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Stability Results on Mild Solution of Impulsive Neutral Fractional Stochastic Integro-Differential Equations Involving Poisson Jumps

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Abstract. This work studies the existence and the $\overline{p}$-th moment asymptotic stability of the mild solution of some neutral fractional stochastic integro-differential equations involving non-instantaneous impulses and Poisson jumps. Sufficient conditions proving existence and asymptotic stability of solutions are obtained utilizing stochastic analysis, resolvent operator and Krasnoselskii-Schaefer type fixed point theorem.

1. Introduction

Fractional differential equations has emerged in numerous engineering and scientific disciplines as the mathematical form of systems and procedures in the areas of chemistry, physics, electrodynamics of complex medium, aerodynamics, rheology, polymer and so forth. The nonlocal property of fractional order differential equation is the main reason behind using such equations in various applications. This means that the state of a system in the future is determined not only by its current state, but also by all of its past states. For more details of differential equations of fractional order, we refer the readers to the monographs [13–15]. Many real world processes experience an abrupt change of state at certain moments of time. These processes, subjected to short-term perturbations whose duration is negligible in comparison with the duration of the whole process, can be modeled by differential equations involving impulsive effects and these equations serve as a natural description of observed evolution phenomena of several real world problems([16, 17]). Shu et al. [28, 29] examined the existence of mild solution of impulsive fractional evolution equations for both ordinary and partial differential equations. However, the above small perturbations could not demonstrate the dynamic change in the development process in totality in pharmacotherapy. We realize that the presentation of new medications in the bloodstream and the consequent absorption for the body are steady and continuous processes. Therefore, the above circumstance has fallen in another impulsive action starting at an arbitrary fixed point and keeping active on a finite time interval. For demonstration of such procedure, the investigation of abstract differential equations involving non-instantaneous impulses was initiated by Hernández and O'Regan [39]. Recently, differential equations on infinite dimensional spaces involving instantaneous/non-instantaneous impulses have received considerable attention of researchers [2, 3, 11, 20, 21, 25, 39–42].

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In recent years, the study of stochastic differential equations has emerged as an important area of research due to their appearance in various applications such as biology, physics, engineering, physics, mathematical finance and in almost all applied sciences (see [36, 37]). Numerous important results on the existence, uniqueness and stability of mild solution have been obtained (see [3–5, 7, 11, 20, 21, 23–26, 38]). Very recently, Guo et al. [30] have studied the existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$. Further, Shu et al. [31] have discussed the approximate controllability and existence of mild solutions for Riemann-Liouville fractional stochastic evolution equations with nonlocal conditions of order.

Poisson jumps have recently turned out to be very popular because it is extensively used to describe many phenomena such as economics, finance, physics, biology, medicine and other sciences. Stochastic differential equations with Poisson jumps are utilized to portray the system jumps from a normal state to a bad state. It is natural and important to include a jump term in any dynamical system to make progressively realistic system. For more details on stochastic differential equations involving Poisson jumps, we refer to [1, 2, 5, 6, 8–10, 12, 19, 22]. After reviewing the past works of stochastic evolution equations involving Poisson jumps, we find that are not very many works that have studied the existence, uniqueness and stability of mild solution to fractional stochastic differential equation with Poisson jumps. This reality is the motivation of the present work. Therefore, in this paper, we are interested in the study of the existence and asymptotic stability of the following nonlinear impulsive fractional stochastic integro-differential equation of the form

$$
\begin{aligned}
{}^{C}D_t^{\alpha}[u(t) + H(t, u(t-\tau_1(t)))] &= A[u(t) + H(t, u(t-\tau_1(t)))] + \int_0^t B(t-s)\Big[u(s) \\
&\quad + H(s, u(s-\tau_1(s)))\Big]ds + \mathbb{J}_t^{1-\alpha}F(t, u(t-\tau_2(t))) + \int_Z h(t, u(t-\tau_4(t)), y)\widetilde{N}(dt, dy) \\
&\quad + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}G(s, u(s-\tau_3(s)))dW(s),\ t \in (s_i, t_{i+1}], \qquad (1)
\end{aligned}
$$

$$
u(t_i^+) = \mathcal{G}_i(t_i, u(t_i^- - \tau_5(t_i^-))), u'(t_i^+) = 0,\quad i = 1, 2, \cdots, k, \qquad (2)
$$

$$
u(t) = \mathcal{G}_i(t, u(t_i^- - \tau_5(t_i^-))),\ u'(t) = 0,\ t \in (t_i, s_i], i = 1, 2, \ldots, k, \qquad (3)
$$

$$
u_0(t) = \phi \in C^b_{\mathcal{F}_0}([-\tau, 0], X),\quad \text{a.s.},\quad u'(0) = 0, \qquad (4)
$$

where $\alpha \in (1, 2)$, ${}^{C}D_t^{\alpha}$ denotes the fractional derivative in Caputo sense, the state $u(\cdot)$ takes values in a separable real Hilbert space X with inner product $\langle\cdot,\cdot\rangle_X$ and norm $\|\cdot\|_X$, A and $(B(t))_{t\geq 0}$ are closed linear operators defined on a common dense domain $D(A)$. The notation $W(t),\ t \geq 0$ stands for a K-valued Wiener process with a covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is generated by the Wiener process W. The functions $H : \mathbb{R}^+ \times X \to X$, $G : \mathbb{R}^+ \times X \to L(K, X)$, $F : \mathbb{R}^+ \times X \to X$, $h : \mathbb{R}^+ \times X \times Z \to X$ and $\mathcal{G}_i : [0, \infty) \times X \to X (i = 1, \cdots, k,\ k \in \mathbb{N})$ are Borel measurable, and t_i, s_i are fixed numbers satisfying $0 = t_0 = s_0 < t_1 \leq s_1 < \cdots < t_N \leq s_k < t_{k+1} < \lim_{m\to\infty} t_m = \infty$, $\tau_j(t) : \mathbb{R}^+ \to [0, \tau] (j = 1, \cdots, 5)$ are continuous functions. For $\tau > 0$, $C([-\tau, 0], X)$ represents a family of all right-continuous functions with left-hand limits x from $[-\tau, 0]$ to X. Denote the norm of $x(t)$ by $\|x\|_C = \sup_{t\in[-\tau,0]} \mathbb{E}\|x(t)\|$. The notation $\widetilde{N}$ and $C^b_{\mathcal{F}_0}$ will be defined in section 2.

This article contains three sections. Section 2 presents some basic notations and preliminaries. Section 3 gives a set of sufficient conditions proving existence and asymptotic behavior of mild solutions to a class of neutral stochastic integro-differential equation involving fractional integral with non-instantaneous impulses with the help of solution operator, stochastic analysis, stability theory and fixed point technique.

2. Preliminaries

Let X and K be two real separable Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_K$, respectively, and $\|\cdot\|_X, \|\cdot\|_K$ be their vector norms.

Let $(\Omega, \mathcal{F}, P; \mathbb{F})(\mathbb{F} = \{\mathcal{F}\}_{t\geq 0})$ be a complete probability space such that $\mathcal{F}_0$ contains all P-null sets. The set $\{e_i\}_{i=1}^{\infty}$ is a complete orthonormal basis of K and $\{W(t) : t \geq 0\}$ is a cylindrical K-valued Brownian motion with a trace class operator $\mathcal{Q}$, defined by $\mathrm{Tr}(\mathcal{Q}) = \sum_{i=1}^{\infty} \mu_i = \mu < \infty$ that satisfies $\mathcal{Q}e_i = \mu_i e_i$. Thus, we can have $W(t) = \sum_{i=1}^{\infty} \sqrt{\mu_i} W_i(t) e_i$, where $\{W_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Brownian motions. Then, the above K-valued stochastic process $W(t)$ is called a $\mathcal{Q}$-Wiener process. The symbol $L(K, X)$ stands for the space of all bounded linear operators from K into X with the usual operator norm $\|\cdot\|_{L(K,X)}$ and we use the notation $L(X)$ when $X = K$. For $\varsigma \in L(K, X)$, we define

$$\|\varsigma\|_{L_2^0}^2 = \mathrm{Tr}(\varsigma\mathcal{Q}\varsigma^*) = \sum_{n=1}^{\infty} \| \sqrt{\mu_n}\varsigma e_n\|^2.$$

If $\|\varsigma\|_{L_2^0}^2 < \infty$, then ς is a $\mathcal{Q}$-Hilbert-Schmidt operator. Let us consider $L_2^0(K, X)$ to be the space of all $\mathcal{Q}$-Hilbert-Schmidt operators $\varsigma : K \to X$. For more details, the reader is referred to [18].

Let us assume $\{\overline{p}(t)\}_{t\geq 0}$ to be a σ-finite stationary $\mathcal{F}_t$-adapted Poisson point process that takes values in a measurable space $(Z, \mathcal{B}(Z))$. The random measure $N_{\overline{p}}$ defined by $N_{\overline{p}}((0, t], \Lambda) := \sum_{s\in(0,t]} 1_\Lambda(\overline{p}(s))$ for $\Lambda \in Z$ is a Poisson random measure which is induced by $\overline{p}(\cdot)$. Therefore, we can define the measure $\widetilde{N}$ by $\widetilde{N}(dt, dy) = N_{\overline{p}}(dt, dy) - \vartheta(dy)dt$, where ϑ is the characteristic measure of $N_{\overline{p}}$, which is called the compensated Poisson random measure. The notation $C^b_{\mathcal{F}_0}([-\tau, 0], X)$ stands for a family of all almost surely bounded, $\mathcal{F}_0$-measurable, continuous random càdlàg functions from $[-\tau, 0]$ to X with norm $\|\phi\|_{D_\tau} := \sup_{t\in[-\tau,0]} \|\phi(t)\|_X$.

Let Υ be the space of all $\mathcal{F}_0$-adapted process $\psi(t, w) : [-\tau, \infty) \times \Omega \to X$ which is almost surely continuous in t for fixed $w \in \Omega$, $\lim_{t\to t_j^-} \psi(t)$ and $\lim_{t\to t_j^+} \psi(t)$ exist, and $\lim_{t\to t_j^-} \psi(t) = \psi(t_j)$, $j = 1, \ldots, k$. Moreover, $\psi(t, w) = \psi(t)$ for all $t \in [-\tau, 0]$ and $\mathbb{E}\|\psi(t, w)\|_X^{\overline{p}} \to 0$ as $t \to \infty$. Υ is also a Banach space equipped with following norm defined by

$$\|\psi\|_\Upsilon = \sup_{t\geq 0} \mathbb{E}\|\psi(t)\|_X^{\overline{p}}.$$

Definition 2.1. *A one-parameter family $\widehat{\mathcal{S}}_\alpha(t)$, $t \geq 0$ of bounded linear operators defined on X is called an α-resolvent operator for*

$$^cD_t^\alpha u(t) = Au(t) + \int_0^t B(t-s)u(s)ds, \tag{5}$$

$$u(0) = u_0 \in X, \quad u'(0) = 0, \tag{6}$$

if the following are fulfilled:

(1) *$\widehat{\mathcal{S}}_\alpha(\cdot) : [0, \infty) \to L(X)$ is continuous in strong sense and $\widehat{\mathcal{S}}_\alpha(0)z = z$ for every $z \in X$ and $\alpha \in (1, 2)$.*

(2) *For $z \in D(A)$, we have $\widehat{\mathcal{S}}_\alpha(\cdot)z \in C([0, \infty), D(A)) \cap C^1((0, \infty), X)$, so that*

$$^cD_t^\alpha \widehat{\mathcal{S}}_\alpha(t)z = A\widehat{\mathcal{S}}_\alpha(t)z + \int_0^t B(t-s)\widehat{\mathcal{S}}_\alpha(s)zds,$$

$$^cD_t^\alpha \widehat{\mathcal{S}}_\alpha(t)z = \widehat{\mathcal{S}}_\alpha(t)Az + \int_0^t \widehat{\mathcal{S}}_\alpha(t-s)B(s)zds, \quad \textit{for every } t \geq 0,$$

i.e., $\widehat{\mathcal{S}}_\alpha(t)z$ is the solution of equation (5).

Next, we consider the following hypotheses:

($P1$) The operator $A : D(A) \subset X \to X$ is a closed, densely linear operator with $[D(A)]$ dense in X. Let $\alpha \in (1, 2)$. For some $\phi_0 \in (0, \pi/2]$ for every $\phi < \phi_0$, there exists a constant $C_0 = C_0(\phi) > 0$ such that $\mu \in \rho(A)$ for each

$$\mu \in \sum_{0,\alpha\eta} = \{\mu \in \mathbb{C}, \ \mu \neq 0, \ |\arg(\mu)| < \alpha\eta\}, \tag{7}$$

where $\eta = \phi + \pi/2$ and $\|R(\mu, A)\| \leq C_0/|\mu|$ for all $\mu \in \sum_{0,\alpha\eta}$.

(*P*2) $B(t) : D(B(t)) \subseteq X \to X$ for $t \geq 0$ is a closed linear operator with $D(A) \subseteq D(B(t))$ and $B(\cdot)z$ is strongly measurable on $(0, \infty)$ for every $z \in D(A)$. For $z \in D(A)$ and $t > 0$, there exists $d(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{d}(\mu)$, Laplace transform of $d(\cdot)$, exists for $\mathrm{Re}(\mu) > 0$ and $\|B(t)z\| \leq d(t)\|z\|_1$. Furthermore, $\widehat{B} : \sum_{0,\pi/2} \to L([D(A)], X)$ has an analytical extension which is denoted by $\widehat{B}$ to $\sum_{0,\eta}$ such that $\|\widehat{B}(\mu)z\| \leq \|\widehat{B}(\mu)\| \cdot \|z\|_1$ for each $z \in D(A)$ and $\|\widehat{B}(\mu)\| = O(\frac{1}{|\mu|})$, $\mu \to \infty$.

(*P*3) There exist positive constant $\mathbb{C}_1$ and a subspace $\widehat{\mathcal{D}} \subseteq D(A)$ dense in $[D(A)]$ such that $A(\widehat{\mathcal{D}}) \subseteq D(A)$, $\widehat{B}(\mu)(\widehat{\mathcal{D}}) \subseteq D(A)$ and $\|A\widehat{B}(\mu)y\| \leq \mathbb{C}_1\|y\|$ for all $y \in \widehat{\mathcal{D}}$ and $\mu \in \sum_{0,\eta}$.

