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Nonlinear Oscillation and Second Order Impulsive Neutral Difference Equations

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Abstract. In this work, the authors have discussed the necessary and sufficient conditions for oscillation and asymptotic behaviour of solutions of second order nonlinear (sublinear/superlinear) neutral impulsive difference equations. The results are illustrated with examples under suitable fixed moments of impulsive effect.

1. Introduction

In [3], Elizabeth et al. have studied the second order nonlinear neutral difference equations

$$\Delta[a(n)(\Delta(x(n)+p(n)x(h(n))))^{\alpha}]+f(n,x(g(n)))=0, \tag{1}$$

where $a(n)>0$, $\alpha>0$ is a ratio of odd positive integers, $-1<p\leq p(n)\leq 0$ and $h(n), g(n)$ are increasing positive sequences in the form of delays. Here, the authors have proved some necessary and sufficient conditions that guarantee that the nonlinear equation (1) either oscillates or converges to zero when the nonlinear function is either strongly sublinear or strongly superliner. But, in the other ranges of the neutral coefficient, the work has been left unanswered as we understand that such equations are arising in the models of electric networks containing lossless transmission lines which are used to interconnect switching circuits in high speed computers (see for e.g. [6, 12]).

Let $m_1, m_2, m_3, \cdots$ be the discrete moments of impulsive effect with the properties $0<m_1<m_2<\cdots, m_j$ and $\lim_{j\to\infty} m_j=+\infty$. If we apply impulse m_j to any solution $x(n)$ of (1), then the impulsive solution $x(m_j)$ could be a solution of another type of neutral difference equations of the form:

$$\underline{\Delta}[a(m_j-1)(\Delta(x(m_j-1)+p(m_j-1)x(h(m_j-1))))^{\alpha}]+f(m_j-1,x(g(m_j-1)))=0, \tag{2}$$

where $\underline{\Delta}x(m_j-1)=x(m_j)-x(m_j-1)$, and (1) and (2) together we call an impulsive system

$$(E^*)\begin{cases}\Delta[a(n)(\Delta(x(n)+p(n)x(h(n))))^{\alpha}]+f(n,x(g(n)))=0, \ n\neq m_j\\ \underline{\Delta}[a(m_j-1)(\Delta(x(m_j-1)+p(m_j-1)x(h(m_j-1))))^{\alpha}]+f(m_j-1,x(g(m_j-1)))=0, \ j\in\mathbb{N}\end{cases}$$

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whose possible solution is given by $x(n) = \lambda^n A^{i(n_0,n)}$ (see for e.g. [15, 17]), where $i(n_0, n)$ denotes the number of impulsive points between n_0 and n and $A, \lambda \in \mathbb{R}$. We notice that without impulse it is so called difference equation (1). Hence, the studies of (E^*) and (1) are comparable.

In [13], [11] and [4], the authors have studied the oscillation properties of solutions of nonlinear neutral difference equations of the form:

$$\Delta[a(n)(\Delta(x(n) - p(n)x(n-\tau)))^{\alpha}] + q(n)f(x(n-\sigma)) = 0, \tag{3}$$

$$\Delta[a(n)(\Delta(x(n) + p(n)x(\tau(n)))^{\alpha}] + q(n)x^{\beta}(\sigma(n)) = 0 \tag{4}$$

and

$$\Delta[a(n)(\Delta(x(n) - p(n)x(n-\tau)))^{\alpha}] + q(n)x^{\beta}(n+1-\sigma) = 0 \tag{5}$$

respectively. Keeping in view of the above facts and with the advantage of our work, we can formulate problems of the type (E^*) for (3), (4) and (5).

In this work, our purpose is to establish the necessary and sufficient conditions for oscillation of the following neutral impulsive difference equations

$$(E)\begin{cases} \Delta[a(n)\Delta(x(n) + p(n)x(n-\tau))] + q(n)F(x(n-\sigma)) = 0,\ n \neq m_j & (6)\\ \underline{\Delta}[a(m_j-1)\Delta(x(m_j-1) + p(m_j-1)x(m_j-\tau-1))] + r(m_j-1)F(x(m_j-\sigma-1)) = 0,\ j \in \mathbb{N}, & (7) \end{cases}$$

where τ, $\sigma > 0$ are integers, a, p, q, r are real valued functions with discrete arguments such that $a(n)$, $q(n)$, $r(m_j-1) > 0$, $|p(n)| < \infty$ for $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots\}$, $F \in C(\mathbb{R}, \mathbb{R})$ satisfying the property $xF(x) > 0$ for $x \neq 0$, and Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$. For additional results, we refer to some of the works [8–10, 14–19], and the references cited therein and we recommend the monographs by Agarwal [1], Agarwal et al. [2] and Lakshmikantham [7].

Definition 1.1. *By a solution of (E) we mean a real valued function $x(n)$ defined on $\mathbb{N}(n_0-\rho)$ which satisfy (E) for $n \geq n_0$ with the initial conditions $x(i) = \phi(i)$, $i = n_0-\rho, \cdots, n_0$, where $\phi(i)$, $i = n_0-\rho, \cdots, n_0$ are given and $\rho = \max\{\tau, \sigma\}$. A nontrivial solution $x(n)$ of (E) is said to be nonoscillatory, if it is either eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory.*

Definition 1.2. *A solution $x(n)$ of (E) is said to be oscillatory, if there exists an integer $N > 0$ such that $x(n+1)x(n) \leq 0$ for all $n \geq N$. Otherwise, it is said to be nonoscillatory.*

Definition 1.3. *[1](Discrete L' Hospital Rule)*
Let $f(n)$ and $g(n)$ be defined on $\mathbb{N}(n_0)$ and $g(n) > 0$, $\Delta g(n) > 0$ for all large $n \in \mathbb{N}(n_0)$. Then $\lim_{n\to\infty} g(n) = \infty$, and

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = l \quad \text{whenever} \quad \lim_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)} = l \quad \text{exists.}$$

Definition 1.4. *[2] A function F is said to be strongly superlinear if there exists a constant $\beta > 1$ such that*

$$\frac{|F(u)|}{|u|^{\beta}} \geq \frac{|F(v)|}{|v|^{\beta}} \quad \text{for } |u| \geq |v|,\ uv > 0, \tag{8}$$

and it is said to be strongly sublinear if there exists a constant $\alpha \in (0, 1)$ such that

$$\frac{|F(u)|}{|u|^{\alpha}} \leq \frac{|F(v)|}{|v|^{\alpha}} \quad \text{for } |u| \geq |v|,\ uv > 0. \tag{9}$$

Lemma 1.5. *[5] Assume that $0 < \gamma < 1$. If u and v are two nonnegative integers such that $u < v$, then*

$$u^{1-\gamma} - v^{1-\gamma} \leq (1-\gamma)v^{-\gamma}(u - v).$$

2. Necessary and Sufficient Conditions

This section deals with the necessary and sufficient conditions for oscillation of all solutions of a class of nonlinear neutral impulsive difference equations of the form (E). Throughout our discussion, we use
(H_0) $A(n) = \sum_{s=n_0}^{n-1} \frac{1}{a(s)}$ *and* $\lim_{n\to\infty} A(n) = \infty$.

