Kang and Kashiwara do not assume that condition.

A year later, these same authors, joint with Oh [150] extended and improved the previous paper [150] and, also in 2015, V. Futorny, J.T. Hartwig and E.A. Wilson studied quantum affine modules for non-twisted affine Kac-Moody algebras [101] and Chen and Zhao represented quantum generalized Kac-Moody algebras with one imaginary simple root [57].

Moving on another subject, in 2016, T. He, P. Mitra and A. Strominger showed that for massless theories at the semiclassical level, the soft gluon theorem is the Ward identity of a holomorphic two-dimensional G-Kac-Moody symmetry acting on two-dimensional correlation functions [120].

The foundation of this theory had been developed in the last five years with many strong relations to Kac-Moody algebras. Indeed, over half a century ago, Bondi, van der Burg, Metzner and Sachs argued that gravitational wave scattering in asymptotically Minkowskian spacetimes is governed by an infinite-dimensional symmetry algebra known as the BMS algebra. The symmetry has consequences for any theory which can be coupled to gravity, and would be expected to play a central role in Minkowskian scattering theory. However, while some important results were obtained, the full import of BMS symmetry has remained elusive. The BMS algebra has two types of elements: first, the global $SL(2;C)$ conformal transformations of the conformal 2-sphere which are related to Lorentz transformations, and second, translations along the null generators of the future null infinity (in the Minkowski space). Two decades later, a mathematically similar problem was studied: $2D$ field theories with a global $SL(2;C)$ symmetry. Now symmetries with analytic singularities are allowed, which leads to two Virasoro algebras. The implications of this infinite symmetry algebra for $2D$ field theory are of course truly extraordinary. In this way, one hopes by analogy that the extended BMS symmetry has extraordinary implications for Minkowski scattering. In 1965, Weinberg showed that photon and graviton amplitudes behave in a universal way when one of the external particles with momentum becomes soft (low energy). See [15] for further information on this topic.

In this same year, 2016, A. Zuevsky dealt with the semi-direct product of Virasoro and affine Kac-Moody Lie algebras and associated Verma modules, coadjoint orbits, Casimir functions, and biHamiltonian systems [281].

In 2017, L. Bao and L. Carbone, in a short survey paper [13], dealt with the construction of Kac-Moody groups over $\mathbb{R}$ and $\mathbb{Z}$ using certain choices of fundamental representations of E_9, E_{10} and E_{11}, that have physical relevance in the context of supergravity theories dimensionally reduced to dimensions less than 3. They constructed Eisenstein series for Kac-Moody groups, which are predicted to encode some perturbative corrections in the framework of string theories.

Previously and related in certain way to this topic, Damour, along with other authors, [71, 72] had conjectured that the spin-extended Weyl group Wspin (E_{10}) could be constructed as a discrete subgroup of a double spin cover Spin(E_{10}) of the subgroup $K(E_{10})$ of elements fixed by the Cartan-Chevalley involution of the split real Kac-Moody group of type E_{10}. Starting from that point, D. Ghatei, M. Horn, R. Köhl and S. Weib [106] constructed non-trivial spin covers of K, thus confirming that conjecture. Their spin covers contained what they called *spin-extended Weyl groups*, which admit a presentation by generators and relations obtained from the one for extended Weyl groups by relaxing the condition on the generators so that only their eighth powers are required to be trivial.

Recently, already in 2018, L. Gilch, S. Müller and J. Parkinson have set limit theorems for random walks on Fuchsian buildings and Kac-Moody groups[107] and Zuevsky, in [282] has proved new theorems to the construction of the shallow water biHamiltonian systems associated to the semi-direct product of Virasoro and affine Kac-Moody Lie algebras. Particularly, he has discuss associated Verma modules, coadjoint orbits, Casimir functions, and bi-Hamiltonian systems.

To finish this subsection, it is convenient to note that recent progress in String Theory and the Theory of Supergravity shows that the generating functions for the quantum degeneracies of supersymmetric black holes are related to reflective Siegel modular forms and mock theta-series. Reflective Siegel forms determine Kac-Moody Lie algebras of Borcherds type which are also important in physics. At this respect, V. Gritsenko organized in 2010 the Program of Automorphic forms, Kac-Moody Lie algebras and Strings with the objective of presenting the current results in this field related to string theory, the theory of automorphic forms and the theory of Kac-Moody Lie algebras (see [115]). Explicit constructions in the theory of automorphic forms and applications have appeared, due to that during the past decade automorphic forms have been successfully used to solve some fundamental problems in algebraic geometry and the theory of Kac–Moody algebras. Thus, special Siegel modular forms and mock theta-series have become an important tool in the theory of supergravity.

2.5. *Kac-Moody algebras: books*

Due the natural importance of Kac-Moody algebras there exist several books dedicated to show a general information of them. Among them, can be cited *Kac-Moody and Virasoro Algebras*, by Goddard and Olive [109] in 1988, *Formules de caractères pour les algèbres de Kac-Moody générales*, by Mathieu [181] in the same year, *Groupes associés aux algèbres de Kac-Moody*, by Tits in 1988-89 [257], *Introduction to Kac-Moody algebras*, by Z.X. Wan (see [267]), in 1991, which supposed an introduction to a rapidly growing subject of modern mathematics (in such a time) and later *Kac-Moody algebra, Encyclopedia of Mathematics*, by M. Hazewinkel [119], in 2001 and *Kac-Moody Groups, their Flag Varieties and Representation Theory*, by Kumar [163], in 2002.

In 2009, the *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)* journal published a special Issue containing 20 papers with the total of 569 pages on Kac-Moody Algebras and Applications. It was devoted to 40th anniversary of Kac-Moody algebras. The Guest Editors for this special issue were R.E. Borcherds, E. Frenkel, V.G. Kac, R.V. Moody, J. Patera, A. Pianzola and P. Ramond. It can be consulted in [36].

More recently, in 2010, two new books on these algebras have been published. The first of them was written by E. Jamsin and J. Palmkvist (see [133]). It is based on lectures given by them at the Fifth International Modave Summer School on Mathematical Physics, held in Modave, Belgium, August 2009. In those lectures, they presented in a self-containing way an introduction to the theory of Lie and Kac-Moody algebras. Throughout the text of that book, they mainly focus on the concepts that are important in applications to mathematical physics, especially in the context of hidden symmetries.

In the book, the introduction to Lie and Kac-Moody algebras is meant to contain tools that are necessary when using this theory in mathematical physics, and in particular in the context of hidden symmetries. The first example of such a symmetry was emphasized by Ehlers in the context of four-dimensional pure gravity. Assuming the existence of one Killing vector and compactifying the metric on the associated circle, one finds a three-dimensional theory that has a global symmetry under the Lie algebra $\mathfrak{sl}_2$ which is bigger than the expected u(1). The appearance of such hidden symmetries could then be generalized to other gravity and supergravity theories in various dimensions. As finite-dimensional simple Lie algebras can be completely classified, one can also completely classify the theories that possess a hidden symmetry under these algebras. A famous example is the E_8 symmetry of $D = 11$ supergravity in the presence of eight commuting Killing vectors. Geroch first noticed the appearance of an infinite-dimensional symmetry when two commuting Killing vectors are assumed in four-dimensional gravity. This symmetry algebra was later identified as the affine extension of $\mathfrak{sl}_2$, the simplest example of an infinite-dimensional Kac-Moody algebra. More generally, one can see that any theory that possesses a hidden symmetry under a finite-dimensional Lie algebra $\mathfrak{g}$ in three dimensions can be seen to be symmetric under the affine extension of $\mathfrak{g}$ in two dimensions.