In continuation, we have,
for $r > 0$ and $\theta \in (\pi/2, \eta)$,

$$\sum_{r,\theta} = \{\mu \in \mathbb{C} : \ \mu \neq 0, \ r < |\mu|, \ \ |\arg(\mu)| < \theta\},$$

and for $\Gamma_{r,\theta}$

$$\begin{aligned} \Gamma^1_{r,\theta} &= \{te^{i\theta} : \ t \geq r\}, \\ \Gamma^2_{r,\theta} &= \{re^{i\zeta} : \ -\theta \leq \zeta \leq \theta\}, \\ \Gamma^3_{r,\theta} &= \{te^{-i\theta} : \ t \geq r\}, \end{aligned} \tag{8}$$

where $\Gamma^i_{r,\theta}$, $i = 1, 2, 3$ are the paths such that $\Gamma_{r,\theta} = \cup_{i=1}^3 \Gamma^i_{r,\theta}$ is oriented counter-clockwise. Moreover, we consider the following sets $\rho_\alpha(\mathcal{G}_\alpha)$ as

$$\rho_\alpha(\mathcal{G}_\alpha) = \{\mu \in \mathbb{C} : \ \mathcal{G}_\alpha(\mu) = \mu^{\alpha-1}(\mu^\alpha I - A - A\widehat{D}(\mu))^{-1} \in L(X)\}. \tag{9}$$

Define the operator family $\widehat{\mathcal{S}}_\alpha(t)$, $t \geq 0$ by

$$\widehat{\mathcal{S}}_\alpha(t) = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\mu t}\mathcal{G}_\alpha(\mu)d\mu, & t > 0, \\ I, & t = 0. \end{cases} \tag{10}$$

Lemma 2.2. *[44] Let $\alpha \in (1, 2)$. Then we define $\widehat{\mathcal{R}}_\alpha(t)$, $t \geq 0$ by*

$$\widehat{\mathcal{R}}_\alpha(t)z = \int_0^t g_{\alpha-1}(t - \zeta)\widehat{\mathcal{S}}_\alpha(\zeta)z d\zeta, \quad t \geq 0, \tag{11}$$

where $g_{\alpha-1}(t) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}$, $t > 0$, $\alpha - 1 \geq 0$.

Lemma 2.3. *[43] There exists a number $r_1 > 0$ such that $\sum_{r_1,\eta} \subseteq \rho_\alpha(\mathcal{G}_\alpha)$ and the map $\mathcal{G}_\alpha : \sum_{r_1,\eta} \to L(X)$ is analytic. Furthermore, we have*

$$\mathcal{G}_\alpha(\mu) = \mu^{\alpha-1}R(\mu^\alpha, A)[I - \widehat{B}(\mu)R(\mu^\alpha, A)]^{-1} \tag{12}$$

and there are constants $\widetilde{M}_i$ for $i = 1, 2$ such that

$$\begin{aligned} \|\mathcal{G}_\alpha(\mu)\| &\leq \frac{\widetilde{M}_1}{|\mu|}, \\ \|A\mathcal{G}_\alpha(\mu)z\| &\leq \frac{\widetilde{M}_2}{|\mu|}\|z\|_1, \ z \in D(A), \\ \|A\mathcal{G}_\alpha(\mu)\| &\leq \frac{\widetilde{M}_2}{|\mu|^{1-\alpha}} \end{aligned} \tag{13}$$

for each $\mu \in \sum_{r_1,\eta}$.

Lemma 2.4. *[43] If the conditions* (*P*1)-(*P*3) *are satisfied, then there exists a unique* α*-resolvent operator for the system (5)-(6).*

For more results on $\widehat{\mathcal{S}}_\alpha(t)$ and $\widehat{\mathcal{R}}_\alpha(t)$, we refer to article [43].

Definition 2.5. A process $\{u(t), t \in [0,T]\}(0 \le T < \infty)$, which is stochastic, is called a mild solution of (1)-(4) if

(*i*) $u(t)$ is adapted to $\mathcal{F}_t, t \ge 0$ and $\int_0^T \|u(t)\|^{\overline{p}} dt < +\infty$ a.s.

(*ii*) $u(t) \in X$ has càdlàg paths on $t \in [0,T]$ a.s and for each $t \in [0,T]$, $u(t)$ fulfills the integral equation

$$u(t) = \begin{cases} \widehat{\mathcal{S}}_\alpha(t)[\phi(0) + H(0,\phi(-\tau_1(0)))] - H(t,u(t-\tau_1(t))) \\ + \int_0^t \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds + \int_0^t \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s) \\ + \int_0^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s,u(s-\tau_4(s)),y)\widetilde{N}(ds,dy),\ t \in [0,t_1], \\ \widehat{\mathcal{S}}_\alpha(t-s_i)[\mathcal{G}_i(s_i,u(t_i^- - \tau_5(t_i))) + H(s_i,u(s_i-\tau_1(s_i)))] - H(t,u(t-\tau_1(t))) \\ + \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds + \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s) \\ + \int_{s_i}^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s,u(s-\tau_4(s)),y)\widetilde{N}(ds,dy), \quad t \in (s_i,t_{i+1})], \\ \mathcal{G}_i(t,u(t_i^- - \tau_5(t_i))), \quad t \in (t_i,s_i],\ i = 1,\cdots,k, \end{cases} \tag{14}$$

and $u(t_i^+) = \mathcal{G}_i(t_i, u(t_i^- - \tau_5(t_i^-))), i = 1,\cdots,k.$

Definition 2.6. *[18]* Equation (14) is called exponentially stable in the $\overline{p}$-th moment if for any initial data ϕ, there exists a pair of positive constants $\mu > 0$ and $\mathbb{B}$ such that

$$\mathbb{E}\|u(t)\|^{\overline{p}} \le \mathbb{B} \times \|\phi\|_0^{\overline{p}} e^{-\mu t}, \quad t \ge 0,$$

where $\mathbb{E}$ represents the expectation with respect to the probability measure $\mathbb{P}$, $\overline{p} \ge 2$ is an integer and $u(t)$ is the mild solution of equation (1).

Definition 2.7. Equation (14) is called asymptotically stable in the $\overline{p}$-th moment if it is stable in the $\overline{p}$-th moment and for any $\phi \in C([-\tau,0],X)$,

$$\lim_{T\to+\infty} \mathbb{E}\Big[\sup_{t\ge T} \|u(t)\|_X^{\overline{p}}\Big] = 0.$$

Lemma 2.8. *[46] For any* $\overline{p} \ge 1$ *and for arbitrary* $L_2^0(K,X)$*-valued predictable process* $\varphi(\cdot)$*, the following holds:*

$$\sup_{s\in[0,t]} \mathbb{E}\Big\| \int_0^s \varphi(v)dW(v)\Big\|_X^{2\overline{p}} \le C_{\overline{p}} \times \Big(\int_0^t (\mathbb{E}\|\varphi(s)\|_{L_2^0}^{2\overline{p}})^{1/\overline{p}} ds\Big)^{\overline{p}}, \quad t \ge 0,$$

where $C_{\overline{p}} = (\overline{p}(2\overline{p}-1))^{\overline{p}}$.

Lemma 2.9. *[1] Let* $\overline{p} \ge 2$*. Then, there exists* $c_{\overline{p}} > 0$ *such that*

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,t]} \| \int_0^s \int_Z H(v,y)\widetilde{N}(dv,dy)\|^{\overline{p}} \quad &\le \quad c_{\overline{p}} \times \Big\{ \mathbb{E}\Big[\Big(\int_0^s \int_Z \|H(v,y)\|^2 \vartheta dydv\Big)^{\overline{p}/2}\Big] \\ &\quad + \mathbb{E}\Big[\int_0^s \int_Z \|H(v,y)\|^{\overline{p}} \vartheta dydv\Big]\Big\}. \end{aligned}$$

Lemma 2.10. *[46] Let* $\widehat{\Phi}_1, \widehat{\Phi}_2$ *be two operators such that*

(a) $\widehat{\Phi}_1$ *is a contraction map, and*

(b) $\widehat{\Phi}_2$ *is completely continuous.*

Then, either

(i) the operator $\widehat{\Phi}_1 x + \widehat{\Phi}_2 x$ *has a solution, or*

(ii) the set $\Upsilon = \{x \in X : \lambda\widehat{\Phi}_1(\frac{x}{\lambda}) + \lambda\widehat{\Phi}_2 x = x\}$ *is unbounded for* $\lambda \in (0, 1)$.

3. Main results

In this section, we present our result on asymptotic stability in the p-th moment of mild solutions of system (1)-(4). Now, we make the following hypotheses:

($B1$) The operator families $\widehat{\mathcal{S}}_\alpha(t)$ and $\widehat{\mathcal{R}}_\alpha(t)$ are compact for all $t > 0$, and there exist constants $M > 0, \delta > 0$ such that $\|\widehat{\mathcal{S}}_\alpha(t)\|_{L(X)} \leq Me^{-\delta t}$ and $\|\widehat{\mathcal{R}}_\alpha(t)\|_{L(X)} \leq Me^{-\delta t}$ for every $t \geq 0$.

($B2$) The function $H : [0, \infty) \times X \to X$ is continuous and there exists $L_H > 0$ such that

$$E\|H(t, \psi_1) - H(t, \omega_1)\|_X^{\overline{p}} \leq L_H[\|\psi_1 - \omega_1\|_X^{\overline{p}}],$$
$$E\|H(t, \psi_1)\|_X^{\overline{p}} \leq L_H[\|\psi_1\|_X^{\overline{p}}],$$

for each $t \geq 0, \psi_1, \omega_1 \in X$ with $H(t, 0) = 0$.

(B3) The function $F : [0, \infty) \times X \to X$ is continuous satisfying the following condition:
There exist a continuous function $m_F : [0, \infty) \to [0, \infty)$ and a continuous increasing function $\Theta_F : [0, \infty) \to (0, \infty)$ such that

$$\mathbb{E}\|F(t, \psi_1)\|_X^{\overline{p}} \leq m_F(t)\Theta_F(\mathbb{E}\|\psi_1\|_X^{\overline{p}}), \quad t \geq 0, \ \psi_1 \in X.$$

($B4$) The function $G : [0, \infty) \times X \to L(K, X)$ is continuous satisfying the following condition:
There exist a continuous function $m_G : [0, \infty) \to [0, \infty)$ and a continuous increasing function $\Theta_G : [0, \infty) \to (0, \infty)$ such that

$$\mathbb{E}\|G(t, \psi_1)\|_X^{\overline{p}} \leq m_G(t)\Theta_G(\mathbb{E}\|\psi_1\|_X^{\overline{p}}), \ t \geq 0, \psi_1 \in X,$$

with

$$\int_1^\infty \frac{1}{\Theta_H(s) + \Theta_F(s)} ds = \infty. \tag{15}$$

($B5$) The functions $\mathcal{G}_i : [0, \infty) \times X \to X$ are equi-continuous and there are constants $L_{\mathcal{G}_i}$, $i = 1, 2, \ldots k$ such that $\mathbb{E}\|\mathcal{G}_i(t_1, u_1) - \mathcal{G}_i(t_2, u_2)\|_X^{\overline{p}} \leq L_{\mathcal{G}_i}[|t_1 - t_2| + \mathbb{E}\|u_1 - u_2\|_X^{\overline{p}}]$ and $\mathbb{E}\|\mathcal{G}_i(t_1, u_1)\|_X^{\overline{p}} \leq L_{\mathcal{G}_i}\|u_1\|_X^{\overline{p}}$, for every $u_1, u_2 \in X$, $t_1, t_2 \in [0, T]$, $T < \infty$ and $\mathcal{G}_i(t, 0) = 0$.

($B6$) The function $h : [0, \infty) \times X \times U \to X$ satisfies the Lipschitz condition and there are constants $L_h > 0$, $\widetilde{L}_h > 0$ such that

$$\begin{aligned} \int_Z \mathbb{E}\|h(t, u(t-), y)\|^{\overline{p}} \vartheta dy &\leq L_h \mathbb{E}\|u\|_X^{\overline{p}}, \\ \int_Z \mathbb{E}\|h(t, u(-), y) - h(t, v(-), y)\|^{\overline{p}} \vartheta dy &\leq \widetilde{L}_h \mathbb{E}\|u - v\|_X^{\overline{p}} \end{aligned} \tag{16}$$

for all $u, v \in X$ and $y \in Z \subset U$.