Theorem 2.1. *Let* $-1 \le p \le p(n) \le 0$ *and* F *is strongly sublinear. In addition to* (H_0), *let's assume that*
(H_1) $F(-u) = -F(u),\ u \in \mathbb{R}$.
Then every unbounded solution of (E) *oscillates if and only if*
(H_2) $\sum_{n=\sigma}^{\infty} q(n)F(CA(n-\sigma) + \sum_{j=1}^{\infty} r(m_j-1)F(CA(m_j-\sigma-1) = \infty$
for every constant $C > 0$.

Proof. Suppose that (H_2) holds. On the contrary, let $x(n)$ be an unbounded nonoscillatory solution of (E) for $n \ge n_0 > 1+\rho$. Without loss of generality and due to (H_1), we assume that $x(n) > 0$, $x(n-\tau) > 0$ and $x(n-\sigma) > 0$ for $n \ge n_1 > n_0$. Setting

$$\begin{cases} y(n) = x(n) + p(n)x(n-\tau), \\ y(m_j-1) = x(m_j-1) + p(m_j-1)x(m_j-\tau-1). \end{cases} \tag{10}$$

in (E), we get

$$(E_1)\begin{cases} \Delta[a(n)\Delta y(n)] = -q(n)F(x(n-\sigma)) < 0,\ n \ne m_j \\ \underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] = -r(m_j-1)F(x(m_j-\sigma-1)) < 0,\ j \in \mathbb{N} \end{cases}$$

for $n \ge n_2 > n_1$. Therefore, $a(n)\Delta y(n)$ and $y(n)$ are monotonic for $n \ge n_2$. If $a(n)\Delta y(n) < 0$ for $n \ge n_2$, then we can find $\gamma > 0$ and $n_3 > n_2 + 1$ such that $a(n)\Delta y(n) \le -\gamma$ for $n \ge n_3$ and thus $a(m_j-1)\Delta y(m_j-1) \le -\gamma$ for $n \ge n_3$. Taking sum to the inequality $\Delta y(n) \le -\frac{\gamma}{a(n)}$ from n_3 to $n-1$, it follows that

$$y(n) - y(n_3) - \sum_{n_3 \le m_j - 1 \le n-1} \Delta y(m_j-1) \le -\gamma \sum_{s=n_3}^{n-1} \frac{1}{a(s)},$$

that is,

$$y(n) \le y(n_3) - \gamma \left[\sum_{s=n_3}^{n-1} \frac{1}{a(s)} + \sum_{n_3 \le m_j-1 \le n-1} \frac{1}{a(m_j-1)} \right] \to -\infty \ \ as\ n \to \infty.$$

Hence, we can find an $n_4 > n_3$ such that $y(n) < 0$ for $n \ge n_4$. Consequently,

$$x(n) < -p(n)x(n-\tau) \le x(n-\tau) \le x(n-2\tau) \le x(n-3\tau) \cdots \le x(n_4)$$

and

$$x(m_j-1) \le x(m_j-\tau-1) \le x(m_j-2\tau-1) \le x(m_j-3\tau-1) \cdots \le x(n_4)$$

due to nonimpulsive points $m_j-1, m_j-\tau-1, m_j-2\tau-1, \cdots$ implies that $x(n)$ is bounded for all non-impulsive point n and m_j-1, a contradiction. Therefore, $a(n)\Delta y(n) > 0$ for $n \ge n_2$. Using the above argument, we conclude that $y(n) > 0$ for $n \ge n_2$. Indeed, $y(n) \le x(n)$ for $n \ge n_2$ and hence (E) reduces to

$$(E_2)\begin{cases} \Delta[a(n)\Delta y(n)] + q(n)F(y(n-\sigma)) \le 0,\ n \ne m_j, \\ \underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + r(m_j-1)F(y(m_j-\sigma-1)) \le 0, j \in \mathbb{N} \end{cases}$$

for $n \ge n_3 > n_2 + 1$. Summing (E_2) from n to $l-1 (l \ge n+1)$, we obtain

$$a(l)\Delta y(l) - a(n)\Delta y(n) - \sum_{n \le m_j-1 \le l-1} \underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + \sum_{s=n}^{l-1} q(s)F(y(s-\sigma)) \le 0,$$

that is,

$$\sum_{s=n}^{\infty} q(s)F(y(s-\sigma)) + \sum_{n\le m_j-1<\infty} r(m_j-1)F(y(m_j-\sigma-1)) \le a(n)\Delta y(n)$$

which then implies that

$$\Delta y(n) \ge \frac{1}{a(n)}\left[\sum_{s=n}^{\infty} q(s)F(y(s-\sigma)) + \sum_{n\le m_j-1<\infty} r(m_j-1)F(y(m_j-\sigma-1))\right]. \tag{11}$$