Secondly, A. Ros Camacho[218] has published the notes of the talk entitled *Kac-Moody Algebras* which she gave in 2010 in the seminar on Cohomology of Lie algebras, under the supervision of J-Prof. Christoph Wockel. As herself comments, *Kac-Moody algebras are a generalization of the finite-dimensional semi-simple Lie algebras which preserve almost the full original structure. In the last decades, they have attracted the attention of physicists as they revealed themselves as a useful tool in theoretical physics. They arise in e.g. dimensional reductions of gravity and supergravity theories, such as in dynamical symmetries in string theory, conformal field theory and the theory of exactly solvable models*.

One of the last books published has been *An Introduction to Kac-Moody Groups over Fields*, by T. Marquis [179]. The author provides an accessible, intuitive, reader-friendly and self-contained introduction to Kac-Moody groups and algebras over arbitrary fields. In its introduction, he says that pretends to *clean* the foundations and to provide a unified treatment of the theory.

In any case, the basic reference in Kac-Moody algebras for the researcher is the book by Kac itself in 1990 *Infinite dimensional Lie algebras* [141], although for an faster and easier introduction, try the cited Wan [267] and for a different approach, see the book by Moody and Pianzola [188].

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References

[1] X. Agrafiotou, Nilpotent Lie algebras of maximal rank and of Kac-Moody type: $D_4^{(1)}$, manuscript.
[2] X. Agrafiotou, Nilpotent Lie algebras of maximal rank and of Kac-Moody type $D_5^{(1)}$, manuscript.
[3] X. Agrafiotou, G. Tsagas, Nilpotent Lie algebras of maximal rank and of type E_6, E_7, E_8, manuscript.
[4] D. Allcock, Root systems for Lorentzian Kac-Moody algebras in rank 3, Bulletin of the London Mathematical Society 47:2 (2015) 325–342.
[5] J.M. Ancochea, R. Campoamor, Characteristically nilpotent Lie algebras: A survey, Extracta Mathematicae 16:2 (2001) 153–210.
[6] J.M. Ancochea, R. Campoamor-Stursberg, Classification of solvable real rigid Lie algebras with a nilradical of dimension $n \leq 6$, Linear Algebra and its Applications 471 (2015) 54–75.
[7] J.M. Ancochea, M. Goze, Classificatiòn des algèbres de Lie filiformes de dimension 8, Archiv der Mathematik 50 (1988) 511–525.
[8] J.M. Ancochea, M. Goze, Classificatiòn des algèbres de Lie nilpotentes de dimension 7, Archiv der Mathematik 52 (1989) 175–185.
[9] J.M. Ancochea, M. Goze, On the varieties of nilpotent Lie algebras of dimension 7 and 8, Journal of Pure and Applied Algebra 77:2 (1992) 131–140.
[10] N. Andruskiewitsch, The representation theory of inner almost compact forms of Kac-Moody algebras, Tohoku Mathematical Journal 43:2-3 (1991) 425–445.
[11] S. Azam, A new characterization of Kac-Moody-Malcev superalgebras, Journal of Algebra and Its Applications 16:8 (2017) 1750144, 15 pp.
[12] V. Back-Valente, N. Bardy-Panse, H. Ben Messaoud, G. Rousseau, Formes presque-dèployées des algèbres de Kac-Moody: classification et racines relatives, Journal of Algebra 171:1 (1995) 43–96.
[13] L. Bao, L. Carbone, Kac-Moody groups and automorphic forms in low dimensional supergravity theories, In: Lie algebras, vertex operator algebras, and related topics, Contemporary Mathematics 695, American Mathematical Society, 2017, pp. 29–40.
[14] Y. Barnea, A. Shalev, E. Zelmanov, Graded subalgebras of affine Kac-Moody algebras, Israel Journal of Mathematics 104 (1998) 321–334.
[15] G. Barnich, B. Oblak, Notes on the BMS group in three dimensions: I. Induced representations, Journal of High Energy Physics 6 (2014) 129, 27 pp.
[16] C. Bartolone, A. di Bartolo, G. Falcone, Nilpotent Lie algebras with 2-dimensional commutator ideals, Linear Algebra and its Applications 434:3 (2011) 650–656.

