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SS-Discrete Modules

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Abstract. In this paper, we define (strongly) $ss-$discrete, semi-$ss-$discrete and quasi-$ss-$discrete modules as a strongly notion of (strongly) discrete, semi-discrete and quasi-discrete modules with the help of $ss-$supplements in [3]. We examined the basic properties of these modules and included characterization of strongly $ss-$discrete modules over semi-perfect rings.

1. Introduction

In this study, R is used to show a ring which is associative and has an identity. All mentioned modules will be unital left $R-$modules. Let M be an $R-$module. The notation $A \leq M$ means that A is a submodule of M. Any submodule A of an $R-$module M is called *small* in M and showed by $A \ll M$ whenever $A + C \neq M$ for all proper submodule C of M. The Jacobson radical of M denoted by $Rad(M)$. Dually, a submodule A of a $R-$module M is called to be *essential* in M which is showed by $A \trianglelefteq M$ if $A \cap K \neq 0$ for each non-zero submodule K of M. The socle of M which is the sum of all simple submodules of M is denoted by $Soc(M)$. A non-zero module M is called *hollow* if every proper submodule of M is small in M and is called *local* providing that the sum of all proper submodules of M is also a proper submodule of M. A submodule N of M is called *coclosed* in M if whenever $\frac{N}{K} \ll \frac{M}{K}$ for a submodule K of M with $K \subseteq N$, $N = K$.

Let A and B be submodules of a module M. Then A is called a *supplement* of B in M when A is minimal with the property $M = A + B$; in other words, $M = A + B$ and $A \cap B \ll A$. M is said to be *supplemented* if every submodule of M has a supplement in M. Two submodules A and B of M are called *mutual supplements* in M if, $M = A + B$, $A \cap B \ll A$ and $A \cap B \ll B$ [1]. There are a lot of papers related with supplemented modules such as [7, 8]. If M is supplemented and self-projective, then M is called *strongly discrete*. The module M is called *amply supplemented* if for any submodules A and B of M with $M = A + B$, there exists a supplement X of A such that $X \subseteq B$.

In [7], a module M is called *lifting* if for every submodule A of M lies over a direct summand, that is, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$, $A \cap M_2 \ll M_2$. By [8], M is lifting iff M is amply supplemented and every supplement submodule of M is a direct summand of it.

Following [9], the sum of all simple submodules of M which are small in M is named with $Soc_s(M)$, that is, $Soc_s(M) = \sum \{A \ll M | A \text{ is simple}\}$. Note that $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$. In [3], a module M is called *strongly local* providing that M is local and $Rad(M) \subseteq Soc(M)$. In the same paper, a ring R is called *left strongly local ring* if ${}_RR$ is a strongly local module.

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According to [3], $ss-$supplemented modules was examined and founded as a strong notion of supplemented modules. Let M be a module and A, $B \leq M$. If $M = A + B$ and $A \cap B \subseteq Soc_s(B)$, then B is an *ss−supplement* of A in M. Any module M is named *ss−supplemented* if each submodule A of M has a $ss-$supplement B in M. As a result of this definition, any finitely generated module is $ss-$supplemented iff it is supplemented and $Rad(M) \subseteq Soc(M)$. In the same paper, amply $ss-$supplemented modules were defined. A submodule A of a module M has ample $ss-$supplements in M if A contains an $ss-$supplement of B in M with $M = A + B$. M is called *amply ss−supplemented* if every submodule of M has ample $ss-$supplements in M.

According to [2], a module M is called *semisimple lifting or briefly ss−*lifting if for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$, $A \cap M_2 \ll M$ and $A \cap M_2$ is semisimple. Some new fundamental properties of $ss-$lifting modules will be examined in this paper.

Let c be a cardinal number. The module M is said to have the c-internal exchange property if every decomposition $M = \bigoplus_I M_i$ with $card(I) \leq c$ is exchangeable. A module M has the *(finite) internal exchange property* if it has the c-internal exchange property for every (finite) cardinal c [1, 11.34]. A lifting module with the finite internal exchange property is called a *semi-discrete module*. The module M is called *discrete* if M is lifting and satisfies the following condition:

(D_2) : If $N \subseteq M$ such that $\frac{M}{N}$ is isomorphic to a direct summand of M, then N is a direct summand of M.