Theorem 3.1. *Assume that the conditions* (B1)-(B6) *hold and* $\overline{p} \geq 2$ *is an integer. Then the fractional impulsive stochastic differential equation (1)-(4) is asymptotically stable in the* $\overline{p}$*-th moment, provided that*

$$4^{\overline{p}-1} \max_{i\in\{1,2,\cdots,k\}} L_{\mathcal{G}_i}\Big[M^{\overline{p}}L_{\mathcal{G}_i} + \|(-A)^{-\beta}\|^{\overline{p}}M^{\overline{p}}L_H + \|(-A)^{-\beta}\|^{\overline{p}}L_H + M^{\overline{p}}Tc_{\overline{p}}(\widetilde{L}_h^{\overline{p}/2} + \widetilde{L}_h)\Big] < 1. \tag{17}$$

Proof. We define the nonlinear operator $\Psi : \Upsilon \to \Upsilon$ as $(\Psi u)(t) = \phi(t)$ for $t \in [-\tau, 0]$ and for $t \geq 0$,

$$(\Psi u)(t) = \begin{cases} \widehat{\mathcal{S}}_\alpha(t)[\phi(0) + H(0, \phi(-\tau_1(0)))] - H(t, u(t-\tau_1(t))) \\ + \int_0^t \widehat{\mathcal{S}}_\alpha(t-s)F(s, u(s-\tau_2(s)))ds + \int_0^t \widehat{\mathcal{S}}_\alpha(t-s)G(s, u(s-\tau_3(s)))dW(s) \\ + \int_0^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy), \ t \in [0, t_1], \\ \widehat{\mathcal{S}}_\alpha(t-s_i)[\mathcal{G}_i(s_i, u(t_i-\tau_5(t_i))) + H(s_i, u(s_i-\tau_1(s_i)))] - H(t, u(t-\tau_1(t))) \\ + \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)F(s, u(s-\tau_2(s)))ds + \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)G(s, u(s-\tau_3(s)))dW(s) \\ + \int_{s_i}^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy), \quad t \in (s_i, t_{i+1}], \\ \mathcal{G}_i(t, u(t_i^- - \tau_5(t_i^-))), \quad t \in (t_i, s_i], \ i = 1, \cdots, k, \end{cases} \tag{18}$$

and $(\Psi)u(t_i^+) = \mathcal{G}_i(t_i, u(t_i^- - \tau_5(t_i)))$, $i = 1, \cdots, k$.
At first, we show that the map Ψ is p-th moment continuous on $[0, \infty)$ and a well-defined map from Υ itself

Y. Let $u \in \mathbb{B}$, $\tilde{t} \geq 0$ and $|\xi|$ be sufficiently small. Then for $\tilde{t} \in [0, t_1]$, by using Hölder's inequality, we have

$$
\begin{aligned}
&\mathbb{E}\|(\Psi u)(\tilde{t}+\xi)-(\Psi u)(\tilde{t})\|_X^{\overline{p}} \\
&\leq 8^{\overline{p}-1}\mathbb{E}\Big\|[\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi)-\widehat{\mathcal{S}}_\alpha(\tilde{t})](\phi(0)+H(0,\phi(-\tau_1(0))))\|_X^{\overline{p}}+8^{\overline{p}-1}\mathbb{E}\Big\|H(\tilde{t}+\xi,u(\tilde{t}+l-\tau_1(\tilde{t}+\xi))) \\
&\quad -H(\tilde{t},u(\tilde{t}-\tau_1(\tilde{t})))\|_X^{\overline{p}}+8^{\overline{p}-1}\mathbb{E}\Big\|\int_0^{\tilde{t}}[\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{S}}_\alpha(\tilde{t}-s)]F(s,u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} \\
&\quad +8^{\overline{p}-1}\mathbb{E}\Big\|\int_{\tilde{t}}^{\tilde{t}+\xi}\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)F(s,u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} \\
&\quad +8^{\overline{p}-1}\mathbb{E}\Big\|\int_0^{\tilde{t}}[\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{S}}_\alpha(\tilde{t}-s)]G(s,u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\quad +8^{\overline{p}-1}\mathbb{E}\Big\|\int_{\tilde{t}}^{\tilde{t}+\xi}\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)G(s,u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\quad +8^{\overline{p}-1}\mathbb{E}\Big\|\int_0^{\tilde{t}}\int_Z[\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{R}}_\alpha(\tilde{t}-s)]h(s,u(s-\tau_4(s)),y)\widetilde{N}(ds,dy)\Big\|_X^{\overline{p}} \\
&\quad +8^{\overline{p}-1}\mathbb{E}\Big\|\int_{\tilde{t}}^{\tilde{t}+\xi}\int_Z\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)h(s,u(s-\tau_4(s)),y)\widetilde{N}(ds,dy)\Big\|_X^{\overline{p}} \\
&\leq 8^{\overline{p}-1}\mathbb{E}\Big\|\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi)-\widehat{\mathcal{S}}_\alpha(\tilde{t})\|^{\overline{p}}\times\mathbb{E}\|(\phi(0)+H(0,\phi(-\tau_1(0))))\|_X^{\overline{p}}+8^{\overline{p}-1}\mathbb{E}\Big\|H(\tilde{t}+\xi,u(\tilde{t}+\xi-\tau_1(\tilde{t}+\xi))) \\
&\quad -H(\tilde{t},u(\tilde{t}-\tau_1(\tilde{t})))\|_X^{\overline{p}}+8^{\overline{p}-1}\mathbb{E}\Big[\int_0^{\tilde{t}}\|[\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{S}}_\alpha(\tilde{t}-s)]F(s,u(s-\tau_2(s)))\|_X ds\Big]^{\overline{p}} \\
&\quad +8^{\overline{p}-1}M^{\overline{p}}\mathbb{E}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi}e^{-\delta(\tilde{t}+\xi-s)}\|F(s,u(s-\tau_2(s)))\|_X ds\Big]^{\overline{p}} \\
&\quad +8^{\overline{p}-1}C_{\overline{p}}\Big[\int_0^{\tilde{t}}(\mathbb{E}\|[\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{S}}_\alpha(\tilde{t}-s)]G(s,u(s-\tau_3(s)))\|_X^{\overline{p}})^{2/\overline{p}}ds\Big]^{\overline{p}/2} \\
&\quad +8^{\overline{p}-1}C_{\overline{p}}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi}(\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)G(s,u(s-\tau_3(s)))\|_X^{\overline{p}})^{2/\overline{p}}ds\Big]^{\overline{p}/2} \\
&\quad +8^{\overline{p}-1}c_{\overline{p}}\mathbb{E}\Big[\int_0^{\tilde{t}}\int_Z\|[\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{R}}_\alpha(\tilde{t}-s)]h(s,u(s-\tau_4(s)),y)\|^2 ds\vartheta dy\Big]^{\overline{p}/2} \\
&\quad +8^{\overline{p}-1}c_{\overline{p}}\mathbb{E}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi}\int_Z\|\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)h(s,u(s-\tau_4(s)),y)\|^2 ds\vartheta dy\Big]^{\overline{p}/2} \\
&\quad +8^{\overline{p}-1}c_{\overline{p}}\mathbb{E}\Big[\int_0^{\tilde{t}}\int_Z\|[\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{R}}_\alpha(\tilde{t}-s)]h(s,u(s-\tau_4(s)),y)\|^{\overline{p}} ds\vartheta dy\Big] \\
&\quad +8^{\overline{p}-1}c_{\overline{p}}\mathbb{E}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi}\int_Z\|\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)h(s,u(s-\tau_4(s)),y)\|^{\overline{p}} ds\vartheta dy\Big] \\
&\leq 8^{\overline{p}-1}\mathbb{E}\Big\|\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi)-\widehat{\mathcal{S}}_\alpha(\tilde{t})\|^{\overline{p}}\times\mathbb{E}\|(\phi(0)+H(0,\phi(-\tau_1(0))))\|_X^{\overline{p}}+8^{\overline{p}-1}\mathbb{E}\Big\|H(\tilde{t}+\xi,u(\tilde{t}+\xi-\tau_1(\tilde{t}+\xi))) \\
&\quad -H(\tilde{t},u(\tilde{t}-\tau_1(\tilde{t})))\|_X^{\overline{p}}+8^{\overline{p}-1}\Big[\int_0^{\tilde{t}}\|\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{S}}_\alpha(\tilde{t}-s)\|_{L(X)}^{(\overline{p}/\overline{p}-1)}ds\Big]^{\overline{p}-1} \\
&\quad \times\int_0^{\tilde{t}}\mathbb{E}\|F(s,u(s-\tau_2(s)))\|_X^p ds+8^{\overline{p}-1}M^{\overline{p}}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi}e^{-(\overline{p}\delta/\overline{p}-1)(\tilde{t}+\xi-s)}ds\Big]^{\overline{p}-1} \\
&\quad \times\int_{\tilde{t}}^{\tilde{t}+\xi}\mathbb{E}\|F(s,u(s-\tau_2(s)))\|_X^{\overline{p}}ds+8^{\overline{p}-1}C_{\overline{p}}\Big[\int_0^{\tilde{t}}(\mathbb{E}\|[\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{S}}_\alpha(\tilde{t}-s)] \\
&\quad \times G(s,u(s-\tau_3(s)))\|_X^{\overline{p}})^{2/\overline{p}}ds\Big]^{\overline{p}/2}+8^{\overline{p}-1}C_{\overline{p}}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi}(\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)G(s,u(s-\tau_3(s)))\|_X^{\overline{p}})^{2/\overline{p}}ds\Big]^{\overline{p}/2} \\
&\quad +8^{\overline{p}-1}c_{\overline{p}}\Big[\int_0^{\tilde{t}}\int_Z\|\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{R}}_\alpha(\tilde{t}-s)\|^2\mathbb{E}\|h(s,u(s-\tau_4(s)),y)\|^2 ds\vartheta dy\Big]^{\overline{p}/2} \\
&\quad +8^{\overline{p}-1}c_{\overline{p}}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi}\int_Z\|\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)\|^2\mathbb{E}\|h(s,u(s-\tau_4(s)),y)\|^2 ds\vartheta dy\Big]^{\overline{p}/2} \\
&\quad +8^{\overline{p}-1}c_{\overline{p}}\Big[\int_0^{\tilde{t}}\int_Z\|\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)-\widehat{\mathcal{R}}_\alpha(\tilde{t}-s)\|^{\overline{p}}\mathbb{E}\|h(s,u(s-\tau_4(s)),y)\|^{\overline{p}} ds\vartheta dy\Big]
\end{aligned}
$$

Similarly, for any $\tilde{t} \in (t_i, t_{i+1}]$, $i = 1, \cdots, k$, we have

$$
\begin{aligned}
&\mathbb{E}\|(\Psi x)(\tilde{t}+\xi) - (\Psi x)(\tilde{t})\|_X^{\overline{p}} \\
&\le 8^{\overline{p}-1}\mathbb{E}\Big\|[\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s_i) - \widehat{\mathcal{S}}_\alpha(\tilde{t}-s_i)][\mathcal{G}_i(s_i, u(t_i^- - \tau_5(t_i))) + H(s_i, u(s_i-\tau_1(s_i)))]\|_X^{\overline{p}} \\
&\quad + 8^{\overline{p}-1}\mathbb{E}\Big\|H(\tilde{t}+\xi, u(\tilde{t}+\xi-\tau_1(\tilde{t}+\xi))) - H(\tilde{t}, u(\tilde{t}-\tau_1(\tilde{t})))\|_X^{\overline{p}} \\
&\quad + 8^{\overline{p}-1}\mathbb{E}\Big\| \int_{s_i}^{\tilde{t}} [\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s) - \widehat{\mathcal{S}}_\alpha(\tilde{t}-s)]F(s, u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} \\
&\quad + 8^{\overline{p}-1}\mathbb{E}\Big\| \int_{\tilde{t}}^{\tilde{t}+\xi} \widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)F(s, u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} \\
&\quad + 6^{\overline{p}-1}\mathbb{E}\Big\| \int_{s_i}^{\tilde{t}} [\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s) - \widehat{\mathcal{S}}_\alpha(\tilde{t}-s)]G(s, u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\quad + 8^{\overline{p}-1}\mathbb{E}\Big\| \int_{\tilde{t}}^{\tilde{t}+\xi} \widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)G(s, u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\quad + 8^{\overline{p}-1}\mathbb{E}\Big\| \int_{s_i}^{\tilde{t}} \int_Z [\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s) - \widehat{\mathcal{R}}_\alpha(\tilde{t}-s)]h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy)\Big\|_X^{\overline{p}} \\
&\quad + 8^{\overline{p}-1}\mathbb{E}\Big\| \int_{\tilde{t}}^{\tilde{t}+\xi} \int_Z \widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy)\Big\|_X^{\overline{p}} \\
&\le 8^{\overline{p}-1}\mathbb{E}\Big\|\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s_i) - \widehat{\mathcal{S}}_\alpha(\tilde{t}-s_i)\|_X^{\overline{p}} \times \mathbb{E}\|[\mathcal{G}_i(s_i, u(t_i-\tau_5(t_i))) + H(s_i, u(s_i-\tau_1(s_i)))]\|_X^{\overline{p}} \\
&\quad + 8^{\overline{p}-1}\mathbb{E}\Big\|H(\tilde{t}+\xi, u(\tilde{t}+\xi-\tau_1(\tilde{t}+\xi))) - H(\tilde{t}, u(\tilde{t}-\tau_1(\tilde{t})))\Big\|_X^{\overline{p}} \\
&\quad + 8^{\overline{p}-1}\Big[\int_{s_i}^{\tilde{t}} \|\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s) - \widehat{\mathcal{S}}_\alpha(\tilde{t}-s)\|_{L(X)}^{(\overline{p}/(\overline{p}-1))}ds\Big]^{\overline{p}-1} \times \int_{s_i}^{\tilde{t}} \mathbb{E}\|F(s, u(s-\tau_2(s)))\|_X^{\overline{p}}ds \\
&\quad + 8^{\overline{p}-1}M^{\overline{p}}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi} e^{(\overline{p}\delta/(\overline{p}-1))(\tilde{t}+\xi-s)}ds\Big]^{\overline{p}-1} \times \int_{\tilde{t}}^{\tilde{t}+\xi} \mathbb{E}\|F(s, u(s-\tau_2(s)))\|_X^{\overline{p}}ds \\
&\quad + 8^{\overline{p}-1}C_{\overline{p}}\Big[\int_{s_i}^{\tilde{t}} (\mathbb{E}\|[\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s) - \widehat{\mathcal{S}}_\alpha(\tilde{t}-s)]G(s, u(s-\tau_3(s)))\|_X^{\overline{p}})^{2/\overline{p}}ds\Big]^{\overline{p}/2} \\
&\quad + 8^{\overline{p}-1}C_{\overline{p}}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi} (\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(\tilde{t}+\xi-s)G(s, u(s-\tau_3(s)))\|_X^{\overline{p}})^{2/\overline{p}}ds\Big]^{\overline{p}/2} \\
&\quad + 8^{\overline{p}-1}c_{\overline{p}}\Big[\int_{s_i}^{\tilde{t}} \int_Z \|\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s) - \widehat{\mathcal{R}}_\alpha(\tilde{t}-s)\|^2\mathbb{E}\|h(s, u(s-\tau_4(s)), y)\|^2ds\vartheta dy\Big]^{\overline{p}/2} \\
&\quad + 8^{\overline{p}-1}c_{\overline{p}}\Big[\int_{\tilde{t}}^{\tilde{t}+\xi} \int_Z \|\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)\|^2\mathbb{E}\|h(s, u(s-\tau_4(s)), y)\|^2ds\vartheta dy\Big]^{\overline{p}/2} \\
&\quad + 8^{\overline{p}-1}c_{\overline{p}}[\int_{s_i}^{\tilde{t}} \int_Z \|\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s) - \widehat{\mathcal{R}}_\alpha(\tilde{t}-s)\|^{\overline{p}}\mathbb{E}\|h(s, u(s-\tau_4(s)), y)\|^{\overline{p}}ds\vartheta dy] \\
&\quad + 8^{\overline{p}-1}c_{\overline{p}}[\int_{\tilde{t}}^{\tilde{t}+\xi} \int_Z \|\widehat{\mathcal{R}}_\alpha(\tilde{t}+\xi-s)\|^{\overline{p}}\mathbb{E}\|h(s, u(s-\tau_4(s)), y)\|^{\overline{p}}ds\vartheta dy] \\
&\to 0
\end{aligned}
$$

as $\xi \to 0$. Thus, for all $u(\tilde{t}) \in \Upsilon$, $t \ge 0$, by using the equi-continuity of $\mathcal{G}_i$, we have

$$\mathbb{E}\|(\Psi u)(\tilde{t}+\xi) - (\Psi u)(\tilde{t})\|_X^{\overline{p}} \to 0, \quad \text{as } \xi \to 0. \tag{19}$$

Thus, Ψ is continuous in the $\overline{p}$-th moment on $[0, \infty)$. Next, we show that $\Psi(\Upsilon) \subset \Upsilon$.