We notice that

$$\sum_{s=n_3}^{n-1}\frac{1}{a(s)} = \sum_{s=n_0}^{n-1}\frac{1}{a(s)} - \sum_{s=n_0}^{n_3-1}\frac{1}{a(s)} = A(n) - A(n_3) = \lambda(n)A(n),$$

where $\lambda(n) = 1 - \frac{A(n_3)}{A(n)}$. Due to (H_0), we have $\lim_{n\to\infty}\lambda(n) = 1$. So for a given $\lambda^* \in (0,1)$, there exists $n_4 > n_3$ such that $\lambda(n) \ge \lambda^*$, that is,

$$A(n) - A(n_3) \ge \lambda^* A(n) \text{ for } n \ge n_4. \tag{12}$$

Summing (11) from n_3 to $n-1$, we get

$$\begin{aligned} y(n) - y(n_3) &\ge \sum_{s=n_3}^{n-1}\frac{1}{a(s)}\left[\sum_{t=n}^{\infty} q(t)F(y(t-\sigma)) + \sum_{n\le m_j-1<\infty} r(m_j-1)F(y(m_j-\sigma-1))\right] \\ &= (A(n)-A(n_3))\left[\sum_{t=n}^{\infty} q(t)F(y(t-\sigma)) + \sum_{n\le m_j-1<\infty} r(m_j-1)F(y(m_j-\sigma-1))\right] \end{aligned}$$

and because of (12),

$$y(n) \ge \lambda^* A(n)\left[\sum_{t=n}^{\infty} q(t)F(y(t-\sigma)) + \sum_{n\le m_j-1<\infty} r(m_j-1)F(y(m_j-\sigma-1))\right] \tag{13}$$

for $n \ge n_4$. Since $a(n)\Delta y(n) > 0$ is nonincreasing, then we can find a constant $C > 0$ and $n_5 > n_4$ such that $a(n)\Delta y(n) \le C$ for $n \ge n_5$ and thus

$$y(n) \le y(n_5) + C\sum_{s=n_5}^{n-1}\frac{1}{a(s)} \le C\sum_{s=n_5}^{n-1}\frac{1}{a(s)} \le CA(n).$$

Consequently, F is strongly sublinear implies that

$$F(y(n-\sigma)) \ge \frac{F(CA(n-\sigma))}{C^\alpha A^\alpha(n-\sigma)} y^\alpha(n-\sigma)$$

for $n \ge n_5$ and hence (13) becomes

$$\begin{aligned} y(n) &\ge \frac{\lambda^* A(n)}{C^\alpha}\left[\sum_{t=n}^{\infty} q(t)\frac{F(CA(t-\sigma))}{A^\alpha(t-\sigma)} y^\alpha(t-\sigma) + \sum_{n\le m_j-1<\infty} r(m_j-1)\frac{F(CA(m_j-\sigma-1))}{A^\alpha(m_j-\sigma-1)} y^\alpha(m_j-\sigma-1)\right] \\ &= A(n)z(n) \end{aligned} \tag{14}$$

if we define

$$z(n)=\frac{\lambda^*}{C^\alpha}\left[\sum_{t=n}^{\infty}q(t)\frac{F(CA(t-\sigma))}{A^\alpha(t-\sigma)}y^\alpha(t-\sigma)+\sum_{n\le m_j-1<\infty}r(m_j-1)\frac{F(CA(m_j-\sigma-1))}{A^\alpha(m_j-\sigma-1)}y^\alpha(m_j-\sigma-1)\right]$$

for $n\ge n_5$. From (14), we have that $\frac{y(n)}{A(n)}\ge z(n)$ for $n\ge n_5$. Now,

$$\begin{aligned}\Delta z(n)=z(n+1)-z(n)&=\frac{\lambda^*}{C^\alpha}\left[\sum_{t=n+1}^{\infty}q(t)\frac{F(CA(t-\sigma))}{A^\alpha(t-\sigma)}y^\alpha(t-\sigma)-\sum_{t=n}^{\infty}q(t)\frac{F(CA(t-\sigma))}{A^\alpha(t-\sigma)}y^\alpha(t-\sigma)\right]\\&=-\frac{\lambda^*}{C^\alpha}q(n)\frac{F(CA(n-\sigma))}{A^\alpha(n-\sigma)}y^\alpha(n-\sigma)\le 0\end{aligned}$$

and

$$\begin{aligned}\underline{\Delta}z(m_j-1)&=z(m_j)-z(m_j-1)\\&=\frac{\lambda^*}{C^\alpha}\left[\sum_{t=m_j}^{\infty}q(t)\frac{F(CA(t-\sigma))}{A^\alpha(t-\sigma)}y^\alpha(t-\sigma)-\sum_{t=m_j-1}^{\infty}q(t)\frac{F(CA(t-\sigma))}{A^\alpha(t-\sigma)}y^\alpha(t-\sigma)\right]\\&+\frac{\lambda^*}{C^\alpha}\Big[\sum_{m_j<m_i-1<\infty}r(m_i-1)\frac{F(CA(m_i-\sigma-1))}{A^\alpha(m_i-\sigma-1)}y^\alpha(m_i-\sigma-1)\\&-\sum_{m_j-1\le m_i-1<\infty}r(m_i-1)\frac{F(CA(m_i-\sigma-1))}{A^\alpha(m_i-\sigma-1)}y^\alpha(m_i-\sigma-1)\Big]\\&=-\frac{\lambda^*}{C^\alpha}q(m_j-1)\frac{F(CA(m_j-\sigma-1))}{A^\alpha(m_j-\sigma-1)}y^\alpha(m_j-\sigma-1)\\&-\frac{\lambda^*}{C^\alpha}r(m_j-1)\frac{F(CA(m_j-\sigma-1))}{A^\alpha(m_j-\sigma-1)}y^\alpha(m_j-\sigma-1)\\&\le-\frac{\lambda^*}{C^\alpha}r(m_j-1)\frac{F(CA(m_j-\sigma-1))}{A^\alpha(m_j-\sigma-1)}y^\alpha(m_j-\sigma-1)\le 0\end{aligned}$$

implies that $z(n)$ is nonincreasing for $n\ge n_6>n_5+1$ and so, $\lim_{n\to\infty}z(n)$ exists. Upon using Lemma 1.5, it follows that

$$\begin{aligned}\Delta[z^{1-\alpha}(n)]=z^{1-\alpha}(n+1)-z^{1-\alpha}(n)&\le(1-\alpha)z^{-\alpha}(n)[z(n+1)-z(n)]\\&=(1-\alpha)z^{-\alpha}(n)\Delta z(n)\\&=-\frac{\lambda^*}{C^\alpha}(1-\alpha)z^{-\alpha}(n)q(n)\frac{F(CA(n-\sigma))}{A^\alpha(n-\sigma)}y^\alpha(n-\sigma)\\&\le-\frac{\lambda^*}{C^\alpha}(1-\alpha)z^{-\alpha}(n)q(n)F(CA(n-\sigma))z^\alpha(n-\sigma),\end{aligned}$$

that is,

$$\Delta[z^{1-\alpha}(n)]\le-\frac{\lambda^*}{C^\alpha}(1-\alpha)q(n)F(CA(n-\sigma)) \tag{15}$$

for $n\ge n_6$. By a similar argument, we obtain

$$\underline{\Delta}[z^{1-\alpha}(m_j-1)]\le-\frac{\lambda^*}{C^\alpha}(1-\alpha)r(m_j-1)F(CA(m_j-\sigma-1)). \tag{16}$$