The module M is called *quasi-discrete* if M is lifting and satisfies the following condition;

(D_3) : If N and K are direct summands of M such that $M = N + K$, then $N \cap K$ is a direct summand of M (See [7]).

By [7, Lemma 4.6], (D_2) implies (D_3). In [1, 4.29], the notion of $\cap$-direct projective modules is defined as an equivalent condition to the property (D_3). By [1, 4.21], a module M is direct projective if and only if M has the property (D_2).

In the first part of this study, we define semi-$ss-$discrete and quasi-$ss-$discrete modules based on the definition of $ss-$lifting module. We give examples of these modules. We show that every quasi-$ss-$discrete module is $ss-$lifting and amply $ss-$supplemented. The factor module of a quasi-$ss-$discrete module is showed to be quasi-$ss-$discrete again under special conditions. In addition, theorems related with the decomposition of quasi-$ss-$discrete modules are obtained. In the second part, we define (strongly) $ss-$discrete modules and determine their relationship with $ss-$supplemented modules.

2. Semi-SS-Discrete and Quasi-SS-Discrete Modules

In this section, semi-ss-discrete modules and quasi-ss-discrete modules are defined and some of the basic features of these modules are obtained.

Definition 2.1. If M is an ss-lifting module with finite internal exchange property, then M is called a *semi-ss-discrete module*. If M is both a π-projective and ss-supplemented module, then M is called a *quasi-ss-discrete module*. Let N be any submodule of M. Any submodule K of M is called *N-ss-lifting* if every homomorphism $M \longrightarrow \frac{M}{N \cap K}$ where $N \cap K$ is semisimple lifts to an endomorphism of M. If K is a ss-supplement of N of M, then K is called a *N-lifting ss-supplement* in M.

Recall from [1] that a module K is said to be *generalized M-projective* if, for any epimorphism $g : M \longrightarrow X$ and homomorphism $f : K \longrightarrow X$, there exist decompositions $K = K_1 \oplus K_2$, $M = M_1 \oplus M_2$, a homomorphism $h_1 : K_1 \longrightarrow M_1$ and an epimorphism $h_2 : M_2 \longrightarrow K_2$, such that $g \circ h_1 = f_{|_{K_1}}$ and $f \circ h_2 = g_{|_{M_2}}$.

Proposition 2.2. *The following statements are equivalent for M:*

1. *M is semi-ss-discrete;*
2. *M is ss-supplemented, every ss-supplement in M is a direct summand and $K \cap L$ are relatively generalized projective, for every decomposition $M = K \oplus L$,*
3. *M is ss-lifting and K, L are relatively generalized projective, for every decomposition $M = K \oplus L$.*

Proof. (1) ⇒ (2) Since M is ss-lifting, it is ss-supplemented and every ss-supplement is a direct summand by [2, Theorem 1]. Let $M = N + K$. Then N contains an ss-supplement N' of K which is a direct summand of M. So, we have $M = N' \oplus L' \oplus K'$ with $L' \subseteq L$ and $K' \subseteq K$ since M has the finite internal exchange property. Thus L is generalized K-projective by [1, 4.42]. Similarly, it is easy to see that K is generalized L-projective.

(2) ⇒ (3) It is enough to prove that M is ss-lifting. Let $N \subseteq M$. By the hypothesis, N has an ss-supplement K which is a direct summand of M, that is $M = L \oplus K$. Then L is generalized K-projective and so $M = N' \oplus L' \oplus K' = N' + K$, where $N' \subseteq N$, $K' \subseteq K$ and $L' \subseteq L$ by [1, 4.42] since $M = N + K$. From here $N = N' + (N \cap K)$. Since $N \cap K \ll K$ and $N \cap K$ is semisimple, we have M is an ss-lifting module.