$\mathbb{E}\|\Psi u(t)\|_X^{\overline{p}}$

$$\leq 6^{\overline{p}-1}\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(t)\phi(0)\|_X^{\overline{p}} + 6^{\overline{p}-1}\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(t)H(0,\phi(-\tau_1(0)))]\|_X^{\overline{p}} + 6^{\overline{p}-1}\mathbb{E}\|H(t,u(t-\tau_1(t)))\|_X^{\overline{p}}$$
$$+ 6^{\overline{p}-1}\mathbb{E}\|\int_0^t \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds\|_X^{\overline{p}} + 6^{\overline{p}-1}\mathbb{E}\|\int_0^t \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s)\|_X^{\overline{p}}$$
$$+ 6^{\overline{p}-1}\mathbb{E}\|\int_0^t\int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s,u(s-\tau_4(s)),y)\widetilde{N}(ds,dy)\|^{\overline{p}}$$
$$\leq 6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + 6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}\|(-A)^{-\beta}\|^{\overline{p}}L_H\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}} + 6^{\overline{p}-1}\|(-A)^{-\beta}\|^{\overline{p}}L_H$$
$$\times E\|u(t-\tau_1(t))\|_X^{\overline{p}} + 6^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\times\int_0^t e^{-\delta(t-s)}m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s))\|_X^{\overline{p}})ds$$
$$+ 6^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\times\int_0^t e^{-\delta(t-s)}m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})ds$$
$$+ 6^{\overline{p}-1}c_{\overline{p}}\Big\{M^{\overline{p}}L_h^{\overline{p}/2}\Big(\frac{(\overline{p}-2)}{2\delta(\overline{p}-1)}\Big)^{\frac{\overline{p}-2}{2}}\int_0^t e^{-\delta(t-s)}\mathbb{E}\|u(s-\tau_4(s))\|_X^{\overline{p}}ds + M^{\overline{p}}L_h\Big(\frac{(\overline{p}-2)}{2\delta(\overline{p}-1)}\Big)^{\frac{\overline{p}-2}{2}}$$
$$\times\int_0^t e^{-\delta(t-s)}\mathbb{E}\|u(s-\tau_4(s))\|_X^{\overline{p}}ds\Big\}.$$

Similarly, for any $t\in(s_i,t_{i+1}],\ i=1,\cdots,k$, we have
$\mathbb{E}\|\Psi u(t)\|_X^{\overline{p}}$

$$\leq 6^{\overline{p}-1}\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(t-s_i)\mathcal{G}_i(s_i,u(t_i-\tau_5(t_i)))\|_X^{\overline{p}} + 6^{\overline{p}-1}\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(t-s_i)H(s_i,u(s_i-\tau_1(s_i)))\|_X^{\overline{p}}$$
$$+ 6^{\overline{p}-1}\mathbb{E}\|H(t,u(t-\tau_1(t)))\|_X^{\overline{p}} + 6^{\overline{p}-1}\mathbb{E}\|\int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds\|_X^{\overline{p}}$$
$$+ 6^{\overline{p}-1}\mathbb{E}\|\int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s)\|_X^{\overline{p}} + 6^{\overline{p}-1}\mathbb{E}\|\int_{s_i}^t\int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s,u(s-\tau_4(s)),y)\widetilde{N}(ds,dy)\|^{\overline{p}}$$
$$\leq 6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}(L_{\mathcal{G}_i}\mathbb{E}\|u(t_i-\tau_5(t_i))\|_X^{\overline{p}}) + 5^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}\|(-A)^{-\beta}\|^{\overline{p}}L_H\|u(s_i-\tau_1(s_i))\|_X^{\overline{p}}$$
$$+ 6^{\overline{p}-1}\|(-A)^{-\beta}\|^{\overline{p}}L_G\mathbb{E}\|u(t-\tau_1(t))\|_X^{\overline{p}} + 6^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\int_{s_i}^t e^{-\delta(t-s)}m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s))\|_X^{\overline{p}})ds$$
$$+ 6^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\int_{s_i}^t e^{-\delta(t-s)}m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})ds$$
$$+ 6^{\overline{p}-1}c_{\overline{p}}\Big\{M^{\overline{p}}L_h^{\overline{p}/2}\Big(\frac{(\overline{p}-2)}{2\delta(\overline{p}-1)}\Big)^{\frac{\overline{p}-2}{2}}\int_{s_i}^t e^{-\delta(t-s)}\mathbb{E}\|u(s-\tau_4(s))\|_X^{\overline{p}}ds + M^{\overline{p}}L_h\Big(\frac{(\overline{p}-2)}{2\delta(\overline{p}-1)}\Big)^{\frac{\overline{p}-2}{2}}$$
$$\times\int_{s_i}^t e^{-\delta(t-s)}\mathbb{E}\|u(s-\tau_4(s))\|_X^{\overline{p}}ds\Big\}.$$

For $t\in(t_i,s_i],\ i=1,\cdots,k$, we have

$$
\begin{aligned}
&\mathbb{E}\|\mathcal{G}_i(t,u(t_i-\tau_5(t_i)))\|_X^{\overline{p}}\\
&= \|\mathcal{G}_i(t,\widehat{\mathcal{S}}_\alpha(t_i-s_{i-i})[\mathcal{G}_{i-1}(s_{i-1},u(t_{i-1}-\tau_5(t_{i-1}))+H(s_{i-1},u(s_{i-1}-\tau_1(s_{i-1})))]\\
&\quad -H(t_i,u(t_i-\tau_1(t_i)))+\int_{s_{i-1}}^{t_i}\widehat{\mathcal{S}}_\alpha(t_i-s)F(s,u(s-\tau_2(s)))ds\\
&\quad +\int_{s_{i-1}}^{t_i}\widehat{\mathcal{S}}_\alpha(t_i-s)G(s,u(s-\tau_3(s)))dW(s)+\int_{s_{i-1}}^{t_i}\int_Z\widehat{\mathcal{R}}_\alpha(t_i-s)h(s,u(s-\tau_4(s)),y)\widetilde{N}(ds,dy))\|^{\overline{p}}\\
&\leq L_{\mathcal{G}_i}\|\widehat{\mathcal{S}}_\alpha(t_i-s_{i-i})[\mathcal{G}_{i-1}(s_{i-1},u(t_{i-1}-\tau_5(t_{i-1}))+H(s_{i-1},u(s_{i-1}-\tau_1(s_{i-1})))]\\
&\quad -H(t_i,u(t_i-\tau_1(t_i)))+\int_{s_{i-1}}^{t_i}\widehat{\mathcal{S}}_\alpha(t_i-s)F(s,u(s-\tau_2(s)))ds\\
&\quad +\int_{s_{i-1}}^{t_i}\widehat{\mathcal{S}}_\alpha(t_i-s)G(s,u(s-\tau_3(s)))dW(s)+\int_{s_{i-1}}^{t_i}\int_Z\widehat{\mathcal{R}}_\alpha(t_i-s)h(s,u(s-\tau_4(s)),y)\widetilde{N}(ds,dy))\|\\
&\to 0, \quad \text{as } t\to\infty. \qquad (20)
\end{aligned}
$$

Thus, for all $u(t)\in\Upsilon$ and $t\in[0,\infty)$, we get

$$
\begin{aligned}
&\mathbb{E}\|\Psi u(t)\|_X^{\overline{p}}\\
&\leq 6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}\mathbb{E}\|\phi(0)\|_X^{\overline{p}}+6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}\|(-A)^{-\beta}\|^{\overline{p}}L_H\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}}\\
&+6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}(L_{\mathcal{G}_i}\mathbb{E}\|u(s_i-\tau_5(s_i))\|_X^{\overline{p}})+6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}\|(-A)^{-\beta}\|^{\overline{p}}L_H\mathbb{E}\|u(s_i-\tau_1(s_i))\|_X^{\overline{p}}\\
&+6^{\overline{p}-1}\|(-A)^{-\beta}\|^{\overline{p}}L_G\mathbb{E}\|u(t-\tau_1(t))\|_X^{\overline{p}}+6^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\int_0^t e^{\delta(t-s)}m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s))\|_X^{\overline{p}})ds\\
&+6^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\int_0^t e^{-\delta(t-s)}m_G(s)\Theta_G(E\|u(s-\tau_3(s))\|_X^{\overline{p}})ds\\
&+6^{\overline{p}-1}c_{\overline{p}}\Big\{M^{\overline{p}}L_h^{\overline{p}/2}\Big(\frac{(\overline{p}-2)}{2\delta(\overline{p}-1)}\Big)^{\frac{\overline{p}-2}{2}}\int_0^t e^{-\delta(t-s)}\mathbb{E}\|u(s-\tau_4(s))\|_X^{\overline{p}}ds\\
&+M^{\overline{p}}L_h\Big(\frac{(\overline{p}-2)}{2\delta(\overline{p}-1)}\Big)^{\frac{\overline{p}-2}{2}}\times\int_0^t e^{-\delta(t-s)}\mathbb{E}\|u(s-\tau_4(s))\|_X^{\overline{p}}ds\Big\}.
\end{aligned}
$$

Furthermore, for any $\epsilon>0$, there exists a $\widetilde{\tau}>0$ such that $\mathbb{E}\|u(t-\tau_i(t))\|_X^{\overline{p}}<\epsilon$ for $t\geq\widetilde{\tau}$. Thus, we get $\mathbb{E}\|\Psi u(t)\|_X^{\overline{p}}$

$$
\begin{aligned}
&\leq 6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}\mathbb{E}\|\phi(0)\|_X^{\overline{p}}+6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}\|(-A)^{-\beta}\|^{\overline{p}}L_H\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}}\\
&+6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}(L_{\mathcal{G}_i}\mathbb{E}\|u(t_i-\tau_5(t_i))\|_X^{\overline{p}})+6^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta t}\times\|(-A)^{-\beta}\|^{\overline{p}}L_H\mathbb{E}\|u(s_i-\tau_1(s_i))\|_X^{\overline{p}}\\
&+6^{\overline{p}-1}\|(-A)^{-\beta}\|^{\overline{p}}L_G\mathbb{E}\|u(t-\tau_1(t))\|_X^{\overline{p}}+6^{\overline{p}-1}M^{\overline{p}}\delta^{\overline{p}-1}e^{-\delta t}\int_0^{\widetilde{\tau}}e^{\delta s}m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s))\|_X^{\overline{p}})ds\\
&+6^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\Theta_F(\epsilon)\times\sup_{t\geq0}\int_{\widetilde{\tau}}^t e^{\delta(t-s)}m_F(s)ds+6^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\\
&\times e^{-\delta t}\int_0^{\widetilde{\tau}}e^{\delta s}m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})ds+C_{\overline{p}}M^{\overline{p}}\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\Theta_G(\epsilon)\sup_{t\geq0}\int_{\widetilde{\tau}}^t e^{-\delta(t-s)}m_G(s)ds\\
&+6^{\overline{p}-1}c_{\overline{p}}M^{\overline{p}}\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}[L_h^{\overline{p}}+\widehat{L}_h^{\overline{p}}]e^{-\delta t}\int_0^{\widetilde{\tau}}e^{\delta s}\mathbb{E}\|u(s-\tau_4(s))\|_X^{\overline{p}}ds+6^{\overline{p}-1}c_{\overline{p}}M^{\overline{p}}\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\\
&\times[L_h^{\overline{p}/2}+L_h]\epsilon\times\sup_{t\geq0}\int_{\widetilde{\tau}}^t e^{-\delta(t-s)}ds.
\end{aligned}
$$

Therefore, we conclude that $\mathbb{E}\|\Psi u(t)\|^{\overline{p}} \to 0$ as $t \to \infty$. Hence, Ψ maps Υ into itself. Next, we are going to show that there exists a fixed point of the operator Ψ which is a mild solution of the problem (1)-(4). To this end, we introduce the decompositions of map Ψ as $\Psi_1 + \Psi_2$ for $t \in [0, T]$, where

$$(\Psi_1 u)(t) = \begin{cases} \widehat{\mathcal{S}}_\alpha(t)H(0, \phi(-\tau_1(0))) - H(t, u(t-\tau_1(t))) \\ + \int_0^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy), \ t \in [0, t_1], \\ \widehat{\mathcal{S}}_\alpha(t-s_i)[\mathcal{G}_i(s_i, u(t_i-\tau_5(t_i))) + H(s_i, u(s_i-\tau_1(s_i)))] - H(t, u(t-\tau_1(t))) \\ + \int_{s_i}^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy), \quad t \in (s_i, t_{i+1}], \\ \mathcal{G}_i(t, u(t_i-\tau_5(t_i))), \quad t \in (t_i, s_i], \ i = 1, \cdots, k., \end{cases}$$

and

$$(\Psi_2 u)(t) = \begin{cases} \widehat{\mathcal{S}}_\alpha(t)\phi(0) + \int_0^t \widehat{\mathcal{S}}_\alpha(t-s)F(s, u(s-\tau_2(s)))ds \\ + \int_0^t \widehat{\mathcal{S}}_\alpha(t-s)G(s, u(s-\tau_3(s)))dW(s), \ t \in [0, t_1], \\ \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)F(s, u(s-\tau_2(s)))ds + \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)G(s, u(s-\tau_3(s)))dW(s), \quad t \in (s_i, t_{i+1}], \\ 0, \quad t \in (t_i, s_i], \ i = 1, \cdots, k. \end{cases}$$

Now, we will prove that Ψ_1 is a contraction while Ψ_2 is a completely continuous operator. We prove the result in the following steps.

Step 1. To show that Ψ_1 is a contraction on Υ.