Summing (15) from n_6 to $n-1$, we get

$$z^{1-\alpha}(n) - z^{1-\alpha}(n_6) - \sum_{n_6 \le m_j - 1 \le n-1} \underline{\Delta}[z^{1-\alpha}(m_j - 1)] \le -\frac{\lambda^*}{C^\alpha}(1-\alpha) \sum_{s=n_6}^{n-1} q(s)F(CA(s-\sigma))$$

and using (16), it follows that

$$\frac{\lambda^*}{C^\alpha}(1-\alpha)\left[\sum_{s=n_6}^{n-1} q(s)F(CA(s-\sigma)) + \sum_{n_6 \le m_j - 1 \le n-1} r(m_j - 1)F(CA(m_j - \sigma - 1))\right] \le z^{1-\alpha}(n_6) < \infty,$$

a contradiction to (H_2). Hence, (E) is oscillatory.

Conversely, let us assume that (H_2) do not hold. Then there exists $n^* > \rho^* > \rho + 1$ and $c > 0$ such that $c \le C(1+2p)$ and

$$\sum_{n=n^*}^{\infty} q(n)F(CA(n-\sigma)) + \sum_{n^* \le m_j - 1 < \infty} r(m_j - 1)F(CA(m_j - \sigma - 1)) \le \frac{c}{2}.$$

Set

$$\Omega = \{x : x(n) = 0 \text{ for } n^* - \rho^* \le n \le n^* \text{ and } \frac{C}{2}[A(n) - A(n^*)] \le x(n) \le C[A(n) - A(n^*)] \text{ for } n > n^*\}.$$

Define a map

$$(\mathcal{T}x)(n) = \begin{cases} \mathcal{T}x(n^*), n^* - \rho^* \le n \le n^*, \\ -p(n)x(n-\tau) + \sum_{s=n^*}^{n-1} \frac{1}{a(s)}\Big[\frac{C}{2} + \sum_{t=s}^{\infty} q(t)F(x(t-\sigma)) \\ \qquad + \sum_{n^* \le m_j - 1 < \infty} r(m_j - 1)F(x(m_j - \sigma - 1))\Big], n > n^*. \end{cases}$$

For $x \in \Omega$ and $n \ge n^*$, we have

$$\begin{aligned}(\mathcal{T}x)(n) &\ge \sum_{s=n^*}^{n-1} \frac{1}{a(s)}\left[\frac{C}{2} + \sum_{t=s}^{\infty} q(t)F(x(t-\sigma)) + \sum_{n^* \le m_j - 1 < \infty} r(m_j - 1)F(x(m_j - \sigma - 1))\right] \\ &\ge \frac{C}{2}\sum_{s=n^*}^{n-1} \frac{1}{a(s)} = \frac{C}{2}[A(n) - A(n^*)]\end{aligned}$$

and using $x(n) \le CA(n)$, we get

$$\begin{aligned}(\mathcal{T}x)(n) &\le -p(n)x(n-\tau) + \frac{C}{2}\sum_{s=n^*}^{n-1} \frac{1}{a(s)} + \frac{c}{2}\sum_{s=n^*}^{n-1} \frac{1}{a(s)} \\ &\le -pC[A(n-\tau) - A(n^*)] + \frac{C}{2}[A(n) - A(n^*)] + \frac{c}{2}\sum_{s=n^*}^{n-1} \frac{1}{a(s)} \\ &\le -pC[A(n) - A(n^*)] + \frac{C}{2}[A(n) - A(n^*)] + \frac{c}{2}[A(n) - A(n^*)] \\ &= (-pC + \frac{C}{2} + \frac{c}{2})[A(n) - A(n^*)] \\ &\le C[A(n) - A(n^*)].\end{aligned}$$

Therefore, $(\mathcal{T}x)(n) \in \Omega$. Define $u^k : [n^* - \rho^*, \infty) \to \mathbb{R}$ by the recursive formula

$$u^k(n) = (\mathcal{T}u^{k-1})(n),\ k \geq 1$$

with initial condition

$$u^0(n) = \begin{cases} 0,\ n^* - \rho^* \leq n < n^*) \\ \frac{C}{2}[A(n) - A(n^*)],\ n \geq n^*. \end{cases}$$

By induction, it is easy to see that

$$\frac{C}{2}[A(n) - A(n^*)] \leq u^{k-1}(n) \leq u^k(n) \leq C[A(n) - A(n^*)] \text{ for } n \geq n^*.$$

Hence, $\lim_{k\to\infty} u^k(n) = u(n)$ exists for $n \geq n^* - \rho^*$. By Lebesque's dominated convergence theorem [2], $(\mathcal{T}u)(n) = u(n)$, $u \in \Omega$ which is a positive solution of (E) for $n \geq n^* - \rho^*$. This completes the proof of the theorem. □

Theorem 2.2. *Let $-1 \leq p \leq p(n) \leq 0$. Assume that (H_0) and (H_1) hold. Then every unbounded solution of the system (E) oscillates if and only if (H_2) holds.*