[17] J. Bausch, Ètude et classification des automorphismes d'ordre fini et de première espèce des algèbres de Kac-Moody affines. (French) [Study and classification of automorphisms of finite order and the first kind of affine Kac-Moody algebras] With appendices by Guy Rousseau. Inst. Èlie Cartan, 11, Algèbres de Kac-Moody affines, 5–124, Univ. Nancy, Nancy, 1989.
[18] H. Ben Messaoud, G. Rousseau, Erratum à: "Classification des formes réelles presque compactes des algèbres de Kac–Moody affines" [Journal of Algebra 267:2 (2003) 443–513], Journal of Algebra 279:2 (2004) 850–851.
[19] H. Ben Messaoud, G. Rousseau, Classification des formes réelles presque compactes des algèbres de Kac-Moody affines", Journal of Algebra 267:2 (2003) 443–513.
[20] S. Benayadi, A. Elduque, Classification of quadratic Lie algebras of low dimension, Journal of Mathematical Physics 55:6 (2014) 081703, 17 pp.
[21] J.C. Benjumea, F.J. Castro, M.C. Márquez, J. Núñez, The equations of the sets of filiform Lie algebras of dimension 13 and 14, Actas del 2^o Encuentro de Álgebra Computacional y Aplicaciones (EACA) (1996) 67–75.
[22] G. Benkart, S.K. Kang, K.C. Misra, Indefinite Kac-Moody algebras of special linear type, Pacific Journal of Mathematics 170:2 (1995) 379–404.
[23] G. Benkart, S.K. Kang, K.C. Misra, Weight multiplicity polynomials for affine Kac-Moody algebras of type $A_r^{(1)}$, Compositio Mathematica 104:2 (1996) 153–187.
[24] G. Benkart, A.I. Kostrikin, M.I. Kuznetsov, The simple graded Lie algebras of characteristic three with classical reductive component L_0, Communications in Algebra 24:1 (1996) 223–234.
[25] G. Benkart, J.M. Osborn, Rank one Lie algebras, Annals of Mathematics (Second Series) 119:3 (1984) 437–463.
[26] C. Bennett, Imaginary roots of a Kac-Moody Lie algebra whose reflections preserve root multiplicities, Journal of Algebra 158:1 (1993) 244–267.
[27] A. Berenstein, D. Kazhdan, Geometric and unipotent crystals II: From unipotent bicrystals to crystal bases, In: Quantum groups, Contemporary Mathematics 433, American Mathematical Society, 2007, pp. 13–88.
[28] A. Bilal, Bosonization of Z_N parafermions and $su(2)_N$ Kac-Moody algebra, Physics Letters B 226:3-4 (1989) 272–278.
[29] N. Blackburn, On a special class of p-groups, Acta Mathematica 100:1-2 (1958) 45–92.
[30] T. Bliem, Chopped and sliced cones and representations of Kac-Moody algebras, Journal of Pure and Applied Algebra 214:7 (2010) 1152–1164.
[31] R.E. Block, R.L. Wilson, Classification of the restricted simple Lie algebras, Journal of Algebra 114:1 (1988) 115–259.
[32] R.E. Borcherds, Generalized Kac-Moody algebras, Journal of Algebra 115:2 (1988) 501–512.
[33] R.E. Borcherds, Central extensions of generalized Kac-Moody algebras, Journal of Algebra 140:2 (1991) 330–335.
[34] R.E. Borcherds, Monstrous moonshine and monstrous Lie super-algebras, Inventiones Mathematicae 109 (1992) 405–444.
[35] R.E. Borcherds, A characterization of generalized Kac-Moody algebras, Journal of Algebra 174:3 (1995) 1073–1079.
[36] R.E. Borcherds, E. Frenkel, V.G. Kac, R.V. Moody, J. Patera, A. Pianzola, P. Ramond (Guest Editors), Special Issue on Kac-Moody Algebras and Applications, Symmetry, Integrability and Geometry: Methods and Applications 5, 2009.
[37] L.R. Bosko, On Schur multipliers of Lie algebras and groups of maximal class, International Journal of Algebra and Computation 20:6 (2010) 807–821.
[38] V. Bouchard, H. Skarke, Affine Kac-Moody algebras, CHL strings and the classification of tops, Advances in Theoretical and Mathematical Physics 7:2 (2003) 205–232.
[39] L. Boza, F.J. Echarte, J. Núñez, Classification of complex filiform Lie algebras of dimension 10, Algebra, Groups and Geometries 11:3 (1994) 253–276.
[40] L. Boza, E.M. Fedriani, J. Núñez, Classification of complex filiform Lie algebras of dimension 12, Prepublicación n^{o}38 de la Facultad de Matemáticas, Univiversidad de Sevilla (sección Álgebra, Computación, Geometría y Topología), 1997.
[41] L. Boza, E.M. Fedriani, J. Núñez, A new method for classifying complex filiform Lie algebras, Applied Mathematics and Computation 121:2-3 (2001) 169–175.
[42] L. Boza, E.M. Fedriani, J. Núñez, Complex filiform Lie algebras of dimension 11, Applied Mathematics and Computation 141:2-3 (2003) 611–630.
[43] F. Bratzlavsky, Sur les algèbres admettant un tore d'automorphismes donné, Journal of Algebra 30:1-3 (1974) 305–316.
[44] A. Braverman, M. Finkelberg, Pursuing the double affine Grassmannian I: transversal slices via instantons on A_k-singularities, Duke Mathematical Journal 152:2 (2010) 175–206.
[45] G. Brown, Families of simple Lie algebras of characteristic two, Communications in Algebra 23:3 (1995) 941–954.
[46] J. Brundan, On the definition of Kac-Moody 2-category, Mathematische Annalen 364 (2016) 353–372.
[47] S. de Buyl, A. Kleinschmidt, Higher spin fields from indefinite Kac-Moody algebras, Based on a talk presented by authors at the First Solvay Workshop on Higher-Spin Gauge Theories held in Brussels on May 2004,
https://arxiv.org/pdf/hep-th/0410274
[48] L. Cagliero, F. Szechtman, Classification of linked indecomposable modules of a family of solvable Lie algebras over an arbitrary field of characteristic 0, Journal of Algebra and Its Applications 15:2 (2016) 1650029, 21 pp.
[49] L. Carbone, W. Freyn, K.H. Lee, Dimensions of imaginary root spaces of hyperbolic Kac-Moody algebras, In: Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics, Contemporary Mathematics 623, American Mathematical Society, 2014, pp. 23–40.
[50] A.L. Carey, S.N.M. Ruijsenaars, On fermion gauge groups, current algebras and Kac-Moody algebras, Acta Applicandae Mathematicae 10:1 (1987) 1–86.
[51] D. Casperson, Strictly imaginary roots of Kac-Moody algebras, Journal of Algebra 168:1 (1994) 90–122.
[52] C. Castro, Effective field theory of W (infinity) Strings and Kac-Moody Algebras, Advanced Studies in Theoretical Physics 2:17 (2008) 825–835.
[53] F.J. Castro, J. Núñez, On characteristically nilpotent filiform Lie algebras of dimension 9, Communications in Algebra 23:8 (1995)

3059–3071.
[54] F.J. Castro, J. Núñez, Gröbner basis in the classification of characteristically nilpotent filiform Lie algebras of dimension 10, Progress in Mathematics 143 (1996) 115–133.
[55] M. Ceballos, J. Núñez, A.F. Tenorio, New results in the classification of filiform Lie algebras, Bulletin of the Malaysian Mathematical Sciences Society 40:1 (2017) 409–437.
[56] V. Chari, A. Pressley, Quantum affine algebras at roots of unity, Representation Theory 1 (1997) 280–328.
[57] J. Chen, Z. Zhao, A fundamental representation of quantum generalized Kac-Moody algebras with one imaginary simple root, Frontiers of Mathematics in China 10:5 (2015) 1041–1056.
[58] Y. Chen, Isomorphisms of adjoint Chevalley groups over integral domains, Transactions of the American Mathematical Society 348:2 (1996) 521–541.
[59] Y. Chen, On classification of affine Kac-Moody groups, Journal of Pure and Applied Algebra 145:3 (2000) 285–291.
[60] Z.X. Chen, Y.N. Lin, Tubular algebras and affine Kac-Moody algebras, Science in China Series A: Mathematics 50: 4 (2007) 521–532.
[61] S.J. Cheng, J.H. Kwon, W. Wang, Irreducible characters of Kac-Moody Lie superalgebras, Proceedings of the London Mathematical Society 110:1 (2015) 108–132.
[62] M.K. Chuah, R. Fioresi, Admissible positive systems of affine non-twisted Kac-Moody Lie algebras, Journal of Algebra 453 (2016) 561–577.
[63] J. Chuang, R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, Annals of Mathematics 167 (2008) 245–298.
[64] S. Cicalò, W. de Graaf, C. Schneider, Six-dimensional nilpotent Lie algebras, Linear Algebra and its Applications 436:1 (2012) 163–189.
[65] N.S. Craigie, W. Nahm, K.S. Narain, Realization of the Kac-Moody algebras of 2D QFTs through soliton operators, Physics Letters B 152:3-4 (1985) 203–208.
[66] C. Daboul, J. Daboul, M. de Montigny, Gradings and contractions of affine Kac-Moody algebras, Journal of Mathematical Physics 49 (2008) 063509, 17 pp.
[67] C. Daboul, J. Daboul, M. de Montigny, Embedding of "open" superalgebras into affine Kac-Moody superalgebras, Journal of Mathematical Physics 51 (2010) 053521, 15 pp.
[68] J. Daboul, Contraction of Broken Symmetries via Kac-Moody Formalism, Journal of Mathematical Physics 47 (2006) 083507, 13 pp.
[69] J. Daboul, P. Slodowy, C. Daboul, The hydrogen algebra as a centerless twisted Kac-Moody algebra, Physics Letters B 317:3 (1993) 321–328.
[70] T. Damour, Cosmological singularities, Einstein billiards and Lorentzian Kac-Moody algebras, Journal of Hyperbolic Differential Equations 2:2 (2005) 293–330.
[71] T. Damour, C. Hillmann, Fermionic Kac-Moody billiards and supergravity, Journal of High Energy Physics 08 (2009) 100, 55 pp.
[72] T. Damour, P. Spindel, Quantum supersymmetric cosmological billiards and their hidden Kac-Moody structure, Physical Review D 95:12 (2017) 126011, 23 pp.
[73] H. Darabi, F. Saeedi, M. Eshrati, Capable n-Lie algebras and the classification of nilpotent n-Lie algebras with $s(A) = 3$, Journal of Geometry and Physics 110 (2016) 25–29.
[74] I. Darijani, H. Usefi, The classification of 5-dimensional p-nilpotent restricted Lie algebras over perfect fields, I, Journal of Algebra 464 (2016) 97–140.
[75] E. Díaz, R. Fernández-Mateos, D. Fernández-Ternero, J. Núñez, Graphs associated with nilpotent Lie algebras of maximal rank, Revista Matemática Iberoamericana 19:2 (2003) 325–338.
[76] M.C. Díaz, Modelos σ-no lineales, álgebras de Kac-Moody y la relatividad general con $G^{\perp}_{2-}$, Revista Mexicana de Física 34:1 (1988) 1–17.
[77] M.I. Dillon, Constructing affine Lie algebras, Communications in Algebra 45:8 (2017) 3515–3530.
[78] J. Dixmier, W.G. Lister, Derivations of nilpotent Lie algebras, Proceedings of the American Mathematical Society 8:1 (1957) 155–158.
[79] V.K. Dobrev, A.H. Ganchev, Classification of modular invariant partition functions for the twisted $N = 2$ superconformal algebra, twisted $SU(2)$ Kac-Moody algebra and $D2K$ parafermions, Nuclear Physics B 332:3 (1990) 709–722.
[80] L. Dolan, Kac-Moody algebra is hidden symmetry of chiral models, Physical Review Letters 47:19 (1981) 1371–1374.
[81] L. Dolan, Free fermion constructions of super Virasoro and super Kac-Moody algebras, Advances in Applied Clifford Algebras 4:S1 (1994) 239–246.
[82] L. Dolan, The beacon of Kac-Moody symmetry for Physics, Notices of the American Mathematical Society 42:12 (1995) 1489–1497.
[83] J. Dozias, Sur les derivations des algèbres de Lie, Comptes Rendus de l'Académie des Sciences 259 (1964) 2748–2750.
[84] F.J. Echarte, J.R. Gómez, Classification of complex filiform nilpotent Lie algebras of dimension 9, Rendiconti Seminario Facoltà Scienze Universitá Cagliari 61:1 (1991) 21–29.
[85] S. Eswara Rao, V. Futorny, S.S. Sharma, Weyl modules associated to Kac-Moody Lie algebras, Communications in Algebra 44:12 (2016) 5045–5057.
[86] O.J. Falcón, R.M. Falcón, J. Núñez, A.M. Pacheco, M.T. Villar, Classification of filiform Lie algebras up to dimension 7 over finite fields, Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica 24:2 (2016) 185–204.
[87] G. Favre, Systéme de poids sur une algèbre de Lie nilpotente, Manuscripta Mathematica 9 (1973) 53–90.
[88] G. Favre, L. Santharoubane, Nilpotent Lie algebras of classical simple type, Journal of Algebra 202:2 (1998) 589–610.
[89] G. Favre, G. Tsagas, Nilpotent Lie algebras of maximal rank and of type F_4, manuscript.
[90] A.J. Feingold, H. Nicolai, Subalgebras of hyperbolic Kac-Moody algebras, In: Kac-Moody Lie Algebras and Related Topics, Contemporary Mathematics 343, American Mathematical Society, 2004, pp. 97–114.