Let $t \in [0, t_1]$ and $u_1, u_2 \in \Upsilon$. We have
$\mathbb{E}\|(\Psi_1 u_1)(t) - (\Psi_1 u_2)(t)\|_X^{\overline{p}}$

$$\begin{aligned} &\leq 2^{\overline{p}-1}\mathbb{E}\|H(t, u_1(t-\tau_1(t))) - H(t, u_2(t-\tau_1(t)))\|_X^{\overline{p}} + 2^{\overline{p}-1}\mathbb{E}\| \int_0^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s, u_1(s-\tau_4(s)), y)\widetilde{N}(ds, dy) \\ &\quad - \int_0^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s, u_2(s-\tau_4(s)), y)\widetilde{N}(ds, dy)\|^{\overline{p}} \\ &\leq 2^{\overline{p}-1}\|(-A)^{-\beta}\|^{\overline{p}} L_H \mathbb{E}\|u_1(t-\tau_1(t)) - u_2(t-\tau_1(t))\|_X^{\overline{p}} + 2^{\overline{p}-1} c_{\overline{p}} M^{\overline{p}}[\widetilde{L}_h^{\overline{p}/2} + \widetilde{L}_h] \\ &\quad \times \mathbb{E}\|u_1(t-\tau_4(t)) - u_2(t-\tau_4(t))\|_X^{\overline{p}} \\ &\leq \|(-A)^{-\beta}\|^{\overline{p}} L_H \|u_1 - u_2\|_\Upsilon^{\overline{p}} + 2^{\overline{p}-1} c_{\overline{p}} T M^{\overline{p}}[\widetilde{L}_h^{\overline{p}/2} + \widetilde{L}_h]\|u_1 - u_2\|_\Upsilon^{\overline{p}}. \end{aligned}$$

Similarly, we have for any $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, k$,

$$
\begin{aligned}
&\mathbb{E}\|(\Psi_1 u_1)(t) - (\Psi_1 u_2)(t)\|_X^{\bar{p}} \\
&\quad \le 4^{\bar{p}-1}\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(t-s_i)[\mathcal{G}_i(s_i, u_1(t_i-\tau_5(t_i))) - \mathcal{G}_i(s_i, u_2(t_i-\tau_5(t_i)))]\|_X^{\bar{p}} \\
&\quad + 4^{\bar{p}-1}\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(t-s_i)[H(s_i, u_1(s_i-\tau_1(s_i))) - H(s_i, u_2(s_i-\tau_1(s_i)))]\|_X^{\bar{p}} \\
&\quad + 4^{\bar{p}-1}\mathbb{E}\|H(t, u_1(t-\tau_1(t))) - H(t, u_2(t-\tau_1(t)))\|_X^{\bar{p}} \\
&\quad + 4^{\bar{p}-1}\mathbb{E}\| \int_{s_i}^{t}\int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s, u_1(s-\tau_4(s)), y)\widetilde{N}(ds,dy) - \int_{s_i}^{t}\int_Z \widehat{\mathcal{R}}_\alpha(t-s) \\
&\qquad \times h(s, u_2(s-\tau_4(s)), y)\widetilde{N}(ds,dy)\|^{\bar{p}} \\
&\quad \le 4^{\bar{p}-1}M^{\bar{p}}L_{\mathcal{G}_i}\mathbb{E}\|u_1(t_i-\tau_5(t_i)) - u_2(t_i-\tau_5(t_i))\|_X^{\bar{p}} + 4^{\bar{p}-1}\|(-A)^{-\beta}\|^{\bar{p}}M^{\bar{p}}L_H \\
&\quad \times \mathbb{E}\|u_1(t-\tau_1(t)) - u_2(t-\tau_1(t))\|_X^{\bar{p}} + 4^{\bar{p}-1}\|(-A)^{-\beta}\|^{\bar{p}}L_H\mathbb{E}\|u_1(t-\tau_1(t)) - u_2(t-\tau_1(t))\|_X^{\bar{p}} \\
&\quad + 4^{\bar{p}-1}M^{\bar{p}}Tc_{\bar{p}}[\widetilde{L}_h^{\bar{p}/2} + \widetilde{L}_h]\mathbb{E}\|u_1(t-\tau_4(t)) - u_2(t-\tau_4(t))\|_X^{\bar{p}} \\
&\quad \le 4^{\bar{p}-1}\max_{i\in\{1,2,\cdots,k\}}\Big[M^{\bar{p}}L_{\mathcal{G}_i} + \|(-A)^{-\beta}\|^{\bar{p}}M^{\bar{p}}L_H + \|(-A)^{-\beta}\|^{\bar{p}}L_H + M^{\bar{p}}Tc_{\bar{p}}(\widetilde{L}_h^{\bar{p}/2} + \widetilde{L}_h)\Big] \\
&\quad \times \|u_1 - u_2\|_{\Upsilon}^{\bar{p}}.
\end{aligned}
$$

For $t \in (t_i, s_i]$, $i = 1, \cdots, k$, we have

$$
\begin{aligned}
&E\|(\Psi_1 u_1)(t) - (\Psi_1 u_2)(t)\|_X^{\bar{p}} \\
&\quad \le E\|\mathcal{G}_i(t, u_1(t_i-\tau_5(t_i))) - \mathcal{G}_i(t, u_2(t_i-\tau_5(t_i)))\|_X^{\bar{p}} \\
&\quad \le L_{\mathcal{G}_i}\mathbb{E}\|u_1(t_i-\tau_5(t_i)) - u_2(t_i-\tau_5(t_i))\|_X^{\bar{p}} \\
&\quad \le \max_{i\in\{1,\cdots,k\}} L_{\mathcal{G}_i}4^{\bar{p}-1}\max_{i\in\{1,2,\cdots,k\}}\Big[M^{\bar{p}}L_{\mathcal{G}_i} + \|(-A)^{-\beta}\|^{\bar{p}}M^{\bar{p}}L_H + \|(-A)^{-\beta}\|^{\bar{p}}L_H + M^{\bar{p}}Tc_{\bar{p}}(\widetilde{L}_h^{\bar{p}/2} + \widetilde{L}_h)\Big]\|u_1 - u_2\|_{\Upsilon}^{\bar{p}}.
\end{aligned}
$$

Thus, for all $t \in [0, T]$,

$$
\begin{aligned}
&E\|(\Psi_1 u_1)(t) - (\Psi_1 u_2)(t)\|_X^{\bar{p}} \\
&\quad \le 4^{\bar{p}-1}\max_{i\in\{1,2,\cdots,k\}} L_{\mathcal{G}_i}\Big[M^{\bar{p}}L_{\mathcal{G}_i} + \|(-A)^{-\beta}\|^{\bar{p}}M^{\bar{p}}L_H + \|(-A)^{-\beta}\|^{\bar{p}}L_H + M^{\bar{p}}Tc_{\bar{p}}(\widetilde{L}_h^{\bar{p}/2} + \widetilde{L}_h)\Big] \times \|u_1 - u_2\|_{\Upsilon}^{\bar{p}}.
\end{aligned}
$$

Taking supremum over t, we get

$$
\|\Psi_1 u_1 - \Psi_1 u_2\|_{\Upsilon}^{\bar{p}} \le \mathcal{L}\|u_1 - u_2\|_{\Upsilon}^{\bar{p}}, \tag{21}
$$

where $\mathcal{L} = 4^{\bar{p}-1}\max_{i\in\{1,2,\cdots,k\}} L_{\mathcal{G}_i}\Big[M^{\bar{p}}L_{\mathcal{G}_i} + \|(-A)^{-\beta}\|^{\bar{p}}M^{\bar{p}}L_H + \|(-A)^{-\beta}\|^{\bar{p}}L_H + M^{\bar{p}}Tc_{\bar{p}}(\widetilde{L}_h^{\bar{p}/2} + \widetilde{L}_h)\Big] < 1$. Because of inequality (17), it implies that $\mathcal{L} < 1$. Hence, it can be concluded that Ψ_1 is a contraction on Υ.

Step 2. To show that Ψ_2 maps bounded sets into bounded sets in Υ.

To prove this, it is enough to show that there exists a constant $\mathbb{M} > 0$ such that for each $u \in \mathcal{B}_q = \{u : \|u\|_{\Upsilon}^{\bar{p}} \le q\}$, one has $\|\Psi_2 u\|_{\Upsilon}^{\bar{p}} \le \mathbb{M}$. Now, for $t \in [0, t_1]$, we have

$$
\begin{aligned}
(\Psi_2 u)(t) &= \widehat{\mathcal{S}}_\alpha(t)\phi(0) + \int_0^t \widehat{\mathcal{S}}_\alpha(t-s)F(s, u(s-\tau_2(s)))ds \\
&\quad + \int_0^t \widehat{\mathcal{S}}_\alpha(t-s)G(s, u(s-\tau_3(s)))dW(s).
\end{aligned} \tag{22}
$$

Let $u \in \mathcal{B}_q$. Then, from the definition of Υ, we have

$$
\mathbb{E}\|u(s-\tau_i(s))\|_X^{\bar{p}} \le \sup_{s\in[0,T]}\mathbb{E}\|u(s)\|_X^{\bar{p}} \le q, \quad i = 3, 4, 5.
$$

By assumptions $(B1)$, $(B3)$-$(B4)$, inequality (22) and Hölder's inequality, for $t \in [0, t_1]$, we get

$$\begin{aligned}
&\mathbb{E}\|(\Psi_2 u)(t)\|_X^{\overline{p}} \\
&\leq 3^{\overline{p}-1}\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(t)\phi(0)\|_X^{\overline{p}} + 3^{\overline{p}-1}\mathbb{E}\Big\|\int_0^t \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} \\
&\quad + 3^{\overline{p}-1}\mathbb{E}\Big\|\int_0^t \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\leq 3^{\overline{p}-1}M^{\overline{p}}\mathbb{E}\|\phi(0)\|_H^{\overline{p}} + 3^{\overline{p}-1}M^{\overline{p}}\Big[\int_0^t e^{-\delta(t-s)}ds\Big]^{\overline{p}-1} \\
&\quad \times \int_0^t e^{-\delta(t-s)}\mathbb{E}\|F(s,u(s-\tau_2(s)))\|_X^{\overline{p}}ds \\
&\quad + 3^{\overline{p}-1}C_p M^{\overline{p}}\Big[\int_0^t [e^{-\overline{p}\delta(t-s)}m_G(s)\Theta_G(\mathbb{E}\|u(t-\tau_3(t))\|_X^{\overline{p}})\|_X^{\overline{p}})]^{2/\overline{p}}ds\Big]^{\overline{p}/2} \\
&\leq 3^{\overline{p}-1}M^{\overline{p}}\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + 3^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\int_0^t e^{-\delta(t-s)}m_F(s)\Theta_F(\mathbb{E}\|u(t-\tau_2(t))\|_X^{\overline{p}})ds \\
&+ 3^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}} \times \Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\int_0^t e^{-\delta(t-s)}m_G(s)\Theta_G(\mathbb{E}\|u(t-\tau_3(t))\|_X^{\overline{p}})ds \\
&\leq 3^{\overline{p}-1}M^{\overline{p}}\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + 3^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\Theta_F(q)\int_0^{t_1} e^{-\delta(t-s)}m_F(s)ds \\
&\quad + 3^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Theta_G(q)\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\int_0^{t_1} e^{-\delta(t-s)}m_G(s)ds := \mathbb{M}_0.
\end{aligned}$$

Similarly, for any $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, k$, we obtain

$$\begin{aligned}
&(\Psi_2 u)(t) \\
&= \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds + \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s).
\end{aligned} \tag{23}$$

By assumptions $(B1)$, $(B3)$-$(B4)$, equation (23) and Hölder's inequality, we have for $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, k$,

$$\begin{aligned}
&\mathbb{E}\|(\Psi_2 u)(t)\|_X^{\overline{p}} \\
&\leq 2^{\overline{p}-1}\mathbb{E}\Big\|\int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} \\
&\quad + 2^{\overline{p}-1}\mathbb{E}\Big\|\int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\leq 2^{\overline{p}-1}M^{\overline{p}}\delta^{\overline{p}-1} \times \int_{s_i}^t e^{-\delta(t-s)}m_F(s)\Theta_F(\mathbb{E}\|u(t-\tau_2(t))\|_X^{\overline{p}})ds \\
&\quad + 2^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\int_{s_i}^t e^{-\delta(t-s)}m_G(s)\Theta_G(\mathbb{E}\|u(t-\tau_3(t))\|_X^{\overline{p}})ds \\
&\leq 2^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\Theta_F(q)\int_{s_i}^t e^{-\delta(t-s)}m_F(s)ds + 2^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Theta_G(q)\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2} \\
&\quad \times \int_{s_i}^t e^{-\delta(t-s)}m_G(s)ds := \mathbb{M}_i.
\end{aligned}$$

Let $\mathbb{M} = \max_{0\leq i\leq k}\mathbb{M}_i$. Thus, for each $u \in \mathcal{B}_q$, we conclude that $\|\Psi_2 u\|_Y^{\overline{p}} \leq \mathbb{M}$.

Step 3. To show that $\Psi_2 : \Upsilon \to \Upsilon$ is a continuous map.