(3) ⇒ (1) Suppose $M = K \oplus L$. It is obtained from [2, Theorem 3] that K and L are ss-lifting modules, and so K and L are relatively generalized projective. It follows from [1, 23.10] that M has the 2-internal exchange property. □

Recall from [5] that a module M is called *duo* if for every submodule U of M is fully invariant, i.e. $f(U) \subseteq U$ for every $f \in End(M)$ and $U \subseteq M$.

Proposition 2.3. *Let $M = M_1 \oplus \ldots \oplus M_n$ be a duo module where each M_i is semi-ss-discrete. Then the following statements are equivalent:*

1. *M is semi-ss-discrete;*
2. *M is ss-lifting and $M = M_1 \oplus \ldots \oplus M_n$ is an exchange decomposition;*
3. *For any direct summand K of $\bigoplus_I M_i$ and any direct summand L of $\bigoplus_J M_j$, K and L are relatively generalized projective where I, J non-empty disjoint subsets of $\{1, 2, \ldots, n\}$;*
4. *If M_i' is any direct summand of M_i and T is any direct summand of $\bigoplus_{j \neq i} M_j$, then M_i' and T are relatively generalized projective for any $1 \leq i \leq n$;*

Proof. is clear by [1, 23.14] and [2, Theorem 10]. □

As an immediate consequence of Proposition 2.3, we have the following corollary.

Corollary 2.4. *Let $M = M_1 \oplus \ldots \oplus M_n$ be a duo module where each M_i is a semi-ss-discrete module. If M_i and M_j are relatively generalized projective for each $i \neq j$, then M is semi-ss-discrete.*

Recall from [1, 12.1] that an R-module M is said to be an *LE-module* if its endomorphism ring $End(M)$ is local.

Theorem 2.5. *Let M be an ss-lifting module with an indecomposable decomposition $M = \bigoplus_I M_i$ is a duo module. Then M is a semi-ss-discrete module if one of the following statements is satisfied:*

1. *M_i is an LE-module for all $i \in I$;*
2. *every non-zero direct summand of M contains a non-zero indecomposable direct summand and the decomposition $M = \bigoplus_{i \in I} M_i$ complements maximal direct summands.*

Proof. A module M with an indecomposable exchange decomposition has the internal exchange property. Hence we can apply [1, 24.13, 24.10] to [3, Theorem 30]. □

We can compare quasi-ss-discrete modules, ss-supplemented modules and ss-lifting modules in following lemmas.

Lemma 2.6. *If M is a quasi-ss-discrete module, then M is ss-lifting.*

Proof. Since M is π-projective, it is clear by [1, 20.9] and [2, Theorem 1] that ss-supplements are direct summands in M. So it is enough to prove that M is amply ss-supplemented. Suppose that $M = U + V$ and X is an ss-supplement of U in M. Then for any $f \in End(M)$ with $Im(f) \subseteq V$ and $Im(1 - f) \subseteq U$, we have $M = U + f(X)$ and $U \cap f(X) = f(U \cap X) \ll f(X)$. Since $U \cap X$ is semisimple, $U \cap f(X)$ is semisimple by [8, 20.3]. Thus $f(X)$ is an ss-supplement of U contained in V. □

By the help of [8, 41.15], it can be seen that if the intersection of any pair of mutual ss-supplements is zero in an ss-supplemented module, then ss-supplement submodules of M are direct summands.

Lemma 2.7. *If M is an ss-lifting and π-projective module, then M is amply ss-supplemented and the intersection of any pair of mutual ss-supplements in M is zero.*

Proof. Follows from [2, Theorem 1] and [1, 20.9]. □

Corollary 2.8. *If M is a quasi-ss-discrete module, then M is amply ss-supplemented and the intersection of any pair of mutual ss-supplements in M is zero.*

Proof. Clear by Lemmas 2.6 and 2.7. □

It is clear that every quasi-ss-discrete module is quasi-discrete by Definition 2.1. The following example shows that the converse is not need to be true. So the notion of quasi-ss-discrete module is a stronger than that of quasi-discrete module.