[91] A. Felikson, P. Tumarkin, Hyperbolic subalgebras of hyperbolic Kac-Moody algebras, Transformation Groups 17:1 (2012) 87–122.

[92] D. Fernández-Ternero, Nilpotent Lie algebras of maximal rank and of Kac-Moody type: $D_4^{(3)}$, Journal of Lie Theory 15:1 (2005) 249–260.

[93] D. Fernández-Ternero, J. Núñez, Classification of metabelian Lie algebras of maximal rank, Comptes Rendus de l'Académie des Sciences Paris Série I (Sciences Mathématiques) 332:11 (2001) 969–974.

[94] D. Fernández-Ternero, J. Núñez, Nilpotent Lie algebras of maximal rank and of Kac-Moody type: $F_4^{(1)}$, Communications in Algebra 29:4 (2001) 1551–1570.

[95] D. Fernández-Ternero, J. Núñez, Nilpotent Lie algebras of maximal rank and of Kac-Moody type: $E_6^{(1)}$, Journal of Algebra 250:2 (2002) 664–695.

[96] P. Fiebig, The combinatorics of category O over symmetrizable Kac-Moody algebras, Transformation Groups 11:1 (2006) 29–49.

[97] P. Fiebig, Lusztig's conjecture as a moment graph problem, Bulletin of the London Mathematical Society 42:6 (2010) 957–972.

[98] J. Fröhlich, E. Thiran, Integral quadratic forms, Kac-Moody algebras, and fractional quantum Hall effect. An $ADE-O$ classification, Journal of Statistical Physics 76:1-2 (1994) 209–283.

[99] J. Fuchs, U. Ray, C. Schweigert, Some automorphisms of Generalized Kac-Moody algebras, Journal of Algebra 191:2 (1997) 518–540.

[100] K. Furutani, I. Markina, Complete classification of pseudo H-type Lie algebras: I, Geometriae Dedicata 190:1 (2017) 23–51.

[101] V. Futorny, J.T. Hartwig, E.A. Wilson, Quantum affine modules for non-twisted affine Kac-Moody algebras, Proceedings of the American Mathematical Society 143:12 (2015) 5159–5171.

[102] V. Futorny, I. Kashuba, Structure of parabolically induced modules for affine Kac-Moody algebras, Journal of Algebra 500 (2018) 362–374.

[103] H. Gan, Y. Tan, On a class of generalized Kac-Moody algebras arising from 2-fold affinizations, Journal of Algebra and Its Applications 14:10 (2015) 1550137, 14 pp.

[104] M.A. Gauger, On the classification of metabelian Lie algebras, Transactions of the American Mathematical Society 179 (1973) 293–329.

[105] S. Gaussent, G. Rousseau, Kac-Moody groups, hovels and Littelmann paths, Annales de l'Institut Fourier 58:7 (2008) 2605–2657.

[106] D. Ghatei, M. Horn, R. Köhl, S. Weib, Spin covers of maximal compact subgroups of Kac-Moody groups and spin-extended Weyl groups, Journal of Group Theory 20:3 (2017) 401–504.

[107] L. Gilch, S. Müller, J. Parkinson, Limit theorems for random walks on Fuchsian buildings and Kac-Moody groups, Groups, Geometry and Dynamics 12:3 (2018) 1069–1121.

[108] H. Glöckner, R. Gramlich, T. Hartnick, Final group topologies, Kac-Moody groups and Pontryagin duality, Israel Journal of Mathematics 177:1 (2010) 49–101.

[109] P. Goddard, D. Olive, Kac-Moody and Virasoro Algebras, Advanced Series in Mathematical Physics 3, World Scientific Publishing Company, 1988.

[110] J.R. Gómez, A. Jiménez, Y.B. Khakimdjanov, On the variety of nilpotent Lie algebra laws of dimension 11, Rendiconti del Seminario della Facoltà di scienze dell'Università di Cagliari 66 (1996) 137–142.

[111] M.P. Gong, Classification of Nilpotent Lie Algebras of Dimension 7 (over Algebraically Closed Fields and $\mathbb{R}$). Ph. Doctoral Thesis. University of Waterloo, Ontario, Canada, 1998.

[112] M. Gorelik, V. Kac, On complete reducibility for infinite-dimensional Lie algebras, Advances in Mathematics 226:2 (2011) 1911–1972.

[113] W. de Graaf, Classification of Solvable Lie Algebras, Experimental Mathematics 14:1 (2005) 15–25.

[114] W. de Graaf, Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2, Journal of Algebra 309:2 (2007), 640–653.

[115] V. Gritsenko, Abstracts of Conference Automorphic forms, Kac-Moody Lie algebras and Strings, 2010. http://www.mpim-bonn.mpg.de/node/178

[116] V.A. Gritsenko, V.V. Nikulin, On the classification of Lorentzian Kac-Moody algebras, Russian Mathematical Surveys 57:5 (2002) 921–979.