Let $\{u_n(t)\}_{n=0}^{\infty} \subseteq \Upsilon$ be such that $u_n \to u (n \to \infty)$ in Υ. Then there exists a number $q > 0$ such that $\mathbb{E}\|u_n(t)\|_X^{\overline{p}} \leq q$ for all n and a.e. $t \in [0, T]$, so that $u_n \in \mathcal{B}_q$ and $u \in \mathcal{B}_q$. By assumptions $(B3)$ and $(B4)$, we have

$$\mathbb{E}\|F(s, u_n(s - \tau_2(s))) - F(s, u(s - \tau_2(s)))\|_X^{\overline{p}} \to 0 \quad \text{as } n \to \infty,$$
$$\mathbb{E}\|G(s, u_n(s - \tau_3(s))) - G(s, u(s - \tau_3(s)))\|_X^{\overline{p}} \to 0 \quad \text{as } n \to \infty$$

for each $s \in [0, t]$, and since

$$\mathbb{E}\|F(s, u_n(s - \tau_2(s))) - F(s, u(s - \tau_2(s)))\|_X^{\overline{p}} \leq 2m_F(t)\Theta_F(q),$$
$$\mathbb{E}\|G(s, u_n(s - \tau_3(s))) - G(s, u(s - \tau_3(s)))\|_X^{\overline{p}} \leq 2m_G(t)\Theta_G(q),$$

then by the dominated convergence theorem, for $t \in [0, t_1]$, we have

$$\begin{aligned}
&\mathbb{E}\|(\Psi_2 u_n)(t) - (\Psi_2 u)(t)\|_X^{\overline{p}} \\
&\leq 2^{\overline{p}-1}M^{\overline{p}}\mathbb{E}\Big[\int_0^t e^{-\delta(t-s)}\|F(s, u_n(s - \tau_2(s))) - F(s, u(s - \tau_2(s)))\|_X ds\Big]^{\overline{p}} \\
&\quad + 2^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_0^t (\mathbb{E}\|\mathcal{S}_\alpha(t-s)[G(s, u_n(s - \tau_3(s))) - G(s, u(s - \tau_3(s)))]\|_X^{\overline{p}})^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&\leq 2^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\int_0^t e^{-\delta(t-s)}\mathbb{E}\|F(s, u_n(s - \tau_2(s))) - F(s, u(s - \tau_2(s)))\|_X^{\overline{p}} ds \\
&\quad + 2^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_0^t e^{-2\delta(t-s)}(\mathbb{E}\|G(s, u_n(s - \tau_3(s))) - G(s, u(s - \tau_3(s)))\|_X^{\overline{p}})^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

For any $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, k$,

$$\begin{aligned}
&\mathbb{E}\|(\Psi_2 u_n)(t) - (\Psi_2 u)(t)\|_X^{\overline{p}} \\
&\leq 2^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\int_{s_i}^t e^{-\delta(t-s)}\mathbb{E}\|F(s, u_n(s - \tau_2(s))) - F(s, u(s - \tau_2(s)))\|_X^{\overline{p}} ds \\
&\quad + 2^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_{s_i}^t e^{-2\delta(t-s)}(\mathbb{E}\|G(s, u_n(s - \tau_3(s))) - G(s, u(s - \tau_3(s)))\|_X^{\overline{p}})^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, for all $t \in [0, T]$, we get

$$\|\Psi_2 u_n - \Psi_2 u\|_\Upsilon \to 0 \quad \text{as } n \to \infty. \tag{24}$$

Therefore, Ψ_2 is continuous on $\mathcal{B}_q$.

Step 4. To show that Ψ_2 maps bounded sets into equicontinuous sets of Υ.

Let $0 < \xi_1 < \xi_2 \le t_1$. Then, by using Hölder's inequality, for each $u \in \mathcal{B}_q$, we have

$$
\begin{aligned}
&\mathbb{E}\|(\Psi_2 u)(\xi_2) - (\Psi_2 u)(\xi_1)\|_X^{\overline{p}} \\
&\quad \le 7^{\overline{p}-1}\mathbb{E}\|[\widehat{\mathcal{S}}_\alpha(\xi_2) - \widehat{\mathcal{S}}_\alpha(\xi_1)]\phi(0)\|_X^{\overline{p}} \\
&\quad + 7^{\overline{p}-1}\mathbb{E}\Big[\int_0^{\xi_1-\varepsilon} \|\widehat{\mathcal{S}}_\alpha(\xi_2-s) - \widehat{\mathcal{S}}_\alpha(\xi_1-s)\|_{L(X)}\|F(s,u(s-\tau_2(s)))\|_X ds\Big]^{\overline{p}} \\
&\quad + 7^{\overline{p}-1}\mathbb{E}\Big[\int_{\xi_1-\varepsilon}^{\xi_1} \|\widehat{\mathcal{S}}_\alpha(\xi_2-s) - \widehat{\mathcal{S}}_\alpha(\xi_1-s)\|_{L(X)}\|F(s,u(s-\tau_2(s)))\|_X ds\Big]^{\overline{p}} \\
&\quad + 7^{\overline{p}-1}\mathbb{E}\Big[\int_{\xi_1}^{\xi_2} \|\widehat{\mathcal{S}}_\alpha(\xi_2-s)\|_{L(X)}\|G(s,u(s-\tau_3(s)))\|_X ds\Big]^{\overline{p}} \\
&\quad + 7^{\overline{p}-1}C_{\overline{p}}\Big[\int_0^{\xi_1-\varepsilon} [\|\widehat{\mathcal{S}}_\alpha(\xi_2-s) - \widehat{\mathcal{S}}_\alpha(\xi_1-s)\|_{L(X)}^{p}\Big(\mathbb{E}\|G(s,u(s-\tau_3(s)))\|_X^{\overline{p}}\Big)]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&\quad + 7^{\overline{p}-1}C_{\overline{p}}\Big[\int_{\xi_1-\varepsilon}^{\xi_1} [\|\widehat{\mathcal{S}}_\alpha(\xi_2-s) - \widehat{\mathcal{S}}_\alpha(\xi_1-s)\|_{L(X)}^{\overline{p}} \times \Big(\mathbb{E}\|G(s,u(s-\tau_3(s)))\|_X^{\overline{p}}\Big)]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&\quad + 7^{\overline{p}-1}C_{\overline{p}}\Big[\int_{\xi_1}^{\xi_2} [\|\widehat{\mathcal{S}}_\alpha(\xi_2-s)\|_{L(X)}^{\overline{p}}(\mathbb{E}\|G(s,u(s-\tau_3(s)))\|_X^{\overline{p}})]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&\le 7^{\overline{p}-1}\mathbb{E}\|[\widehat{\mathcal{S}}_\alpha(\xi_2) - \widehat{\mathcal{S}}_\alpha(\xi_1)]\phi(0)\|_X^{\overline{p}} + 7^{\overline{p}-1}T^{\overline{p}}\int_0^{\xi_1-\varepsilon} \|\widehat{\mathcal{S}}_\alpha(\xi_2-s) - \widehat{\mathcal{S}}_\alpha(\xi_1-s)\|_{L(X)}^{\overline{p}} \\
&\quad \times m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s)))\|_X^{\overline{p}})ds + 14^{\overline{p}-1}M^{\overline{p}}\Big[\int_{\xi_1-\varepsilon}^{\xi_1} e^{-\delta(\xi_1-s)}ds\Big]^{\overline{p}-1}\int_{\xi_1-\varepsilon}^{\xi_1} e^{-\delta(\xi_1-s)}m_F(s) \\
&\quad \times \Theta_F(\mathbb{E}\|u(s-\tau_2(s)))\|_X^{\overline{p}})ds + 7^{\overline{p}-1}M^{\overline{p}}\Big[\int_{\xi_1}^{\xi_2} e^{-\delta(\xi_2-s)}ds\Big]^{\overline{p}-1} \\
&\quad \times \int_{\xi_1}^{\xi_2} e^{-\delta(\xi_2-s)}m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s)))\|_X^{\overline{p}})ds + 7^{\overline{p}-1}C_{\overline{p}}\Big[\int_0^{\xi_1-\varepsilon} [\|\widehat{\mathcal{S}}_\alpha(\xi_2-s) - \widehat{\mathcal{S}}_\alpha(\xi_1-s)\|_{L(X)}^{\overline{p}} \\
&\quad \times m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})]^{2/\overline{p}} ds\Big]^{\overline{p}/2} + 14^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_{\xi_1-\varepsilon}^{\xi_1} [e^{-\overline{p}\delta(\xi_1-s)}m_G(s) \\
&\quad \times \Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})]^{2/\overline{p}} ds\Big]^{\overline{p}/2} + 7^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_{\xi_1}^{\xi_2} [e^{-\overline{p}\delta(\xi_2-s)}m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&\le 7^{\overline{p}-1}\mathbb{E}\|[\widehat{\mathcal{S}}_\alpha(\xi_2) - \widehat{\mathcal{S}}_\alpha(\xi_1)]\phi(0)\|_X^{\overline{p}} + 7^{\overline{p}-1}T^{\overline{p}}\Theta_F(q)\int_0^{\xi_1-\varepsilon} \|\widehat{\mathcal{S}}_\alpha(\xi_2-s) - \widehat{\mathcal{S}}_\alpha(\xi_1-s)\|_{L(X)}^{\overline{p}} m_F(s)ds \\
&\quad + 14^{\overline{p}-1}M^{\overline{p}}\Theta_F(q)\delta^{1-\overline{p}}\int_{\xi_1-\varepsilon}^{\xi_1} e^{-\delta(\xi_1-s)}m_F(s)ds + 7^{\overline{p}-1}M^{\overline{p}}\Theta_F(q)\delta^{1-\overline{p}}\int_{\xi_1}^{\xi_2} e^{-\delta(\xi_2-s)}m_F(s)ds \\
&\quad + 7^{\overline{p}-1}C_{\overline{p}}\Theta_G(q)\Big[\int_0^{\xi_1-\varepsilon} [\|\widehat{\mathcal{S}}_\alpha(\xi_2-s) - \widehat{\mathcal{S}}_\alpha(\xi_1-s)\|_{L(X)}^{\overline{p}} m_G(s)]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&\quad + 14^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Theta_G(q)\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\int_{\xi_1-\varepsilon}^{\xi_1} e^{-\delta(\xi_1-s)}m_G(s)ds \\
&\quad + 7^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Theta_G(q)\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\int_{\xi_1}^{\xi_2} e^{-\delta(\xi_2-s)}m_G(s)ds.
\end{aligned}
$$

Similarly, for any $\xi_1, \xi_2 \in (s_i, t_{i+1}]$, $\xi_1 < \xi_2$, $i = 1, \ldots, k$, we have

$$
(\Psi_2 u)(t) = \int_{s_i}^{t} \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds + \int_{s_i}^{t} \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s). \tag{25}
$$

Then, we have

$$
\begin{aligned}
&\mathbb{E}\|(\Psi_2 u)(\xi_2) - (\Psi_2 u)(\xi_1)\|_X^{\overline{p}} \\
&\leq 6^{\overline{p}-1}\mathbb{E}\Big\| \int_{s_i}^{\xi_1-\varepsilon} [\widehat{\mathcal{S}}_\alpha(\xi_2 - s) - \widehat{\mathcal{S}}_\alpha(\xi_1 - s)]F(s, u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} \\
&\quad + 6^{\overline{p}-1}\mathbb{E}\Big\| \int_{\xi_1-\varepsilon}^{\xi_1} [\widehat{\mathcal{S}}_\alpha(\xi_2 - s) - \widehat{\mathcal{S}}_\alpha(\xi_1 - s)]F(s, u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} \\
&\quad + 6^{\overline{p}-1}\mathbb{E}\Big\| \int_{\xi_1}^{\xi_2} \widehat{\mathcal{S}}_\alpha(\xi_2 - s)G(s, u(s-\tau_3(s)))ds\Big\|_X^{\overline{p}} \\
&\quad + 6^{\overline{p}-1}\mathbb{E}\Big\| \int_{s_i}^{\xi_1-\varepsilon} [\widehat{\mathcal{S}}_\alpha(\xi_2 - s) - \widehat{\mathcal{S}}_\alpha(\xi_1 - s)]G(s, u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\quad + 6^{\overline{p}-1}\mathbb{E}\Big\| \int_{\xi_1-\varepsilon}^{\xi_1} [\widehat{\mathcal{S}}_\alpha(\xi_2 - s) - \widehat{\mathcal{S}}_\alpha(\xi_1 - s)]G(s, u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\quad + 6^{\overline{p}-1}\mathbb{E}\Big\| \int_{\xi_1}^{\xi_2} \widehat{\mathcal{S}}_\alpha(\xi_2 - s)G(s, u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\leq 6^{\overline{p}-1}T^{\overline{p}}\Theta_F(q)\int_{s_i}^{\xi_1-\varepsilon} \|\widehat{\mathcal{S}}_\alpha(\xi_2 - s) - \widehat{\mathcal{S}}_\alpha(\xi_1 - s)\|_{L(X)}^{\overline{p}} m_F(s)ds \\
&\quad + 12^{\overline{p}-1}M^{\overline{p}}\Theta_F(q)\delta^{1-\overline{p}}\int_{\xi_1-\varepsilon}^{\xi_1} e^{-\delta(\xi_1-s)}m_F(s)ds \\
&\quad + 6^{\overline{p}-1}M^{\overline{p}}\Theta_F(q)\delta^{1-\overline{p}}\int_{\xi_1}^{\xi_2} e^{-\delta(\xi_2-s)}m_F(s)ds \\
&\quad + 6^{\overline{p}-1}C_{\overline{p}}\Theta_G(q)\Big[\int_{s_i}^{\xi_1-\varepsilon} [\|\widehat{\mathcal{S}}_\alpha(\xi_2 - s) - \widehat{\mathcal{S}}_\alpha(\xi_1 - s)\|_{L(X)}^{\overline{p}} m_G(s)]^{2/p}ds\Big]^{\overline{p}/2} \\
&\quad + 6^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Theta_G(q)\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\int_{\xi_1-\varepsilon}^{\xi_1} e^{-\delta(t-s)}m_G(s)ds \\
&\quad + 6^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Theta_G(q)\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2}\int_{\xi_1}^{\xi_2} e^{-\delta s}m_G(s)ds.
\end{aligned}
$$

The compactness of $\widehat{\mathcal{S}}_\alpha(t)$ for $t > 0$ gives the continuity in the uniform operator topology. Since ε is sufficiently small, the right-hand side of the above inequality is independent of $u \in \mathcal{B}_q$ and tends to zero as $\xi_2 - \xi_1 \to 0$. The equicontinuity for the cases $\xi_1 < \xi_2 \leq 0$ or $\xi_1 \leq 0 \leq \xi_2 \leq T$ are very simple. Thus the set $\{\Psi_2 u : u \in \mathcal{B}_q\}$ is equicontinuous.