Proof. We proceed as in the proof of Theorem 2.1 to obtain (E_1) for $n \geq n_3$. As $y(n)$ is unbounded and monotonically increasing, we use L'Hospital rule to find

$$\lim_{n\to\infty} \frac{y(n)}{A(n)} = \lim_{n\to\infty} \frac{\Delta y(n)}{\Delta A(n)} = \lim_{n\to\infty} a(n)\Delta y(n) = \mu,$$

where $0 \leq \mu < \infty$. If $\mu = 0$, then $\lim_{n\to\infty} A(n) = \infty$ implies that $\lim_{n\to\infty} y(n) < \infty$, a contradiction to our assumption. Ultimately, $\mu \neq 0$. So, there exists a $n_4 > n_3 + 1$ and a constant $C > 0$ such that $\frac{y(n)}{A(n)} \geq C$ for $n \geq n_4$. And for nonimpulsive points $m_j - 1, m_j - \tau - 1, m_j - 2\tau - 1, \cdots$ also, $y(m_j - 1) \geq CA(m_j - 1)$ for $n \geq n_4$. Now, (E_2) becomes

$$\begin{aligned} &\Delta[a(n)\Delta y(n)] + q(n)F(CA(n - \sigma)) < 0,\ n \neq m_j, n \geq n_4, \\ &\underline{\Delta}[a(m_j - 1)\Delta y(m_j - 1)] + r(m_j - 1)F(CA(m_j - \sigma - 1)) < 0,\ j \in \mathbb{N}. \end{aligned}$$

Summing the above impulsive system from n_4 to $n - 1$, we get a contradiction to (H_2). The necessary part is same as in the proof of Theorem 2.1. Thus, the theorem is proved. □

Remark 2.3. *In Theorem 2.2, F could be linear, subliner or superlinear.*

Theorem 2.4. *Let $-1 < p \leq p(n) \leq 0$ and F is strongly sublinear. Assume that (H_0) and (H_1) hold. Then every solution of the system (E) either oscillates or converges to zero if and only if (H_2) holds.*

Proof. Let $x(n)$ be a nonoscillatory solution of (E) for $n \geq n_0$. Proceeding as in Theorem 2.1, it follows that $a(n)\Delta y(n)$ and $y(n)$ are monotonic for $n \geq n_2$. Therefore, we have four possible cases, viz.

1. $a(n)\Delta y(n) < 0,\ y(n) > 0$; 2. $a(n)\Delta y(n) < 0,\ y(n) < 0$,
3. $a(n)\Delta y(n) > 0,\ y(n) > 0$; 4. $a(n)\Delta y(n) > 0,\ y(n) < 0$.

Case 1 and Case 3 follow from Theorem 2.1.

Case 2. In this case, $\lim_{n\to\infty} y(n) = -\infty$. Analogously, $\lim_{j\to\infty} y(m_j - 1) = -\infty$ due to the nonimpulsive points $m_j - 1, m_j - \tau - 1, m_j - 2\tau - 1, \cdots$ and so on. We note that $m_j - 1 < m_j < n$ and by Sandwich theorem, $\lim_{j\to\infty} y(m_j) = -\infty$. Indeed, $y(n) < 0$ implies that

$$x(n) < -p(n)x(n - \tau) < x(n - \tau) < x(n - 2\tau) < x(n - 3\tau) \cdots < x(n_2)$$

and

$$x(m_j-1) < x(m_j-\tau-1) < x(m_j-2\tau-1) < x(m_j-3\tau-1)\cdots < x(n_2).$$

Hence, $x(n)$ is bounded for all n and m_j-1, $j \in \mathbb{N}$. We claim that $x(m_j)$ is bounded for all $j \in \mathbb{N}$. If not, let $\lim_{j\to\infty} x(m_j) = \infty$. Now,

$$y(m_j) = x(m_j) + p(m_j)x(m_j-\tau) \geq x(m_j) - x(m_j-\tau) \geq x(m_j) - b$$

implies that $y(m_j) > 0$ as $j \to \infty$, a contradiction, where $x(m_j-\tau) \leq b$. So, our claim holds and hence $y(n)$ is bounded for all n which is again a contradiction to the fact that $y(n)$ is unbounded.

Case 4. Here also, $x(n)$ is bounded and so also $y(n)$. Thus $\lim_{n\to\infty} y(n)$ exists. Therefore,

$$\begin{aligned}0 \geq \lim_{n\to\infty} y(n) &= \limsup_{n\to\infty}(x(n)+p(n)x(n-\tau)) \geq \limsup_{n\to\infty}(x(n)+px(n-\tau))\\ &\geq \limsup_{n\to\infty} x(n) + \liminf_{n\to\infty}(px(n-\tau)) = (1+p)\limsup_{n\to\infty} x(n).\end{aligned}$$

Because $(1+p) > 0$, then $\limsup_{n\to\infty} x(n) = 0$. Ultimately, $\lim_{n\to\infty} x(n) = 0$ for all nonimpulsive points n and m_j-1, $j \in \mathbb{N}$. Due to $m_j - 1 < m_j < n$ and an application of Sandwich theorem shows that $\lim_{j\to\infty} x(m_j) = 0$. Hence, $\lim_{n\to\infty} x(n) = 0$ for all n. With this we conclude that (H_2) is a sufficient condition. The necessary part is same as in the proof of Theorem 2.1. This completes the proof of the theorem. □

Theorem 2.5. *Let $-1 \leq p \leq p(n) \leq 0$ and $a(n) \geq a(n-\tau)$ for all $n \in \mathbb{N}$. Assume that F is strongly superlinear. If (H_0) and (H_1) hold, then every unbounded solution of the system (E) oscillates if and only if*

(H_3) $\sum_{s=n^*}^{\infty} \frac{1}{a(s)} \left[\sum_{t=s}^{\infty} q(t) + \sum_{j=1}^{\infty} r(m_j-1)\right] = \infty.$

Proof. Let $x(n)$ be an unbounded nonoscillatory solution of (E) for $n \geq n_0$. Then, we proceed as in the proof of Theorem 2.1 to obtain that $y(n) > 0$, $a(n)\Delta y(n) > 0$ and (11) hold for $n \geq n_3$. There exists $n_4 > n_3 + 1$ and a constant $C > 0$ such that $y(n-\sigma) > C$ for $n \geq n_4$. F is strongly superlinear implies that

$$F(y(n-\sigma)) \geq \frac{F(y(n-\sigma))}{y^\beta(n-\sigma)} y^\beta(n-\sigma) \geq \frac{F(C)}{C^\beta} y^\beta(n-\sigma)$$

and hence (11) reduces to

$$a(n)\Delta y(n) \geq \left[\sum_{s=n}^{\infty} q(s)\frac{F(C)}{C^\beta} y^\beta(s-\sigma) + \sum_{n\leq m_j-1<\infty} r(m_j-1)\frac{F(C)}{C^\beta} y^\beta(m_j-\sigma-1)\right],$$