Example 2.9. *For any prime integer p, consider the left $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$. M is supplemented but not ss-supplemented by [3, Example 17]. Since M has the property (D_3), M is quasi-discrete but not quasi-ss-discrete.*

The following corollary is obtained by automatically by Lemma 2.7.

Corollary 2.10. *If M is an ss-lifting module and has the property (D_3), then M is a quasi-ss-discrete module.*

Lemma 2.11. *Let M be a quasi-ss-discrete module, K be a submodule of M and L be an ss−supplement of K. If N is an ss−supplement submodule of M contained in K, then $N \cap L = 0$ and $N \oplus L$ is a direct summand of M.*

Proof. Since M is a quasi-ss-discrete module, M is *ss*−lifting by Lemma 2.6. If we use [2, Theorem 1], it can be concluded that L and N are direct summand of M. Therefore there exists a submodule N_1 of M such that $M = N \oplus N_1$. It is clear that $K = (K \cap N_1) \oplus N$ and so $M = N + L + (K \cap N_1)$. By [2, Theorem 1], $K \cap N_1$ contains an *ss*−supplement X of $N + L$, where X is a direct summand of M. Thus $X \oplus N$ is a direct summand of M due to $X \leq N$. However, we have that $(X \oplus N) \cap L$ is a direct summand of M by [4.14 (4)]. From here $(X \oplus N) \cap L \leq K \cap L \subseteq Soc_s(L)$. Finally we can get $(X \oplus N) \cap L = 0$ and so $M = X \oplus N \oplus L$. □

Proposition 2.12. *If K, L are direct summand of a quasi-ss-discrete module M and L is hollow, then*
(*i*) *$K \cap L = 0$ and $K \oplus L$ is a direct summand of M or*
(*ii*) *$K + L = K \oplus S$ with $S \subseteq Soc_s(M)$ and L is isomorphic to a summand of K.*

Proof. Suppose that T is an *ss*−supplement of $K+L$. Then we have $M = T+(K + L)$ and $T \cap (K + L) \subseteq Soc_s(T)$. By Lemma 2.11, $K \cap T = 0$. Let's complete the proof by evaluating the following two situations.

(1) If $L \nleq K \oplus T$, then $L \cap (K + T) = 0$ and so L is an *ss*−supplement of $K + T$. It follows that $K \cap L = 0$ and $K \oplus L$ is a direct summand of M by Lemma 2.11.

(2) Assume that $L \leq K \oplus T$. Since $M = K + T + L = K + T$ and $K \cap T = 0$, we have $M = K \oplus T$. If we intersect the equality $M = K + T$ with $K + L$, then we can write $K + L = K \oplus S$ where $S = (K + L) \cap T$. Moreover $S \subseteq Soc_s(M)$ by [2, Theorem 1]. Since L is a direct summand of M, there exists a submodule L_1 of M such that $M = L \oplus L_1$. It follows that $M = K + L + L_1 = K + [(K + L) \cap T] + L_1 = K + L_1$ because $(K + L) \cap T \ll M$. Let N_1 be an *ss*−supplement of L_1 contained in K. Then, we get $M = [N_1 \oplus (K \cap L_1)] + L_1 = N_1 \oplus L_1$ and $L \cong N_1$. □

Theorem 2.13. *If M is a quasi-ss-discrete module, then M is ss-lifting and for every decomposition $M = K \oplus L$, K and L are relatively projective.*

Proof. We obtain by Lemmas 2.6 and 2.7 that M is amply ss-supplemented and the intersection of any pair of mutual ss-supplements in M is zero. Since M is ss-supplemented, ss-supplements are direct summands and so M is ss-lifting by [2, Theorem 1]. Suppose that $M = U + V$ where U and V are direct summands of M. Let X be an ss-supplement of V such that $X \subseteq U$. Then $M = X \oplus V$. As $U = X \oplus (U \cap V)$, we get $U \cap V$ is a direct summand of M. Therefore M is $\cap$-direct projective. The rest follows from [1, 4.14(2)]. □

By the definition, every quasi ss-discrete module is semi-ss-discrete. But the converse is not always true as in the following example.