Step 5. To show that the set $\mathbb{W}(t) = \{(\Psi_2 u)(t) : u \in \mathcal{B}_q\}$ is relatively compact in X.
It is clear that $\mathbb{W}(0) = \{(\Psi_2 u)(0) : \ u \in \mathcal{B}_q\}$ is a compact subset of X due to the compactness of the operator $\widehat{\mathcal{S}}_\alpha(t),\ t \geq 0$. Next, we show that $\{(\Psi_2 u)(t) : u \in \mathcal{B}_q\}$ is relatively compact for every $t \in (0, T]$. Let $0 < t \leq s \leq t_1$ be fixed and let ε be a real number such that $0 < \varepsilon < t$. For $u \in \mathcal{B}_q$, we define

$$
\begin{aligned}
(\Psi_2^\varepsilon u)(t)(t) &= \widehat{\mathcal{S}}_\alpha(t)\phi(0) + \int_0^{t-\varepsilon} \widehat{\mathcal{S}}_\alpha(t-s)F(s, u(s-\tau_2(s)))ds \\
&\quad + \int_0^{t-\varepsilon} \widehat{\mathcal{S}}_\alpha(t-s)G(s, u(s-\tau_3(s)))dW(s). \qquad (26)
\end{aligned}
$$

Using the compactness of $\widehat{\mathcal{S}}_\alpha(t)$ for $t > 0$, we can deduce that the set $\mathbb{U}_\varepsilon(t) = \{(\Psi_2^\varepsilon u)(t) : u \in \mathcal{B}_q\}$ is relatively

compact in X for every $\varepsilon, 0 < \varepsilon < t$. Moreover, by using Hölder's inequality, we have, for every $u \in \mathcal{B}_q$

$$\begin{aligned}
&\mathbb{E}\|(\Psi_2 u)(t) - (\Psi_2^{\varepsilon} u)(t)(t)\|_X^{\overline{p}} \\
&\leq 2^{\overline{p}-1}\mathbb{E}\Big\| \int_{t-\varepsilon}^{t} \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} + 2^{\overline{p}-1}\mathbb{E}\Big\| \int_{t-\varepsilon}^{t} \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\leq 2^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}} \int_{t-\varepsilon}^{t} e^{-\delta(t-s)} m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s))\|_X^{\overline{p}})ds \\
&\quad + 2^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_{t-\varepsilon}^{t} [e^{-\overline{p}\delta(t-s)} m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&\leq 2^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\Theta_F(q) \int_{t-\varepsilon}^{t} e^{-\delta(t-s)} m_F(s)ds + 2^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Theta_G(q)\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2} \\
&\quad \times \int_{t-\varepsilon}^{t} e^{-\delta(t-s)} m_G(s)ds.
\end{aligned}$$

For any $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, k$, let $s_i < t \leq s \leq t_{i+1}$ be fixed and let ε be a real number such that $0 < \varepsilon < t$. For $u \in \mathcal{B}_q$, we define

$$(\Psi_2^{\varepsilon} u)(t) = \int_{s_i}^{t-\varepsilon} \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds + \int_{s_i}^{t-\varepsilon} \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_3(s)))dW(s).$$

Using the compactness of $\widehat{\mathcal{S}}_\alpha(t)$ for $t > 0$, we deduce that the set $\mathbb{U}_\varepsilon(t) = \{(\Psi_2^{\varepsilon} u)(t) : u \in \mathcal{B}_q\}$ is relatively compact in X for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for every $u \in \mathcal{B}_q$ we have

$$\begin{aligned}
&\mathbb{E}\|(\Psi_2 u)(t) - (\Psi_2^{\varepsilon} u)(t)\|_X^{\overline{p}} \\
&\leq 2^{\overline{p}-1}\mathbb{E}\Big\| \int_{t-\varepsilon}^{t} \widehat{\mathcal{S}}_\alpha(t-s)F(s,u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} + 2^{\overline{p}-1}\mathbb{E}\Big\| \int_{t-\varepsilon}^{t} \widehat{\mathcal{S}}_\alpha(t-s)G(s,u(s-\tau_4(s)))dw(s)\Big\|_X^{\overline{p}} \\
&\leq 2^{\overline{p}-1}M^{\overline{p}}\delta^{1-\overline{p}}\Theta_F(q) \int_{t-\varepsilon}^{t} e^{-\delta(t-s)} m_F(s)ds + 2^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Theta_G(q)\Big[\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big]^{1-\overline{p}/2} \\
&\quad \times \int_{t-\varepsilon}^{t} e^{-\delta(t-s)} m_G(s)ds.
\end{aligned}$$

There are relatively compact sets which are arbitrarily close to the set $\mathbb{W}(t) = \{(\Psi_2 u)(t) : u \in \mathcal{B}_q\}$, and $\mathbb{W}(t)$ is relatively compact in X. It is easy to see that $\Psi_2(\mathcal{B}_r)$ is uniformly bounded. Since we have shown $\Psi_2(\mathcal{B}_q)$ is an equicontinuous collection, by the Arzelá-Ascoli theorem, we conclude the relatively compactness of $\mathbb{W}$. Therefore, we obtain that operator Ψ_2 is a compact map.

Step 6. We shall show the set $\Upsilon = \{u \in \Upsilon : \lambda_1\Psi_1(\frac{u}{\lambda_1}) + \lambda_1\Psi_2(u) = u \text{ for some } \lambda_1 \in (0,1)\}$ is bounded on $[0,T]$. To prove this, we consider the following nonlinear operator equation:

$$u(t) = \lambda_1\Psi u(t),\ 0 < \lambda_1 < 1, \tag{27}$$

where Ψ is already defined by equation (18). Next, we give a priori estimate for the solution of the above equation. Indeed, let $u \in \Upsilon$ be a possible solution of $u = \lambda_1\Psi(u)$ for some $0 < \lambda_1 < 1$. Therefore, for each

$t \in [0, T]$, we have

$$u(t) = \begin{cases} \lambda_1 \widehat{\mathcal{S}}_\alpha(t)[\phi(0) + H(0, \phi(-\tau_1(0)))] - \lambda_1 H(t, u(t-\tau_1(t))) \\ +\lambda_1 \int_0^t \widehat{\mathcal{S}}_\alpha(t-s) F(s, u(s-\tau_2(s)))ds + \lambda_1 \int_0^t \widehat{\mathcal{S}}_\alpha(t-s) G(s, u(s-\tau_4(s)))dW(s) \\ +\lambda_1 \int_0^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s) h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy), \ t \in [0, t_1] \\ \lambda_1 \widehat{\mathcal{S}}_\alpha(t-s_i)[\mathcal{G}_i(s_i, u(t_i - \tau_5(t_i))) + H(s_i, u(s_i - \tau_1(s_i)))] - \lambda_1 H(t, u(t-\tau_1(t))) \\ +\lambda_1 \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s) F(s, u(s-\tau_2(s)))ds + \lambda_1 \int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s) G(s, u(s-\tau_3(s)))dW(s) \\ +\lambda_1 \int_0^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s) h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy), \quad t \in (s_i, t_{i+1}] \\ \lambda_1 \mathcal{G}_i(t, u(t_i - \tau_5(t_i))), \quad t \in (t_i, s_i], \ i = 1, \cdots, k. \end{cases} \tag{28}$$

By using Hölder's inequality and Lemma 2.8, we have, for $t \in [0, t_1]$

$$\begin{aligned}
&\mathbb{E}\|u(t)\|_X^{\overline{p}} \\
&\le 5^{\overline{p}-1}\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(t)[\phi(0) + H(0, \phi(-\tau_1(0)))]\|_X^{\overline{p}} + 5^{\overline{p}-1}\mathbb{E}\|H(t, u(t-\tau_1(t)))\|_X^{\overline{p}} \\
&+ 5^{\overline{p}-1}\mathbb{E}\Big\| \int_0^t \widehat{\mathcal{S}}_\alpha(t-s) F(s, u(s-\tau_3(s)))ds\Big\|_X^{\overline{p}} + 5^{\overline{p}-1}\mathbb{E}\Big\| \int_0^t \widehat{\mathcal{S}}_\alpha(t-s) G(s, u(s-\tau_4(s)))dW(s)\Big\|_X^{\overline{p}} \\
&+ 5^{\overline{p}-1}\mathbb{E}\| \int_0^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s) h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy)\|^{\overline{p}} \\
&\le 10^{\overline{p}-1}M^{\overline{p}}[\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + \mathbb{E}\|H(0, \phi(-\tau_1(0)))\|_X^{\overline{p}}] + 5^{\overline{p}-1}\mathbb{E}\|H(t, u(t-\tau_1(t)))\|_X^{\overline{p}} \\
&\quad + 5^{\overline{p}-1}M^{\overline{p}}\mathbb{E}\Big[\int_0^t e^{-\delta(t-s)}\|F(s, u(s-\tau_2(s)))\|_X ds\Big]^{\overline{p}} + 5^{\overline{p}-1}\Big[\int_0^t [e^{-\overline{p}\delta(t-s)}(\mathbb{E}\|G(s, u(s-\tau_3(s)))\|_X^{\overline{p}})]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&+ 5^{\overline{p}-1}c_{\overline{p}}\Big\{\mathbb{E}\Big(\int_0^t \int_Z \|\widehat{\mathcal{R}}_\alpha(t-s) h(s, u(s-\tau_4(s)), y)\|^2 ds\vartheta dy\Big)^{\overline{p}/2} \\
&\quad + \mathbb{E}\int_0^t \int_Z \|\widehat{\mathcal{R}}_\alpha(t-s) h(s, u(s-\tau_4(s)), y)\|^{\overline{p}} ds\vartheta dy\Big\} \\
&\le 10^{\overline{p}-1}M^{p}[\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + L_H\|(-A)^{-\beta}\|^{\overline{p}}\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}}] + 5^{\overline{p}-1}L_H\|(-A)^{-\beta}\|^{\overline{p}}(\mathbb{E}\|u(t-\tau_1(t))\|_X^{\overline{p}}) \\
&+ 5^{\overline{p}-1}M^{\overline{p}}\Big[\int_0^t e^{-(\overline{p}\delta/\overline{p}-1)(t-s)}ds\Big]^{\overline{p}-1}\int_0^t m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s))\|_X^{\overline{p}})ds \\
&\quad + 5^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_0^t [e^{-\overline{p}\delta(t-s)}m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&+ 5^{\overline{p}-1}c_{\overline{p}}\Big\{\Big(\int_0^t \|\widehat{\mathcal{R}}_\alpha(t-s)\|^2 \int_Z \mathbb{E}\|h(s, u(s-\tau_4(s)), y)\|^2 \vartheta dyds\Big)^{\overline{p}/2} \\
&\quad + \int_0^t \|\widehat{\mathcal{R}}_\alpha(t-s)\|^{\overline{p}} \int_Z \mathbb{E}\|h(s, u(s-\tau_4(s)), y)\|^{\overline{p}}\vartheta dyds\Big\} \\
&\le 10^{\overline{p}-1}M^{\overline{p}}[\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + 2^{\overline{p}-1}L_H\|(-A)^{-\beta}\|^{\overline{p}}\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}}] + 5^{\overline{p}-1}L_H\|(-A)^{-\beta}\|^{\overline{p}}(\mathbb{E}\|u(t-\tau_1(t))\|_X^{\overline{p}}) \\
&+ 5^{\overline{p}-1}M^{\overline{p}}\Big[\int_0^t e^{-(\overline{p}\delta/\overline{p}-1)(t-s)}ds\Big]^{\overline{p}-1}\int_0^t m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s))\|_X^{\overline{p}})ds \\
&\quad + 5^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_0^t [e^{-\overline{p}\delta(t-s)}m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \\
&+ 5^{\overline{p}-1}c_{\overline{p}}M^{\overline{p}}\Big\{\Big(\frac{2\overline{p}\delta}{(\overline{p}-2)}\Big)^{(2-\overline{p})/2}TL_h^{\overline{p}/2}\mathbb{E}\|u(t-\tau_4(t))\|^{\overline{p}} + \Big(\frac{\overline{p}\delta}{\overline{p}-1}\Big)^{1-\overline{p}}TL_h\mathbb{E}\|u(t-\tau_4(t))\|^{\overline{p}}\Big\}.
\end{aligned}$$

For any $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, k$, we have

$$
\begin{aligned}
&\mathbb{E}\|u(t)\|_X^{\overline{p}} \\
&\leq 5^{\overline{p}-1}\mathbb{E}\|\widehat{\mathcal{S}}_\alpha(t-s_i)[\mathcal{G}_i(s_i, u(t_i-\tau_5(t_i))) + H(s_i, u(s_i-\tau_1(s_i)))]\|_X^{\overline{p}} + 5^{\overline{p}-1}\mathbb{E}\|H(t, u(t-\tau_1(t)))\|_X^{\overline{p}} \\
&\quad + 5^{\overline{p}-1}\mathbb{E}\Big\|\int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)F(s, u(s-\tau_2(s)))ds\Big\|_X^{\overline{p}} + 5^{\overline{p}-1}\mathbb{E}\Big\|\int_{s_i}^t \widehat{\mathcal{S}}_\alpha(t-s)G(s, u(s-\tau_3(s)))dW(s)\Big\|_X^{\overline{p}} \\
&\quad + 5^{\overline{p}-1}\mathbb{E}\|\int_{s_i}^t \int_Z \widehat{\mathcal{R}}_\alpha(t-s)h(s, u(s-\tau_4(s)), y)\widetilde{N}(ds, dy)\|^{\overline{p}} \\
&\leq 10^{\overline{p}-1}M^{\overline{p}}\Big[L_{\mathcal{G}_i}\|u(t_i-\tau_5(t_i))\|_X^{\overline{p}} + \|(-A)^{-\beta}\|^{\overline{p}}L_H\|u(s_i-\tau_1(s_i))\|_X^{\overline{p}}\Big] + 5^{\overline{p}-1}L_H\mathbb{E}\|u(t-\tau_1(t))\|_X^{\overline{p}} \\
&\quad + 5^{\overline{p}-1}M^{\overline{p}}\Big[\int_{s_i}^t e^{-(\overline{p}\delta/\overline{p}-1)(t-s)}ds\Big]^{\overline{p}-1}\int_{s_i}^t m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_2(s))\|_X^{\overline{p}})ds \\
&\quad + 5^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_{s_i}^t [e^{-\overline{p}\delta(t-s)}m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})]^{2/\overline{p}}ds\Big]^{\overline{p}/2} + 5^{\overline{p}-1}c_{\overline{p}}M^{\overline{p}} \\
&\times\Big\{\Big(\frac{2\overline{p}\delta}{(\overline{p}-2)}\Big)^{(2-\overline{p})/2}TL_h^{\overline{p}/2}\mathbb{E}\|u(t-\tau_4(t))\|^{\overline{p}} + \Big(\frac{\overline{p}\delta}{\overline{p}-1}\Big)^{1-\overline{p}}L_h \times T\ \mathbb{E}\|u(t-\tau_4(t))\|^{\overline{p}}\Big\}.
\end{aligned}
$$