that is,

$$a(n-\sigma)\Delta y(n-\sigma) \geq \frac{F(C)}{C^\beta}\left[\sum_{s=n}^{\infty} q(s)y^\beta(s-\sigma) + \sum_{n\leq m_j-1<\infty} r(m_j-1)y^\beta(m_j-\sigma-1)\right]$$

for $n \geq n_5 > n_4$. Consequently,

$$\begin{aligned}\Delta y(n-\sigma) &\geq \frac{F(C)}{C^\beta a(n-\sigma)}\left[\sum_{s=n}^{\infty} q(s)y^\beta(s-\sigma) + \sum_{n\leq m_j-1<\infty} r(m_j-1)y^\beta(m_j-\sigma-1)\right]\\ &\geq \frac{F(C)}{C^\beta a(n)}\left[\sum_{s=n+1}^{\infty} q(s)y^\beta(s-\sigma) + \sum_{n+1<m_j-1<\infty} r(m_j-1)y^\beta(m_j-\sigma-1)\right]\\ &\geq \frac{F(C)y^\beta(n+1-\sigma)}{C^\beta a(n)}\left[\sum_{s=n+1}^{\infty} q(s) + \sum_{n+1<m_j-1<\infty} r(m_j-1)\right]\end{aligned}$$

which implies that

$$\frac{\Delta y(n-\sigma)}{y^{\beta}(n+1-\sigma)} \geq \frac{F(C)}{C^{\beta}a(n)}\left[\sum_{s=n+1}^{\infty} q(s) + \sum_{n+1<m_j-1<\infty} r(m_j-1)\right]$$

for $n \geq n_5$. If $y(n-\sigma) \leq u \leq y(n+1-\sigma)$ for $n \geq n_5$, then $\frac{1}{y^{\beta}(n-\sigma)} \geq \frac{1}{u^{\beta}} \geq \frac{1}{y^{\beta}(n+1-\sigma)}$. Therefore, the last inequality can be written as

$$\frac{F(C)}{C^{\beta}}\frac{1}{a(s)}\Big[\sum_{s=n+1}^{\infty} q(s) + \sum_{n+1<m_j-1<\infty} r(m_j-1)\Big] \leq \frac{\Delta y(n-\sigma)}{y^{\beta}(n+1-\sigma)} \leq \int_{y(n-\sigma)}^{y(n+1-\sigma)} \frac{du}{u^{\beta}}.$$

Summing the last inequality from n_5 to $n-1$, we get

$$\frac{F(C)}{C^{\beta}}\sum_{s=n_5}^{n-1}\frac{1}{a(s)}\Big[\sum_{s=n+1}^{\infty} q(s) + \sum_{n+1<m_j-1<\infty} r(m_j-1)\Big] \leq \sum_{s=n_5}^{n-1}\int_{y(s-\sigma)}^{y(s+1-\sigma)} \frac{du}{u^{\beta}} = \int_{y(n_5-\sigma)}^{y(n-\sigma)} \frac{du}{u^{\beta}} < \infty$$

as $n \to \infty$, a contradiction to (H_3).

Conversely, we suppose that (H_3) does not hold. Then there exists $n^* > \rho^* > \sigma + 1$ and $C > 0$ such that

$$F(C)\sum_{s=n^*}^{\infty}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty} q(n) + \sum_{j=1}^{\infty} r(m_j-1)\right] \leq \frac{C}{5}.$$

Set

$$\Omega = \{x : x(n) = \frac{C}{5} \text{ for } n^*-\sigma \leq n \leq n^* \text{ and } \frac{C}{5} \leq x(n) \leq C \text{ for } n \geq n^*\}.$$

Define a map

$$(\mathcal{T}x)(n) = \begin{cases} \frac{C}{5}, n^*-\sigma \leq n \leq n^*, \\ -p(n)x(n-\tau) + \frac{C}{5} + \sum_{s=n^*}^{n-1}\frac{1}{a(s)}\Big[\sum_{t=s}^{\infty} q(t)F(x(t-\sigma)) + \sum_{j=1}^{\infty} r(m_j-1)F(x(m_j-\sigma-1))\Big], n > n^*. \end{cases}$$

For $x \in \Omega$ and $n \geq n^*$, we have

$$(\mathcal{T}x)(n) \geq \frac{C}{5}$$

and

$$\begin{aligned}(\mathcal{T}x)(n) &\leq -p(n)x(n-\tau) + \frac{C}{5} + F(C)\sum_{s=n^*}^{n-1}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty} q(t) + \sum_{j=1}^{\infty} r(m_j-1)\right] \\ &\leq -pC + \frac{C}{5} + \frac{C}{5} = \left(\frac{2}{5}-p\right)C \leq C, \text{ if } \left(\frac{2}{5}-p\right) \leq 1.\end{aligned}$$

Therefore, $(\mathcal{T}x)(n) \in \Omega$. Define $u^k : [n^*-\rho^*, \infty) \to \mathbb{R}$ by the recursive formula

$$u^k(n) = (\mathcal{T}u^{k-1})(n),\ k \geq 1$$

with the initial condition

$$u^0(n) = \begin{cases} \frac{C}{5},\ n^*-\rho^* \leq n \leq n^* \\ C,\ n \geq n^*. \end{cases}$$

By induction, it is easy to see that

$$\frac{C}{5} \leq u^{k-1}(n) \leq u^k(n) \leq C, \text{ for } n \geq n^*.$$

Therefore, $\lim_{k\to\infty} u^k(n) = u(n)$ exists for $n \geq n^* - \rho^*$. By Lebesgue's dominated convergence theorem [2], $(\mathcal{T}u)(n) = u(n)$ and $u \in \Omega$ which is a positive solution of the impulsive system (E) for $n \geq n^* - \rho^*$. This completes the proof of the theorem. □

Theorem 2.6. *Let $-1 < p \leq p(n) \leq 0$ and $a(n) \geq a(n-\tau)$ for all $n \in \mathbb{N}$. Assume that F is strongly superlinear. Furthermore, assume that (H_0) and (H_1) hold. Then every solution of (E) oscillates or converges to zero if and only if (H_3) holds.*

Proof. The proof of the theorem follows from the proofs of Theorem 2.4 and Theorem 2.5. Hence, the details are omitted. □

Theorem 2.7. *Let $0 \leq p(n) \leq p < 1$ and $a(n) \geq a(n-\sigma)$ for all $n \in \mathbb{N}$. Assume that F is strongly superlinear. If (H_0) and (H_1) hold, then every solution of the system (E) oscillates if and only if (H_3) holds.*

Proof. Let $x(n)$ be a nonoscillatory solution of (E). Then proceeding as in Theorem 2.4, we have two possible cases:

1. $a(n)\Delta y(n) < 0, y(n) > 0$; 2. $a(n)\Delta y(n) > 0, y(n) > 0$.