Example 2.14. *Consider the $\mathbb{Z}$-module $U = \frac{\mathbb{Z}}{p\mathbb{Z}}$ and $V = \frac{\mathbb{Z}}{p^2\mathbb{Z}}$ where p is prime. Then U and V are relatively generalised projective but U is not V-projective. So M is not a quasi ss-discrete module although M is an ss-lifting module. Since $M = U \oplus V$ is a ss-lifting module with the finite internal exchange property, M is semi-ss-discrete.*

Now we can obtain properties of quasi ss-discrete modules.

Proposition 2.15. *Let M be a quasi-ss-discrete module. Then every direct summand of M is quasi-ss-discrete and every ss-supplement submodule of it is a direct summand.*

Proof. Let N be a direct summand of M. Since M is ss-lifting and π-projective, every ss-supplement submodule of M is a direct summand by [2, Theorem 1]. Since every direct summand of a π-projective module is again π-projective, N is ss-supplemented by [3, Corollary 38]. Therefore N is quasi-ss-discrete module. □

Since *ss*-supplemented modules are supplemented, proofs of the following facts are clear by [8, 41.16-(2,3)].

Lemma 2.16. *Let M be a quasi ss-discrete module and $S = End(M)$. Let $e \in S$ be an idempotent and N be a direct summand of M. If $(1-e)(N) \ll (1-e)(M)$, then $N \cap (1-e)(M) = 0$ and $N \oplus (1-e)(M)$ is a direct summand in M.*

Proposition 2.17. *Let M be a quasi-ss-discrete module. If $\{N_i\}_{i \in I}$ is a directed family of direct summands of M with respect to inclusion, then $\bigcup_{i \in I} N_i$ is also a direct summand in M.*

Recall from [3, Proposition 16] that an *ss*−supplemented hollow module is strongly local.

Lemma 2.18. *Let M be a quasi-ss-discrete module. Then for every $0 \neq m \in M$, there is a decomposition $M = M_1 \oplus M_2$ such that $m \notin M_1$ and M_2 is strongly local.*

Proof. Given $0 \neq m \in M$. Let's define the set $S = \{T \subset M \mid T \text{ is direct summand and } m \notin T\}$. This set is non-empty and inductive with respect to inclusion by Proposition 2.17 and has a maximal element M_1 by Zorn's Lemma. Since M_1 is a direct summand, there exists a submodule M_2 of M such that $M = M_1 \oplus M_2$. By Proposition 2.15 and Lemma 2.6, M_2 is a quasi-ss-discrete module and M_2 is *ss*−lifting. Therefore M_2 must be strongly local. If M_2 is not hollow, then there is a proper non-superfluous submodule in M_2, say U. It follows that there exists a nontrivial decomposition $M_2 = V \oplus V_1$ with $V \subset U$ and $U \cap V_1 \subseteq Soc_s(V_1)$ for some submodule V, V_1 of M_2. Then we can write $M = M_1 \oplus M_2 = M_1 \oplus V \oplus V_1$. By the maximality of M_1, we get $m \in M_1 \oplus V$ and $m \in M_1 \oplus V_1$. But this means $m \in M_1$ contradicting the choice of M_1. Therefore all proper submodules in M_2 are superfluous, i.e. M_2 is hollow. By [3, Proposition 16], we deduce that M_2 is strongly local. □

Observe from [3, Lemma 13] that an ss-supplemented and radical module is zero. Using this fact we prove that the following fact:

Theorem 2.19. *Let M be a quasi-ss-discrete module. Then M has a decomposition $M = \bigoplus_{i \in I} H_i$, where each H_i is strongly local. In particular, if N is a direct summand of M, there exists a subset $J \subset I$ such that $M = \left(\bigoplus_J H_i\right) \oplus N$.*