For $t \in (t_i, s_i]$, $i = 1, \cdots, k$, we have

$$\mathbb{E}\|u(t)\|_X^{\overline{p}} \leq L_{\mathcal{G}_i}\mathbb{E}\|u(t_i-\tau_5(t_i))\|_X^{\overline{p}}. \tag{29}$$

Thus, for all $t \in [0, T]$, we have

$$
\begin{aligned}
&\mathbb{E}\|u(t)\|_X^{\overline{p}} \\
&\leq 10^{\overline{p}-1}M^{\overline{p}}[\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + L_H\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}}\Big] + 10^{\overline{p}-1}M^{\overline{p}}\Big[L_{\mathcal{G}_i}\mathbb{E}\|u(t_i-\tau_5(t_i))\|_X^{\overline{p}} + \|(-A)^{-\beta}\|^{\overline{p}} \\
&\quad \times L_H\mathbb{E}\|u(s_i-\tau_1(s_i))\|_X^{\overline{p}}\Big] + 5^{\overline{p}-1}\|(-A)^{-\beta}\|^{\overline{p}}L_H\mathbb{E}\|u(t-\tau_1(t))\|_X^{\overline{p}} + 5^{\overline{p}-1}M^{\overline{p}}\Big[\int_0^t e^{-(\overline{p}\delta/\overline{p}-1)(t-s)}ds\Big]^{\overline{p}-1} \\
&\quad \times \int_0^t m_F(s)\Theta_F(\mathbb{E}\|u(s-\tau_3(s))\|_X^{\overline{p}})ds + 4^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_0^t [e^{-\overline{p}\delta(t-s)}m_G(s)\Theta_G(\mathbb{E}\|u(s-\tau_4(s))\|_X^{\overline{p}})]^{2/\overline{p}}ds\Big]^{\overline{p}/2} \\
&\quad + 5^{\overline{p}-1}c_{\overline{p}}M^{\overline{p}}\Big\{\Big(\frac{2\overline{p}\delta}{\overline{p}-2}\Big)^{(2-\overline{p})/2}TL_h^{\overline{p}/2}\mathbb{E}\|u(t-\tau_4(t))\|^{\overline{p}} + \Big(\frac{\overline{p}\delta}{\overline{p}-1}\Big)^{1-\overline{p}}L_h \times T\ \mathbb{E}\|u(t-\tau_4(t))\|^{\overline{p}}\Big\}.
\end{aligned}
$$

By the definition of Υ, it follows that

$$\mathbb{E}\|u(s-\tau_j(s))\|_X^{\overline{p}} \leq \sup_{s\in[0,t]}\mathbb{E}\|u(s)\|_X^{\overline{p}},\ j = 1, 2, 3, 4, 5.$$

Let $\mu(t) = \sup_{s\in[0,t]}\mathbb{E}\|u(s)\|_X^{\overline{p}}$. We obtain that

$$
\begin{aligned}
\mu(t) &\leq 10^{\overline{p}-1}M^{\overline{p}}[\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + L_H\|(-A)^{-\beta}\|^{\overline{p}}\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}}\Big] + 10^{\overline{p}-1}M^{\overline{p}}\Big[L_{\mathcal{G}_i}\mu(t) + L_H\|(-A)^{-\beta}\|^{\overline{p}}\mu(t)\Big] \\
&\quad + 5^{\overline{p}-1}\|(-A)^{-\beta}\|^{\overline{p}}L_H\mu(t) + 5^{\overline{p}-1}M^{\overline{p}}\Big[\int_0^t e^{-(\overline{p}\delta/\overline{p}-1)(t-s)}ds\Big]^{\overline{p}-1}\int_0^t m_F(s)\Theta_F(\mu(t))ds \\
&\quad + 5^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big[\int_0^t [e^{-\overline{p}\delta(t-s)}m_G(s)\Theta_G(\mu(t))]^{2/\overline{p}}ds\Big]^{\overline{p}/2} + 5^{\overline{p}-1}c_{\overline{p}}M^{\overline{p}}\Big\{\Big(\frac{2\overline{p}\delta}{(\overline{p}-2)}\Big)^{(2-\overline{p})/2}TL_h^{\overline{p}/2} \\
&\quad + \Big(\frac{\overline{p}\delta}{\overline{p}-1}\Big)^{1-\overline{p}}L_hT\Big\}\mu(t).
\end{aligned}
$$

Also we have

$$\Big[\int_0^t [e^{-\overline{p}\delta(t-s)} m_G(s)\Theta_G(\mu(s))]^{2/\overline{p}} ds\Big]^{\overline{p}/2} \leq \Big[\int_0^t e^{-[\frac{2\overline{p}}{\overline{p}-2}]\delta(t-s)} ds\Big]^{\overline{p}/2-1} \int_0^t m_G(s)\Theta_G(\mu(s))ds$$
$$\leq [\frac{2\delta\overline{p}}{\overline{p}-2}]^{1-\overline{p}/2} \int_0^t m_G(s)\Theta_G(\mu(s))ds.$$

Thus, we obtain

$$\mu(t) \leq 10^{\overline{p}-1}M^{\overline{p}}[\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + L_H\|(-A)^{-\beta}\|^{\overline{p}}\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}}] + 10^{\overline{p}-1}M^{\overline{p}}\Big[L_{\mathcal{G}_i} + L_H\|(-A)^{-\beta}\|^{\overline{p}}\Big]\mu(t)$$
$$+ 5^{\overline{p}-1}\|(-A)^{-\beta}\|^{\overline{p}}L_H \times \mu(t) + 5^{\overline{p}-1}M^{\overline{p}}\Big(\frac{\overline{p}\delta}{\overline{p}-1}\Big)^{1-\overline{p}} \int_0^t m_F(s)\Theta_F(\mu(s))ds$$
$$+ 8^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big(\frac{2\delta\overline{p}}{\overline{p}-2}\Big)^{1-\overline{p}/2} \int_0^t m_G(s)\Theta_G(\mu(s))ds + 5^{\overline{p}-1}c_{\overline{p}}M^{\overline{p}}\Big\{\Big(\frac{2\overline{p}\delta}{(\overline{p}-2)}\Big)^{(2-\overline{p})/2} TL_h^{\overline{p}/2}$$
$$+ \Big(\frac{\overline{p}\delta}{\overline{p}-1}\Big)^{1-\overline{p}} L_h T\Big\}\mu(t).$$

Since

$$\widetilde{\mathbb{L}} = \max_{1\leq i\leq k}\Big\{10^{\overline{p}-1}M^{\overline{p}}[L_{\mathcal{G}_i} + L_H\|(-A)^{-\beta}\|^{\overline{p}}] + 5^{\overline{p}-1}\|(-A)^{-\beta}\|^{\overline{p}}L_H + 5^{\overline{p}-1}c_{\overline{p}}M^{\overline{p}}$$
$$\times\Big\{\Big(\frac{2\overline{p}\delta}{(\overline{p}-2)}\Big)^{(2-\overline{p})/2} TL_h^{\overline{p}/2} + \Big(\frac{\overline{p}\delta}{\overline{p}-1}\Big)^{1-\overline{p}} L_h T\Big\}\Big\} < 1, \tag{30}$$

therefore, we get

$$\mu(t) \leq \frac{1}{1-\widetilde{\mathbb{L}}}\Big[10^{\overline{p}-1}M^{p}[\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + L_H\|(-A)^{-\beta}\|^{\overline{p}}\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}}] + 5^{\overline{p}-1}M^{\overline{p}}\Big(\frac{\overline{p}\delta}{\overline{p}-1}\Big)^{1-\overline{p}}$$
$$\times \int_0^t m_F(s)\Theta_F(\mu(s))ds + 5^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big(\frac{2\delta\overline{p}}{\overline{p}-2}\Big)^{1-\overline{p}/2} \int_0^t m_G(s)\Theta_G(\mu(s))ds\Big].$$

Denoting by $\xi(t)$ the right-hand side of the above inequality, we have

$$\mu(t) \leq \xi(t) \quad \text{for all } t \in [0,T],$$
$$\xi(0) = \frac{1}{1-\widetilde{\mathbb{L}}}\Big[10^{\overline{p}-1}M^{\overline{p}}[\mathbb{E}\|\phi(0)\|_X^{\overline{p}} + L_H\|(-A)^{-\beta}\|^{\overline{p}}\mathbb{E}\|\phi(-\tau_1(0))\|_X^{\overline{p}}]\Big],$$
$$\xi'(t) = \frac{1}{1-\widetilde{\mathbb{L}}}\Big[5^{\overline{p}-1}M^{\overline{p}}[\frac{\overline{p}\delta}{\overline{p}-1}]^{1-\overline{p}} m_F(t)\Theta_F(\mu(t)) + 5^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}[\frac{2\delta\overline{p}}{\overline{p}-2}]^{1-\overline{p}/2} m_G(t)\Theta_G(\mu(t))\Big]$$
$$\leq \frac{1}{1-\widetilde{\mathbb{L}}}\Big[5^{\overline{p}-1}M^{\overline{p}}[\frac{\overline{p}\delta}{\overline{p}-1}]^{1-\overline{p}} m_F(t)\Theta_F(\xi(t)) + 5^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}[\frac{2\delta\overline{p}}{\overline{p}-2}]^{1-\overline{p}/2} m_G(t)\Theta_G(\xi(t))\Big]$$
$$\leq m^*(t)[\Theta_F(\xi(t)) + \Theta_G(\xi(t))],$$

where

$$m^*(t) = \max\Big\{\frac{1}{1-\widetilde{\mathbb{L}}}5^{\overline{p}-1}M^{\overline{p}}\Big(\frac{\overline{p}\delta}{\overline{p}-1}\Big)^{1-\overline{p}} m_F(t),\ \frac{1}{1-\widetilde{\mathbb{L}}}8^{\overline{p}-1}C_{\overline{p}}M^{\overline{p}}\Big(\frac{2\delta\overline{p}}{\overline{p}-2}\Big)^{1-\overline{p}/2} m_G(t)\Big\}.$$

This implies that

$$\int_{\xi(0)}^{\xi(t)} \frac{d\theta}{\Theta_F(\theta) + \Theta_G(\theta)} \leq \int_0^T m^*(s)ds < \infty.$$

The above inequality provides that there is a constant $\widetilde{C}$ such that $\xi(t) \leq \widetilde{C}, t \in [0,T]$, and hence $\|u\|_{\Upsilon} \leq \xi(t) \leq \widetilde{C}$, where $\widetilde{C}$ depends only on $M, C_{\overline{p}}, c_{\overline{p}}, \delta, \overline{p}$, and on the functions $m_G(\cdot), m_F(\cdot), \Theta_F(\cdot)$ and $\Theta_G(\cdot)$. It implies that the map Ψ is bounded on $[0,T]$. Consequently, by Lemma 2.9, we conclude that $\Psi_1 + \Psi_2$ has a fixed point $u(\cdot) \in \Upsilon$ and that fixed point is a mild solution of the system (1)-(4) with $u(s) = \phi(s)$ on $[-\tau, 0]$ and $\mathbb{E}\|u(t)\|_X^{\overline{p}} \to 0$ as $t \to \infty$. This gives the asymptotic stability of the mild solution of (1)-(4). In fact, let $\varepsilon > 0$ be given and choose $\tilde{\delta} > 0$ such that $\tilde{\delta} < \varepsilon$ and satisfies

$$\Big[10^{\overline{p}-1}M^{\overline{p}} + 5^{\overline{p}-1}M^{\overline{p}}[\delta^{1-\overline{p}} \times \mathbb{L}_F + C_{\overline{p}} \times (\frac{2\delta(\overline{p}-1)}{\overline{p}-2})^{1-2/\overline{p}}\mathbb{L}_G]\Big]\tilde{\delta} + \widetilde{L}\varepsilon < \varepsilon,$$

where $\mathbb{L}_F = \sup_{t\geq 0}\int_{\widetilde{\tau}}^{t} e^{-\delta(t-s)}m_F(s)ds$, $t \geq \widetilde{\tau}$ and $\mathbb{L}_G = \sup_{t\geq 0}\int_{\widetilde{\tau}}^{t} e^{-\delta(t-s)}m_G(s)ds$. If $u(t) = u(t,w)$ is a mild solution of (1)-(4), with $\|w\|_X^{\overline{p}} + L_H\mathbb{E}\|w(-\tau_1(0))\|_X^{\overline{p}} < \tilde{\delta}$, then $(\Psi u)(t) = u(t)$ such that $\mathbb{E}\|u(t)\|_X^{\overline{p}} < \varepsilon$ for every $t \geq 0$. It is noticed that $\mathbb{E}\|u(t)\|_X^{\overline{p}} < \varepsilon$ on $t \in [-\tau, 0]$. If there is a $\tilde{t}$ such that $\mathbb{E}\|u(\tilde{t})\|_X^{\overline{p}} = \varepsilon$ and $\mathbb{E}\|u(s)\|_X^{\overline{p}} < \varepsilon$ for $s \in [-\tau, \tilde{t}]$, then we have

$$\begin{aligned}
&\mathbb{E}\|u(t)\|_X^{\overline{p}} \\
&\leq \Big[10^{\overline{p}-1}M^{\overline{p}}e^{-\overline{p}\delta\tilde{t}} + 5^{\overline{p}-1}M^{\overline{p}}\Big[\delta^{1-\overline{p}} \times L_F + C_{\overline{p}} \times \Big(\frac{2\delta(\overline{p}-1)}{\overline{p}-2}\Big)^{1-2/\overline{p}}L_G\Big]\Big]\tilde{\delta} + \widetilde{L}\varepsilon < \varepsilon,
\end{aligned}$$

which contradicts the definition of $\tilde{t}$. Therefore, the mild solution of (1)-(4) is asymptotically stable in the $\overline{p}$-th moment. Thus the proof is complete. □

4. Conclusion

This work studies the asymptotic stability in the p-th moment of the mild solution of an impulsive neutral fractional stochastic differential equation driven by Poisson jumps. The asymptotic stability of the stochastic system with Poisson jumps is obtained by virtue of resolvent operator theory via Krasnoselskii-Schaefer fixed point theorem and these results generalize the past results on asymptotic stability of stochastic differential equation with non-instantaneous conditions. The stability of the mild solution of a neutral fractional differential equation with impulsive conditions driven by fractional Brownian motion will be investigated via fixed point technique in our future work.

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