Proof for Case 1 follows from the proof of Theorem 2.4. Consider **Case 2**. Using the fact that $y(n)$ is nondecreasing, we have

$$\begin{aligned}(1-p)y(n) &\leq y(n) - p(n)y(n-\tau)\\ &= x(n) + p(n)x(n-\tau) - p(n)x(n-\tau) - p(n)p(n-\tau)x(n-2\tau)\\ &= x(n) - p(n)p(n-\tau)x(n-2\tau) \leq x(n)\end{aligned}$$

$n \geq n_2$. Since $y(n)$ is nondecreasing, then we can find an $n_3 > n_2$ and a constant $C > 0$ such that $y(n) \geq C$ for $n \geq n_3$. Using F as strongly superlinear, it follows that

$$\begin{aligned}F((1-p)y(n-\sigma)) &= \frac{F((1-p)y(n-\sigma))}{(1-p)^\beta y^\beta(n-\sigma)}(1-p)^\beta y^\beta(n-\sigma)\\ &\geq \frac{F(C(1-p))}{C^\beta(1-p)^\beta}(1-p)^\beta y^\beta(n-\sigma) = \frac{F(C(1-p))}{C^\beta}y^\beta(n-\sigma)\end{aligned}$$

for $n \geq n_3$. So, (E) can be written as

$$\begin{cases}\Delta[a(n)\Delta y(n)] + \frac{F(C(1-p))}{C^\beta}q(n)y^\beta(n-\sigma) \leq 0, \ n \neq m_j\\ \underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + \frac{F(C(1-p))}{C^\beta}r(m_j-1)y^\beta(m_j-\sigma-1) \leq 0, \ j \in \mathbb{N}.\end{cases}$$

Summing the impulsive system from n to $l-1$ ($l \geq n+1$, we get

$$\frac{F(C(1-p))}{C^\beta}\left[\sum_{s=n}^{\infty} q(s)y^\beta(s-\sigma) + \sum_{n\leq m_j-1<\infty} r(m_j-1)y^\beta(m_j-\sigma-1)\right] \leq a(n)\Delta y(n).$$

The rest proof of this case follows from Theorem 2.5.

Conversely, let us assume that (H_3) do not hold. Let $X = l_\infty^{n_1}$ be the Banach space of all real valued bounded sequence $x(n)$ for $n \geq n_1$ with the norm defined by

$$\|x\| = \sup\{|x(n)| : n \geq n_1\}.$$

Consider a closed subset Ω of X such that

$$\Omega = \{x \in X : \beta_1 \leq x(n) \leq \beta_2, n \geq n_1\},$$

where $\beta_1, \beta_2 > 0$ are so chosen that $\beta_1 < (1-p)\beta_2$. Let $\beta_1 + p\beta_2 \leq \gamma < \beta_2$ be such that

$$\sum_{s=n_1}^{\infty} \frac{1}{a(s)} \left[\sum_{t=s}^{\infty} q(t) + \sum_{j=1}^{\infty} r(m_j - 1) \right] < \frac{\beta_2 - \gamma}{M}, \tag{17}$$

where $M = \max\{F(x) : \beta_1 \leq x \leq \beta_2\}$. For $x \in \Omega$ and $n \geq n_1$, we define two maps

$$(\mathcal{T}_1 x)(n) = \begin{cases} \mathcal{T}_1 x(n_1), \ n_1 - \rho \leq n \leq n_1, \\ \gamma - p(n)x(n-\tau), \ n > n_1 \end{cases}$$

and

$$(\mathcal{T}_2 x)(n) = \begin{cases} \mathcal{T}_2 x(n_1), \ n_1 - \rho \leq n \leq n_1, \\ \sum_{s=n}^{\infty} \frac{1}{a(s)} \Big[\sum_{t=s}^{\infty} q(t) F(x(t-\sigma)) + \sum_{j=1}^{\infty} r(m_j - 1) F(x(m_j - \sigma - 1)) \Big], n > n_1. \end{cases}$$

Indeed,

$$\begin{aligned} \mathcal{T}_1 x(n) + \mathcal{T}_2 x(n) &= \gamma - p(n)x(n-\tau) + \sum_{s=n}^{\infty} \frac{1}{a(s)} \left[\sum_{t=s}^{\infty} q(t) F(x(t-\sigma)) + \sum_{j=1}^{\infty} r(m_j - 1) F(x(m_j - \sigma - 1)) \right] \\ &\leq \gamma + \sum_{s=n}^{\infty} \frac{1}{a(s)} \left[\sum_{t=s}^{\infty} q(t) M + \sum_{j=1}^{\infty} r(m_j - 1) M \right] \\ &\leq \gamma + M \sum_{s=n}^{\infty} \frac{1}{a(s)} \left[\sum_{t=s}^{\infty} q(t) + \sum_{j=1}^{\infty} r(m_j - 1) \right] \leq \beta_2 \end{aligned}$$

and

$$\mathcal{T}_1 x(n) + \mathcal{T}_2 x(n) \geq \gamma - p(n)x(n-\tau) \geq \beta_1 + p\beta_2 - p\beta_2 = \beta_1.$$

Thus, $\beta_1 \leq \mathcal{T}_1 x + \mathcal{T}_2 x \leq \beta_2$ for $n \geq n_1$. For $x_1, x_2 \in \Omega$ and $n \geq n_1$, we have

$$|\mathcal{T}_1 x_1(n) - \mathcal{T}_1 x_2(n)| \leq |p(n)||x_1(n-\tau) - x_2(n-\tau)| \leq p|x_1(n-\tau) - x_2(n-\tau)|$$

for which

$$\|\mathcal{T}_1 x_1 - \mathcal{T}_1 x_2\| \leq p\|x_1 - x_2\|,$$

that is, $\mathcal{T}_1$ is a contraction mapping with contraction constant $p < 1$.