[117] V. Gritsenko, V.V. Nikulin, Lorentzian Kac-Moody algebras with Weyl groups of 2-reflections, Proceedings of the London Mathematical Society 116:3 (2018) 485–533.

[118] M.B. Halpern, N.A. Obers, Graph theory, SO(n) current algebra and the Virasoro master equation, Communications in Mathematical Physics 138:1 (1991) 63–105.

[119] M. Hazewinkel, Kac–Moody algebra, In: Encyclopaedia of Mathematics 3, Springer, 1989, pp. 374–376.

[120] T. He, P. Mitra, A. Strominger, 2D Kac-Moody symmetry of 4D Yang-Mills theory, Journal of High Energy Physics 10 (2016) 137, 19 pp.

[121] A.S. Hegazi, H. Abdelwahab, A.J. Calderon Martin, The classification of n-dimensional non-Lie Malcev algebras with $(n-4)$-dimensional annihilator, Linear Algebra and its Applications 505 (2016) 32–56.

[122] D. Hill, W. Wang, Categorification of quantum Kac-Moody superalgebras, Transactions of the American Mathematical Society 367:2 (2015) 1183–1216.

[123] G. Höhn, N.R. Scheithauer, A generalized Kac-Moody algebra of rank 14, Journal of Algebra 404 (2014) 222–239.

[124] B. Hou, S. Yang, Generalized McKay quivers, root system and Kac-Moody algebras, Journal of the Korean Mathematical Society 52:2 (2015) 239–268.

[125] C. Hoyt, Regular Kac-Moody superalgebras and integrable highest weight modules, Journal of Algebra 324:12 (2010) 3308–3354.

[126] C. Hoyt, V. Serganova, Classification of finite-growth general Kac-Moody superalgebras, Communications in Algebra 35:3 (2007) 851–874.

[127] K. Iohara, Y. Koga, Enright functors for Kac-Moody superalgebras, Abhandlungen aus dem Mathematischen Seminar der

Universität Hamburg 82 (2012) 205–226.
[128] M. Ishii, Path model for representations of generalized Kac-Moody algebras, Journal of Algebra 379 (2013) 277–300.
[129] M. Ishii, Demazure character formulas for generalized Kac-Moody algebras, Communications in Algebra 43:7 (2015) 2785–2801.
[130] V.D. Ivashchuk, V.N. Melnikov, Black brane solutions related to non-singular Kac-Moody algebras, Gravitation and Cosmology 17:1 (2011) 7–17.
[131] N. Jacobson, Abstract derivation and Lie algebras, Transactions of the American Mathematical Society 42:2 (1937) 206–224.
[132] N. Jacobson, Classes of restricted Lie algebras of characteristic p, I, American Journal of Mathematics 63:3 (1941) 481–515.
[133] E. Jamsin, J. Palmkvist, Lie and Kac-Moody algebras, Lecture Notes. Universite Libre de Bruxelles and International Solvay Institutes 101, 2010.
[134] K. Jeong, S.J. Kang, M. Kashiwara, Crystal bases for quantum generalized Kac-Moody algebras, Proceedings of the London Mathematical Society 90:2 (2005) 395–438.
[135] K. Jeong, S.J. Kang, M. Kashiwara, D.U. Shin, Abstract crystals for quantum generalized Kac-Moody algebras, International Mathematics Research Notices 2007 (2007) rnm001, 18 pp.
[136] K. Jeong, S.J. Kang, J.A. Kim, D.U. Shin, Crystals and Nakajima monomials for quantum generalized Kac-Moody algebras, Journal of Algebra 319:9 (2008) 3732–3751.
[137] A. Joseph, P. Lamprou, A Littelmann path model for crystals of generalized Kac-Moody algebras, Advances in Mathematics 221:6 (2009) 2019–2058.
[138] V.G. Kac, Simple graded algebras of finite growth, Functional Analysis and Its Applications 1:4 (1967) 328–329.
[139] V.G. Kac, Simple irreducible graded Lie algebras of finite growth, Mathematics of the USSR-Izvestiya 2:6 (1968) 1271–1311.
[140] V.G. Kac, On the classification of simple Lie algebras over a field of nonzero characteristic, Mathematics of the USSR-Izvestiya 4:2 (1970) 391–413.
[141] V.G. Kac, Infinite dimensional Lie algebras, 3rd edition, Cambridge University Press, 1990.
[142] V. Kac, Introduction to Lie algebras. Lecture 4: Nilpotent and solvable Lie algebras, 2004. https://math.mit.edu/classes/18.745/Notes/Lecture_4_Notes.pdf
[143] S. Kanagavel, Nilpotent Lie algebras of type $A_2^{(1)}$, $B_2^{(1)}$, $G_2^{(1)}$, manuscript.
[144] S.J. Kang, M. Kashiwara, S.J. Oh, Categorification of highest weight modules over quantum generalized Kac-Moody algebras, Moscow Mathematical Journal 13:2 (2013) 315–343.
[145] S.J. Kang, M. Kashiwara, S.J. Oh, Supercategorification of quantum Kac-Moody algebras II, Advances in Mathematics 265 (2014) 169–240.
[146] S.J. Kang, M. Kashiwara, O. Schiffmann, Geometric construction of crystal bases for quantum generalized Kac-Moody algebras, Advances in Mathematics 222:3 (2009) 996–1015.
[147] S.J. Kang, M. Kashiwara, O. Schiffmann, Geometric construction of highest weight crystals for quantum generalized Kac–Moody algebras, Mathematische Annalen 354:1 (2012) 193–208.
[148] S.J. Kang, D.I. Lee, E. Park, H. Park, Generalized Castelnuovo-Mumford regularity for affine Kac-Moody algebras, Journal of Algebra 341 (2011) 13–22.
[149] S.J. Kang, K.H. Lee, K. Lee, A combinatorial approach to root multiplicities of rank 2 hyperbolic Kac-Moody algebras, Communications in Algebra 45:11 (2017) 4785–4800.
[150] S.J. Kang, S.J. Oh, E. Park, Perfect bases for integrable modules over generalized Kac-Moody algebras, Algebras and Representation Theory 14:3 (2011) 571–587.
[151] S.J. Kang, S.J. Oh, E. Park, Categorification of Quantum Generalized Kac-Moody Algebras and Crystal Bases, International Journal of Mathematics 23:11 (2012) 1250116, 51 pp.
[152] I. Kaplansky, Lie algebras of characteristic p, Transactions of the American Mathematical Society 89:1 (1958) 149–183.
[153] E. Khamseh, S.A. Niri, Classification of pair of nilpotent Lie algebras by their Schur multipliers, Mathematical Reports (Bucuresti) 20(70):2 (2018) 177–185.
[154] M. Khovanov, A. Lauda, A diagrammatic approach to categorification of quantum groups III, Quantum Topology 1:1 (2010) 1–92.
[155] M. Khovanov, A. Lauda, A diagrammatic approach to categorification of quantum groups II, Transactions of the American Mathematical Society 363:5 (2011) 2685–2700.
[156] A.Kh. Khudoyberdyev, Z.Kh. Shermatova, Description of solvable Leibniz algebras with four-dimensional nilradical, In: Topics in Functional Analysis and Algebra, Contemporary Mathematics 672, American Mathematical Society, 2016, pp. 217–224.
[157] A. Kleinschmidt, I. Schnakenburg, P. West, Very Extended Kac-Moody Algebras and their Interpretation at Low Levels, Classical and Quantum Gravity 21:9 (2004) 2493–2525.
[158] A.I. Kostrikin, Irreducible graded Lie algebras with the component $L_0 = W_1$, Ural Gosudarstvenny Universitet, Matematicheskie Zapiski 7:3 (1969/70) 92–103.
[159] A.I. Kostrikin, I.R. Safarevic, Cartan's pseudogroups and the p-algebras of Lie, Doklady Akademii Nauk SSSR 168:4 (1966) 740–742.
[160] A.I. Kostrikin, I.R. Safarevic, Graded Lie algebras of finite characteristic, Mathematics of the USSR-Izvestiya 3:2 (1969) 237–304.
[161] J. Kübel, A Jantzen sum formula for restricted Verma modules over affine Kac-Moody algebras at the critical level, Pacific Journal of Mathematics 269:2 (2014) 371–384.
[162] S. Kumar, Bernstein-Gelfand-Gelfand resolution for arbitrary Kac-Moody algebras, Mathematische Annalen 286:4 (1990) 709–729.
[163] S. Kumar, Kac-Moody Groups, their Flag Varieties and Representation Theory, 1st edition, Progress in Mathematics 204, Springer, Birkhäuser, 2002.
[164] R.A. Laamara, A. Belhaj, L.J. Boya, L. Medari, A. Segui, On F-theory quiver models and Kac-Moody algebras, International Journal of Geometric Methods in Modern Physics 7:6 (2010) 989–999.