Proof. We indicate by Ω the set of all strongly local submodules in M and take into account $\Phi = \{\wp \subset \Omega | \sum_{H\in\wp} H$ is a direct sum and a direct summand in $M\}$. Then, since M is a quasi-*ss*-discrete module, M has a strongly local submodule that is a direct summand of its by [3, Lemma 13] and Lemma 2.6. So this set is non-empty and inductive with respect to inclusion by Proposition 2.17 has a maximal element $\wp$ by Zorn's Lemma. By indexing the elements in $\wp$ with i, let $L = \bigoplus_{i\in I} H_i$. Since L is a direct summand, there exists a submodule K of M such that $M = L \oplus K$. If we prove that $K = \{0\}$, then the proof will be completed. Suppose that $K \neq \{0\}$. Then, there is an element a of K with $a \neq 0$. Moreover, K is a quasi-*ss*-discrete module by Proposition 2.15. We get that a decomposition $K = K_1 \oplus K_2$ such that $a \notin K_1$ and K_2 is strongly local by Lemma 2.18. Then we have $M = L \oplus K = L \oplus K_1 \oplus K_2 = (L \oplus K_2) \oplus K_1$ and so $K_2 \neq \{0\}$ because of $a \notin K_1$. Therefore, the direct summand $L \oplus K_2$ of M is properly larger than L. This contradicts the maximality of L. Consequently $K = 0$ and we deduce that $M = \bigoplus_{i\in I} H_i$.

Suppose that N is a direct summand of M. Let's define $S = \{\Lambda \subset I | \ N \cap \left(\bigoplus_{\Lambda} H_\lambda\right) = \{0\}$ and $N \cap \left(\bigoplus_{\Lambda} H_\lambda\right)$ is a direct summand in $M\}$. By using Proposition 2.17 and Zorn's Lemma, we can say that S has a maximal element J. Assume that $L = N \cap \left(\bigoplus_{J} H_i\right)$. We must prove that $M = L$. Assume that $L \neq M$. Therefore there exists an element $a \in M \backslash L$. Then by Lemma 2.18, we have a decomposition $M = K \oplus H$ with $L \subset K$ and H is strongly local. If we show that $H = \{0\}$, then the proof is completed. Suppose that $H \neq \{0\}$. We consider the canonicial projection $p : M \to H$. It is clear that if $p\left(H_j\right) = H$ holds for some $j \in I$, then $M = K + H_j$. If $K \cap H_j = H_j$, then $M = K$ and so $H = \{0\}$. Because of $K \cap H_j \neq H_j$, we get that $K \cap H_j \ll H_j$. Since M is $\pi-$projective, we have $K \cap H_j = \{0\}$, i.e. $M = K \oplus H_j$. $L \oplus H_j$ is a direct summand of M because L is a direct summand of M. Since $j \notin J$, this is a contradiction to the maximality of J. It follows from $p(H_i) \neq H$ for every $i \in I$. From here, if we say $T = H_{i_1} \oplus H_{i_2} \oplus ..... \oplus H_{i_n}$ for every finite $i_1, i_2, ..., i_n \in I$, then $p(T) = p(H_{i_1}) \oplus p(H_{i_2}) \oplus ..... \oplus p(H_{i_n}) \ll H$. Moreover, for the canonicial projection $e : M \to K$, we get that $p = I_M - e$ and $p(T) = (I_M - e)(T) \ll H = (I_M - e)(M)$. Then we have $T \cap H = 0$ by Lemma 2.16. This situation is valid for every finite $i_1, i_2, ..., i_n$ we obtain $\left(\bigoplus_{I} H_i\right) \cap H = \{0\}$ and so $H = M \cap H = \{0\}$. It is a contradiction to the $H \neq \{0\}$. Hence $H = \{0\}$, this means $M = L$. □

Recall that a module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M. A ring R is called *left max* if every non-zero R-module has a maximal submodule. Note that if R is a left max ring, then every R-module is coatomic.

Corollary 2.20. *Let M be a quasi-ss-discrete. Then M is coatomic and Rad(M) is semisimple.*

Proof. It follows from Theorem 2.19 and [3, Theorem 27]. □

Proposition 2.21. *The following statements are equivalent for an amply ss-supplemented module M.*

1. *M is quasi-ss-discrete;*
2. *M is π-projective.*

Proof. Clear by [8, 41.15] and [3, Proposition 26]. □

Recall from [1, 4.13] that any factor module $\frac{M}{N}$ of a π-projective module M by a fully invariant submodule N is π-projective.