[165] D.V. Lagno, A characterization of imaginary roots of hyperbolic Kac-Moody algebras, Russian Mathematical Surveys 54:3 (1999) 636–637.
[166] T.F. Lam, M. Shimozono, Dual graded graphs for Kac-Moody algebras, Algebra and Number Theory 1:4 (2007) 451–488.
[167] M. Lanini, On the stable moment graph of an affine Kac-Moody algebra, Transactions of the American Mathematical Society 367:6 (2015) 4111–4156.
[168] G. Leger, S. Togo, Characteristically nilpotent Lie algebras, Duke Mathematical Journal 26:4 (1959) 623–628.
[169] C. Lenart, A. Postnikov, A combinatorial model for crystals of Kac-Moody algebras, Transactions of the American Mathematical Society 360:8 (2008) 4349–4381.
[170] N.C. Leung, M. Xu, J. Zhang, Kac-Moody $\widetilde{E_k}$-bundles over elliptic curves and del Pezzo surfaces with singularities of type A, Mathematische Annalen 352:4 (2012) 805–828.
[171] E.E. Levi, Sulla struttura dei gruppi finiti e continui, Atti della Reale Accademia delle Scienze di Torino XL (1905) 551–565.
[172] F. Levstein, A classification of involutive automorphisms of an affine Kac-Moody Lie algebra, Journal of Algebra 114:2 (1988) 489–518.
[173] C. Lu, H. Xu, A sufficient condition for pairing problem of generators in symmetrizable Kac-Moody algebras, Journal of Algebra 260:2 (2003) 570–576.
[174] C. Lu, H. Zhang, The isomorphic realization of nondegenerate solvable Lie algebras of maximal rank based on Kac-Moody algebras, Science China Mathematics 56:5 (2013) 995–1014.
[175] G. Lusztig, Hecke algebras and Jantzens generic decomposition patterns, Advances in Mathematics 37:2 (1980) 121–164.
[176] R. Lv, Y. Tan, Some gim algebras and the associated indefinite Kac-Moody algebras, Journal of Algebra 425 (2015) 42–64.
[177] A.U. Maheswari, Fuzzy decomposition on the affine Kac-Moody algebras, International Journal of Pure and Applied Mathematics 80:1 (2012) 75–91.
[178] A.I. Malcev, Solvable Lie algebras, American Mathematical Society Translations 1950:27 (1950) 228–262 (Translated from Akademii Nauk SSSR. Seriya Matematicheskaya 9:5 (1945) 329–356).
[179] T. Marquis, An Introduction to Kac-Moody Groups over Fields, European Mathematical Society Textbooks in Mathematics 22, 2018.
[180] O. Mathieu, Sur la construction de groupes associès aux algèbres de Kac-Moody, Comptes Rendus de l'Académie des Sciences Paris Série I (Sciences Mathématiques) 299:6 (1984) 161–164.
[181] O. Mathieu, Formules de caractères pour les algèbres de Kac-Moody gènèrales, Astérisque 159-160 (1988) 267 pp.
[182] O. Mathieu, On some modular representations of affine Kac-Moody algebras at the critical level, Compositio Mathematica 102:3 (1996) 305–312.
[183] G.M. Melikyan, Simple Lie algebras of characteristic 5, Uspekhi Matematicheskikh Nauk 35:1 (211) (1980) 203–204.
[184] V.G. Mikhalëv, A generalization of the Kac-Moody algebras with a parameter on an algebraic curve and perturbations of solitons, Communications in Mathematical Physics 134:3 (1990) 633–646.
[185] K. Miki, Determinant of twisted bosons related to the Kac-Moody algebra, Modern Physics Letters A 3:2 (1988) 191–200.
[186] R.V. Moody, Lie Algebras associated with generalized Cartan matrices, Bulletin of the American Mathematical Society 73:2 (1967) 217–221.
[187] R.V. Moody, A new class of Lie algebras, Journal of Algebra 10 (1968) 211–230.
[188] R.V. Moody, A. Pianzola, Lie algebras with triangular decompositions, Canadian Mathematics Series of Monographs and Texts, Wiley-Interscience, New York, 1995.
[189] R.V. Moody, A. Pianzola, Kac-Moody Lie algebras: a survey of their algebraic development over the first 30 years, In: CMS Books in Mathematics 3, Canadian Mathematical Society, 1996, pp. 211–245.
[190] J. Morita, Roots of Kac-Moody Lie algebras, Tsukuba Journal of Mathematics 4:2 (1980) 349–354.
[191] B. Mühlherr, K. Struyve, Extremal elements in Coxeter groups and metric commensurators of Kac-Moody groups, Mathematische Zeitschrift 288 (2018) 287–298.
[192] S. Naito, Kostant's formula for a certain class of generalized Kac-Moody algebras, Tohoku Mathematical Journal 44:4 (1992) 567–580.
[193] S. Naito, Kostant's formula and homology vanishing theorems for generalized Kac-Moody algebras, Journal of Mathematics of Kyoto University 33:3 (1993), 709–722.
[194] T. Nakashima, A. Zelevinsky, Polyhedral realizations of crystal bases for quantized Kac-Moody algebras, Advances in Mathematics 131:1 (1997) 253–278.
[195] R.M. Navarro, Classification of filiform Lie algebras of order 3, Journal of Geometry and Physics 110 (2016) 248–258.
[196] S. Nayak, K.C. Pati, Embedding of simply laced hyperbolic Kac-Moody superalgebras, Journal of Mathematical Physics 54:3 (2013) 033508, 10 pp.
[197] E. Neher, D. Nguyen, Finite gradings of Kac-Moody algebras, Algebras, Groups and Geometries 7:1 (1990) 95–113.
[198] V.V. Nikulin, A theory of Lorentzian Kac-Moody algebras, Journal of Mathematical Sciences 106:4 (2001) 3212–3221.
[199] P. Niroomand, On dimension of the Schur multiplier of nilpotent Lie algebras, Central European Journal of Mathematics 9:1 (2011) 57–64.
[200] S. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons: The Inverse Scattering Method, Monographs in Contemporary Mathematics, Springer US, 1984.
[201] T.A. Nutma, Kac-Moody symmetries and gauged supergravity, PhD thesis, 2010.
https://research.rug.nl/en/publications/kac-moody-symmetries-and-gauged-supergravity
[202] D.I. Olive, Kac-Moody algebras: an introduction for physicists, Proceedings of the Winter School "Geometry and Physics", Rendiconti del Circolo Matematico di Palermo, Serie II, 9 (1985) 177–198.
[203] S. Ojima, A note on Krichever-Novikov-Kac-Moody algebra on Riemann surfaces, Modern Physics Letters A 5:27 (1990) 2215–2221.

[204] K.C. Pati, K.C. Tripathy, Super Kac-Moody algebra: classification theory I, Indian Journal of Pure and Applied Mathematics 21:5 (1990) 485–499.
[205] L. Peng, J. Xiao, Triangulated categories and Kac-Moody algebras, Inventiones Mathematicae 140 (2000) 563–603.
[206] D. Persson, N. Tabti, Lectures on Kac-Moody Algebras with Applications in Super-Gravity, Lecture Notes of the Third International Modave Summer School in Mathematical Physics, Modave (Belgium) 2007.
[207] D.H. Peterson, V.G. Kac, Infinite flag varieties and conjugacy theorems, Proceedings of the National Academy of Sciences of the United States of America 80:6 (1983) 1778–1782.
[208] G. Pezzini, Spherical subgroups of Kac-Moody groups and transitive actions on spherical varieties, Advances in Mathematics 312 (2017) 680–736.
[209] E. Poroshenko, Gröbner-Shirshov bases for Kac-Moody algebras $A_n^{(1)}$ and $B_n^{(1)}$, In: Formal power series and algebraic combinatorics (Proceedings of the 12th International Conference, FPSAC'00, Moscow, Russia, 2000), Springer, Berlin, 2000, pp. 552–563.
[210] A. Premet, A generalization of Wilson's theorem on Cartan subalgebras of simple Lie algebras, Journal of Algebra 167:3 (1994) 641–703.
[211] A. Premet, H. Strade, Simple Lie algebras of small characteristic. VI. Completion of the classification Journal of Algebra 320:9 (2008) 3559–3604.
[212] B. Ransingh, Vogan diagrams of untwisted affine Kac-Moody superalgebras, Asian-European Journal of Mathematics 6:4 (2013) 1350062, 10 pp.
[213] M. Rausch de Traubenberg, M.K. Slupinski, Kac-Moody algebras and Lie algebras of regular vector fields on tori, Journal of Algebra 253:1 (2002) 189–208.
[214] U. Ray, Generalized Kac-Moody algebras and some related topics, Bulletin of the American Mathematical Society 38:1 (2001) 1–42.
[215] B. Ren, L.S. Zhu, Classification of 2-step nilpotent Lie algebras of dimension 8 with 2-dimensional center, Communications in Algebra 39:6 (2011) 2068–2081.
[216] B. Ren, L.S. Zhu, Classification of 2-step nilpotent Lie algebras of dimension 9 with 2-dimensional center, Czechoslovak Mathematical Journal 67(142):4 (2017) 953–965.
[217] P. Revoy, Algèbres de Lie metabeliennes, Annales de la Faculté des sciences de Toulouse: Mathématiques (Série 5) 2:2 (1980) 93–100.
[218] A. Ros Camacho, Kac-Moody algebras, Notes of a talk, 2010.
https://www.math.uni-hamburg.de/home/wockel/teaching/data/cohomology_of_lie_algebras_ros_camacho.pdf
[219] R. Rouquier, 2-Kac-Moody algebras,
https://arxiv.org/abs/0812.5023
[220] S.N.M. Ruijsenaars, Dirac quantization, Virasoro and Kac-Moody algebras, and boson-fermion correspondence in 2N dimensions, In: Proceedings IXth International on Mathematical Physics (Swansea, 1988), pp. 479–482.
[221] M. Sakamoto, Bosonization of bosons in vertex operator representations of affine Kac-Moody algebras, Progress of Theoretical Physics 84:2 (1990) 351–359.
[222] L. Santharoubane, Kac-Moody Lie algebras and the classification of nilpotent Lie algebras of maximal rank, Canadian Journal of Mathematics 34:6 (1982) 1215–1239.
[223] L. Santharoubane, Nilpotent Lie algebras of Kac-Moody affine type, Journal of Algebra 302:2 (2006) 553–585.
[224] A. Savage, Crystals, quiver varieties, and coboundary categories for Kac-Moody algebras, Advances in Mathematics 221:1 (2009) 22–53.
[225] N.R. Scheithauer, Generalized Kac-Moody algebras, automorphic forms and Conways group, Advances in Mathematics 183:2 (2004) 240–270.
[226] N.R. Scheithauer, On the classification of automorphic products and generalized Kac-Moody algebras, Inventiones Mathematicae 164 (2006) 641–678.
[227] J. Scheuneman, Two-step nilpotent Lie algebras, Journal of Algebra 7 (1967) 152–159.
[228] C. Schneider, A computer-based approach to the classification of nilpotent Lie algebras, Experimental Mathematics 14:2 (2005) 153–160.
[229] C. Schneider, H. Usefi, The classification of p-nilpotent restricted Lie algebras of dimension at most 4, Forum Mathematicum 28:4 (2016) 713–727.
[230] C. Seeley, Some nilpotent Lie algebras of even dimensions, Bulletin of the Australian Mathematical Society 45 (1992) 71–77.
[231] A. Shalev, Subgroup structure, fractal dimension, and Kac-Moody algebras, Groups and geometries, In: and Geometries, Trends in Mathematics, Birkhäuser, Basel, 1998, pp. 163–176.
[232] A. Shalev, E.I. Zelmanov, Narrow Lie algebras: A coclass theory and a characterization of the Witt algebra, Journal of Algebra 189:2 (1997) 294–331.
[233] P. Shan, M. Varagnolo, E. Vasserot, Koszul duality of affine Kac-Moody algebras and cyclotomic rational double affine Hecke algebras, Advances in Mathematics 262 (2014) 370–435.
[234] S.S. Sharma, S. Viswanath, The t-analog of the basic string function for twisted affine Kac-Moody algebras, Journal of Algebra 363 (2012) 19–28.
[235] S.S. Sharma, S. Viswanath, The t-analog of the level one string function for twisted affine Kac-Moody algebras, Comptes Rendus de l'Académie des Sciences Series I Mathematics 350:3-4 (2012) 133–136.
[236] D.U. Shin, Polyhedral realization of crystal bases for generalized Kac-Moody algebras, Journal of the London Mathematical Society 77:2 (2008) 273–286.
[237] D.U. Shin, Polyhedral realization of the highest weight crystals for generalized Kac-Moody algebras, Transactions of the American Mathematical Society 360:12 (2008) 6371–6387.