The following proposition can be proven by [3, Proposition 26].

Proposition 2.22. *Let M be a quasi-ss-discrete module and N be a fully invariant submodule of M. Then $\frac{M}{N}$ is quasi-ss-discrete.*

Proposition 2.23. *The following statements are equivalent for any module M.*

1. *M is quasi-ss-discrete;*
2. *M is amply ss-supplemented and all ss-supplements of any coclosed submodule N of M are K-ss-lifting.*

Proof. (1) ⇒ (2) It is clear that M is amply ss-supplemented by [3, Proposition 37]. Let N be a coclosed submodule of M and K be an ss-supplement of N in M. Then N and K are ss-supplements of each other and so $K \cap N = 0$ by [7, Proposition 4.11].

(2) ⇒ (1) It is enough to prove that M is π-projective. Let N and K be submodules of M with $M = N + K$. Since M is amply ss-supplemented, there exists a submodule K' of M such that $M = N + K'$, $N \cap K' \ll K'$, $N \cap K'$ is semisimple, $K' \subseteq K$ and a submodule N' of M such that $M = K' + N'$, $K' \cap N' \ll N'$, $K' \cap N'$ is semisimple and $N' \subseteq N$. Therefore K' and N' are ss-supplements of each other. Define $\varphi : M \longrightarrow \frac{M}{K' \cap N'}$ by $\varphi(k' + n') = k' + (K' \cap N')$ $(k' \in K', n' \in N')$. By the hypothesis, there exists a homomorphism $\theta : M \longrightarrow M$ where $\theta(M) \subseteq K'$ and $(1 - \theta)(M) \subseteq N'$. Hence M is π-projective. □

Lemma 2.24. *Let N be a submodule of M such that* $\frac{M}{N} \cong \frac{M}{N'}$ *with* N' *is a coclosed submodule of M. If K is a N-lifting ss-supplement, then* $M = N \oplus K$.

Proof. Suppose that K is an ss-supplement of N in M. Then we have $M = N + K$, $N \cap K \ll K$ and $N \cap K$ is semisimple, and every homomorphism $\psi : M \longrightarrow \frac{M}{N \cap K}$ lifts to a homomorphism of M. Since $\frac{M}{N} \cong \frac{M}{N'}$, then an isomorphism $\xi : \frac{M}{N'} \longrightarrow \frac{M}{N}$. We can similarly obtain rest of the proof follows from [4, Lemma 2.2]. □

Corollary 2.25. *Let N be a coclosed submodule of M. If K is a N-lifting ss-supplement in M, then* $M = N \oplus K$.

Proof. Clear by Lemma 2.24. □

In the following theorem, we give a characterization of ss-lifting modules via coclosed submodule from renaissance of [4, Theorem 2.4].

Theorem 2.26. *Let M be an amply ss-supplemented module. M is ss-lifting if and only if every coclosed submodule N of M has a N-lifting ss-supplement.*

Proof. Follows from Corollary 2.25 and [2, Theorem 1]. □

3. SS-Discrete Modules and Strongly SS-Discrete Modules

In this section, we define notions of ss-discrete modules and strongly ss-discrete modules, and we obtain some elementary characterizations of these notions.

Definition 3.1. Let M be a ss-supplemented module which is π-projective and direct projective, then M is called a *ss-discrete module*. If M is a ss-supplemented module which is self-projective, then M is called a *strongly ss-discrete module*.

By this definition, we can obtain that if a module M is ss-lifting and has the property (D_2), then M is a ss-discrete module.

Lemma 3.2. *Let N be an ss-supplement in M. N is a direct summand of M if and only if there exists an ss-supplement K of N in M such that K is a direct summand of M and every homomorphism* $f : M \longrightarrow \frac{M}{N \cap K}$ *can be lifted to a homomorphism* $\varphi : M \longrightarrow M$.

Proof. (⇒) Clear.

(⇐) Let K be an ss-supplement of N in M with the stated property and $f : M \longrightarrow \frac{M}{N \cap K}$ be the homomorphism defined by $f(a + b) = a + (N \cap K)$ for every $a \in N$ and $b \in K$. By the hypothesis, there exists a homomorphism $\varphi : M \longrightarrow M$ such that f can be lifted to the homomorphism φ. We have $M = K \oplus K'$ for some submodule K' of M and $K \cap N \ll N$ and $K \cap N$ is semisimple. By [6, Lemma 2.1], we have $M = \varphi(K') \oplus K$. Since $\varphi(K') \leq N$, then $N = \varphi(K') \oplus (N \cap K)$. This implies that $N \cap K = 0$. Thus N is a direct summand of M. □

Now we can characterize ss-lifting modules via the above lemma.

Corollary 3.3. *Let M be an amply ss-supplemented module. M is ss-lifting if and only if for every ss-supplement N in M there is a direct summand ss-supplement K of N in M such that every homomorphism* $f : M \longrightarrow \frac{M}{N \cap K}$ *can be lifted to a homomorphism* $\varphi : M \longrightarrow M$.

Proposition 3.4. *Let M be a module with* $Rad(M) \subseteq Soc(M)$. *If M is a (quasi-)discrete module, then M is a (quasi-)ss-discrete module.*

Proof. Clear by [3, Theorem 20]. □

Proposition 3.5. *Let M be an ss-discrete module. Then every direct summand of M is an ss-discrete module.*

Proof. Let N be a direct summand of M. Since M is direct projective by [1, 4.22], we have N is direct projective, i.e. N has the property (D_2). Since M is ss-supplemented and π-projective, M is ss-lifting by [2, Theorem 2]. Thus N is ss-lifting by [2, Theorem 3] and so N is an ss-discrete module. □

Example 3.6. *Consider the self-projective* $\mathbb{Z}$*-module* $M = \frac{\mathbb{Z}}{4\mathbb{Z}}$. *Since M is ss-supplemented, M is strongly ss-discrete.*

Proposition 3.7. *Let M be a projective module. M is a strongly ss-discrete module if and only if M is a strongly discrete module and* $Rad(M) \subseteq Soc(M)$.

Proof. Since M be a projective module, M is self-projective. The proof is obvious by [3, Theorem 20] □

Proposition 3.8. *Let M be a strongly ss-discrete module. Then every direct summand of M is a strongly ss-discrete module.*

Proof. As self-projective modules are closed under direct summands, the proof clear by [2, Theorem 3]. □

Theorem 3.9. *Let* $\{M_i\}_{i\in I}$ *be any finite family of R-modules and let* $M = \bigoplus_{i\in I} M_i$. *Suppose that for every* $i \in I$, $Rad(M_i) \subseteq Soc(M_i)$. *Then the following statements are equivalent.*

1. *M is strongly ss-discrete;*
2. (*a*) *each* M_i *is strongly discrete;*
   (*b*) *for each* $i \in I$, M_i *is* M_j*-projective for* $j \neq i$.

Proof. The proof similar to these of [1, 27.16] and [3, Theorem 20]. □

In the following corollary, we prove that strongly ss-discrete rings thanks to semiperfect ring.

Corollary 3.10. *The following statements are equivalent for a ring R:*

1. ${}_RR$ *is ss-supplemented;*
2. ${}_RR$ *is semiperfect and* $Rad(R) \subseteq Soc({}_RR)$;
3. *for any finite set I and for each* $i \in I$, *every left R-module* $M = \bigoplus_{i\in I} M_i$ *where* M_i *is a strongly local M-projective module;*
4. ${}_RR$ *is strongly ss-discrete.*

Proof. Follows from [3, Theorem 41]. □

Finally we give the following hierarchy for any module:
strongly ss-discrete ⇒ ss-discrete ⇒ quasi-ss-discrete ⇒ semi-ss-discrete ⇒ ss-lifting

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