[238] D.S. Shirokov, Classification of Lie algebras of specific type in complexified Clifford algebras, Linear and Multilinear Algebra 66:9 (2018) 1870–1887.
[239] P.E. Singer, Products of weight subspaces of Kac-Moody algebras, Communications in Algebra 15:12 (1987) 2511–2528.
[240] P.E. Singer, A digraph and a product of subspaces of a Kac-Moody algebra, In: Modern trends in Lie algebra representation theory (Kingston, ON, 1993), Queen's Papers in Pure and Applied Mathematics 94, Queen's Univiversity, Kingston, ON, 1994, pp. 88–92.
[241] A.K. Singh, Some studies on dynkin diagrams associated with Kac-moody algebra, Ph.D. Doctoral Thesis, 2011. http://ethesis.nitrkl.ac.in/2195/1/ethesis.pdf
[242] S.M. Skryabin, A contragredient Lie algebra of dimension 29 over a field of characteristic 3, Siberian Mathematical Journal 34:3 (1993) 548–554.
[243] W. Slofstra, A Brylinski filtration for affine Kac-Moody algebras, Advances in Mathematics 229:2 (2012) 968–983.
[244] E.M. Souidi, M. Zaoui, Maximal subalgebras of Kac-Moody algebras, Communications in Algebra 32:7 (2004) 2573–2588.
[245] I. Sterling, Soliton geometry, Kac-Moody algebras and the Yang-Baxter equations, In: Proceedings of Workshop on Theoretical and Numerical Aspects of Geometric Variational Problems (Canberra, 1990), Proceedings of the Centre for Mathematics and Its Applications 26, Australian National Univiversity, Canberra, 1991, pp. 225–230.
[246] N. Sthanumoorthy, P.L. Lilly, Complete classifications of generalized Kac-Moody algebras possessing special imaginary roots and strictly imaginary property, Communications in Algebra 35:8 (2007) 2450–2471.
[247] N. Sthanumoorthy, P.L. Lilly, Root multiplicities of some generalized Kac-Moody algebras, Indian Journal of Pure and Applied Mathematics 38:2 (2007) 55–78.
[248] N. Sthanumoorthy, A.U. Maheswari, P.L. Lilly, Extended-hyperbolic Kac-Moody algebras $EHA_2^{(2)}$ structure and root multiplicities, Communications in Algebra 32:6 (2004), 2457–2476.
[249] H. Strade, The classification of the simple modular Lie algebras. VI. Solving the final case, Transactions of the American Mathematical Society 350:7 (1998) 2553–2628.
[250] H. Strade, Simple Lie algebras over fields of positive characteristic. I. Structure theory, de Gruyter Expositions in Mathematics 38, Walter de Gruyter and Company, Berlin, 2004.
[251] H. Strade, Simple Lie algebras over fields of positive characteristic. II. Classifying the absolute toral rank two case, de Gruyter Expositions in Mathematics 42, Walter de Gruyter and Company, Berlin, 2009.
[252] H. Strade, Simple Lie algebras over fields of positive characteristic. III. Completion of the classification, de Gruyter Expositions in Mathematics 57, Walter de Gruyter and Company, Berlin, 2013.
[253] H. Sugawara, A field theory of currents, Physical Review 170 (1968) 1659–1662.
[254] C.P. Sun, Boson realisation of Virasoro and Kac-Moody algebras and their indecomposable representations, Journal of Physics A: Mathematical and General 20:17 (1987) L1157–L1159.
[255] T.A. Terlep, J. Williford, Graphs from generalized Kac-Moody algebras, SIAM Journal on Discrete Mathematics 26:3 (2012) 1112–1120.
[256] H.B. Thacker, Spin chain deformations, lattice fermion masses, and noncritical Virasoro and Kac-Moody algebras, In: Topological and Geometrical Methods in Field Theory, Proceedings of the 2nd International Symposium (Turku, 1991), World Sci. Publ., River Edge, NJ, 1992, pp. 406–422.
[257] J. Tits, Groupes associès aux algèbres de Kac-Moody, Astèrisque 177-178 (1988/89) Séminaire Bourbaki, exp. n^o 700, 7–31.
[258] G. Tsagas, Classification of nilpotent Lie algebras of dimension eight, Journal of Institute of Mathematics & Computer Sciences. Mathematics series 12:3 (1999) 179–183.
[259] G. Tsagas, X. Agrafiotou, Hyperbolic Kac-Moody algebras, In: Proceedings of the Workshop on Global Analysis, Differential Geometry, Lie Algebras (Thessaloniki, 1996), Balkan Society of Geometers, Geometry Balkan Press, Bucharest, 1999, pp. 104–107.
[260] G. Tsagas, A. Kobotis, T. Koukouvinos, Classification of nilpotent Lie algebras of dimension nine whose maximum abelian ideal is of the dimension seven, International Journal of Computer Mathematics 74:1 (2000) 5–28.
[261] K.A. Umlauf, Über die Zusammensetzung der endlichen continuierliechen Transformationsgruppen insbesondere der gruppen vom range null, Ph. D. Thesis, Univ. Leipzig, 1891.
[262] H.J. de Vega, Yang-Baxter and Kac-Moody algebras in field theory with solitons. Solitons and coherent structures, Physica D: Nonlinear Phenomena 18:1-3 (1986) 374–377.
[263] B.Y. Veisfeiler, V.G. Kac, Exponentials in Lie algebras of characteristic p, Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya 35:4 (1971) 762–788.
[264] R. Venkatesh, S. Viswanath, Chromatic polynomials of graphs from Kac-Moody algebras, Journal of Algebraic Combinatorics 41:4 (2015) 1133–1142.
[265] M. Vergne, Sur la variété des algèbres de Lie nilpotentes, Thèse de 3^e Cycle, Paris, 1966.
[266] M. Vergne, Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes, Bulletin de la Société Mathématique de France 98 (1970) 81–116.
[267] Z.X. Wan, Introduction to Kac-Moody algebra, World Scientific Publishing Company, 1991.
[268] D.Y. Wang, Overalgebras of Borel subalgebras in Kac-Moody algebras, Journal of Pure and Applied Mathematics 17:4 (2001) 295–297.
[269] A. Wassermann, Kac-Moody and Virasoro algebras, 2010, https://arxiv.org/pdf/1004.1287.pdf
[270] R.L. Wilson, A structural characterization of the simple Lie algebras of generalized Cartan type over fields of prime characteristic, Journal of Algebra 40:2 (1976) 418–465.
[271] R.L. Wilson, Simple Lie algebras of toral rank one, Transactions of the American Mathematical Society 236 (1978) 287–295.
[272] P. Windey, Super-Kac-Moody algebras and supersymmetric 2d-free fermions, Communications in Mathematical Physics 105:4 (1986) 511–518.

[273] Z. Wu, The weak Hopf algebras related to generalized Kac-Moody algebra, Journal of Mathematical Physics 47:6 (2006) 062108, 13 pp.

[274] J. Xiao, F. Xu, Hall algebras associated to triangulated categories, Duke Mathematical Journal 143:2 (2008) 357–373.

[275] F. Xu, X. Chen, Some remarks on triangulated categories and Kac-Moody algebras, Journal of Algebra 388 (2013) 22–42.

[276] J.H. Yang, Kac-Moody Algebras, the Monstrous Moonshine, Jacobi Forms and Infinite Products, 2006, https://arxiv.org/pdf/math/0612474.pdf

[277] M.V. Zaitsev, Varieties of affine Kac-Moody algebras, Mathematical Notes 62:1 (1997) 80–86.

[278] H. Zassenhaus, Uber Liesche Ringe mit Primzahlcharakteristik, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 13 (1939) 1–100.

[279] Q. Zhang, Z.X. Xia, Special imaginary roots of a class of generalized Kac-Moody algebras, Journal of Hebei Normal University (Natural Science Edition) 24:2 (2000) 166–168.

[280] J.B. Zuber, Classification of modular invariants made of characters of the Virasoro or Kac-Moody algebras, In: Proceedings of Infinite-dimensional Lie algebras and quantum field theory (Varna, 1987), World Sci. Publ., Teaneck, NJ, 1988, pp. 124–133.

[281] A. Zuevsky, Algebraic Properties of the Semi-direct Product of Kac-Moody and Virasoro Lie Algebras and Associated Bi-Hamiltonian Systems, In: Differential and Difference Equations with Applications, ICDDEA 2015, Springer Proceedings in Mathematics & Statistics 164, Springer, 2016, pp. 1–7.

[282] A. Zuevsky, Algebraic proofs for shallow water bi-Hamiltonian systems for three cocycle of the semi-direct product of Kac-Moody and Virasoro Lie algebras, Open Mathematics 16:1 (2018) 1–8.