In order to show that $\mathcal{T}_2$ is completely continuous, we need to show that $\mathcal{T}_2 x$ is continuous and relatively compact. Let $x_k \in \Omega$ be such that $x_k(n) \to x(n)$ as $k \to \infty$, of course $x = x(n) \in \Omega$. For $n \geq n_1$, we have

$$\begin{aligned} |(\mathcal{T}_2 x_k)(n) - (\mathcal{T}_2 x)(n)| \leq \sum_{s=n}^{\infty} \frac{1}{a(s)} \Big[&\sum_{t=s}^{\infty} q(t) |F(x_k(t-\sigma)) - F(x(t-\sigma))| \\ &+ \sum_{j=1}^{\infty} r(m_j - 1) |F(x_k(m_j - \sigma - 1) - F(x(m_j - \sigma - 1))| \Big]. \end{aligned}$$

Since $|F(x_k(n-\sigma)) - F(x(n-\sigma))| \to 0$ as $k \to \infty$, then applying Lebesgue's dominated convergence theorem [2, Lemma 5.3.4] we have that $\lim_{k\to\infty} |(\mathcal{T}_2 x_k)(n) - (\mathcal{T}_2 x)(n)| \to 0$. Therefore, $\mathcal{T}_2 x$ is continuous. To show that

$\mathcal{T}_2x$ is relatively compact, we show that the family of functions $\{\mathcal{T}_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[n_1, \infty)$. Indeed, $\mathcal{T}_2x$ is uniformly bounded. For $n_3 > n_2 > n_1$ and $x \in \Omega$, it follows that

$$\begin{aligned}|\mathcal{T}_2x(n_3) - \mathcal{T}_2x(n_2)| &= \Big|\sum_{s=n_3}^{\infty}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty}q(t)F(x(t-\sigma)) + \sum_{j=1}^{\infty}r(m_j-1)F(x(m_j-\sigma-1))\right]\\ &\quad - \sum_{s=n_2}^{\infty}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty}q(t)F(x(t-\sigma)) + \sum_{j=1}^{\infty}r(m_j-1)F(x(m_j-\sigma-1))\right]\Big|\\ &\le M\sum_{s=n_2}^{n_3}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty}q(t) + \sum_{j=1}^{\infty}r(m_j-1)\right].\end{aligned}$$

For $0 < \epsilon < \frac{\beta_2-\gamma}{M}$, we can find a $\delta > 0$ such that

$$|\mathcal{T}_2x(n_3) - \mathcal{T}_2x(n_2)| < \epsilon \text{ when ever } 0 < n_3 - n_2 < \delta,$$

and this relation continues to hold for every $n_2, n_3 > n_1$. Therefore, $\{\mathcal{T}_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous for $n \ge n_1$ and hence $\mathcal{T}_2x$ is relatively compact. By Krasnoselskii's fixed point theorem, $\mathcal{T}_1 + \mathcal{T}_2$ has a unique fixed point $x \in \Omega$ such that $\mathcal{T}_1x + \mathcal{T}_2x = x$, that is,

$$x(n) = \begin{cases} \mathcal{T}_2x(n_2), \ n_1 < n_2 - \rho \le n \le n_2,\\ \gamma - p(n)x(n-\tau) + \sum_{s=n}^{\infty}\frac{1}{a(s)}\Big[\sum_{t=s}^{\infty}q(t)F(x(t-\sigma)) + \sum_{j=1}^{\infty}r(m_j-1)F(x(m_j-\sigma-1))\Big], \ n > n_2.\end{cases}$$

It is easy to show that $x(n)$ is a positive solution of the impulsive system (E). This completes the proof of the theorem. □

Theorem 2.8. *Let $0 \le p(n) < p < 1$ and $a(n) > a(n-\sigma)$ for all $n \in \mathbb{N}$. Assume that (H_0) and (H_1) hold. Then every bounded solution of the system (E) oscillates if and only if (H_3) holds.*

Proof. The proof of the theorem is same as in the proof of Theorem 2.7 and hence the details are omitted. □

Theorem 2.9. *Let $1 \le p(n) \le p < \infty$, $\sigma \ge \tau$ and $a(n) > a(n-\sigma)$ for all $n \in \mathbb{N}_0$. Assume that F is strongly sublinear. In additiion to (H_0) and (H_1), assume that*
(H_4) *$F(u)F(v) \ge F(uv)$ for $u, v \ge 0; u, v \in \mathbb{R}$*
(H_5) *there exists $\lambda > 0$ such that $F(u) + F(v) \ge \lambda F(u+v)$ for $u, v \ge 0; u, v \in \mathbb{R}$*
and
(H_6) $\sum_{n=\sigma}^{\infty} Q(n)F(CA(n-\sigma)) + \sum_{j=1}^{\infty} R(m_j-1)F(CA(m_j-\sigma-1)) = \infty$
for every $C > 0$ hold, where $Q(n) = \min\{q(n), q(n-\tau)\}$, $R(m_j-1) = \min\{r(m_j-1), r(m_j-\tau-1)\}$, $n \ge \tau$. Then every solution of (E) oscillates.

Proof. Let $x(n)$ be a nonoscillatory solution of (E). Proceeding as in the proof of Theorem 2.7, we get $a(n)\Delta y(n) > 0$ for $n \ge n_1$. Therefore, $\lim_{n\to\infty} a(n)\Delta y(n)$ exists. From and using (6), it follows that

$$\Delta[a(n)\Delta y(n)] + q(n)F(x(n-\sigma)) + F(p)(\Delta[a(n-\tau)\Delta y(n-\tau)] + q(n-\tau)F(x(n-\tau-\sigma))) = 0.$$

Due to (H_4) the above inequality can be written as

$$\Delta[a(n)\Delta y(n)] + F(p)\Delta[a(n-\tau)\Delta y(n-\tau)] + Q(n)F(x(n-\sigma)) + F(px(n-\tau-\sigma)) \le 0$$

which in turn

$$\Delta[a(n)\Delta y(n)] + F(p)\Delta[a(n-\tau)\Delta y(n-\tau)] + \lambda Q(n)F(x(n-\sigma) + px(n-\tau-\sigma)) \le 0$$

due to (H_5). Using $y(n-\sigma) \le x(n-\sigma) + px(n-\tau-\sigma)$ in the last inequality, we find

$$\Delta[a(n)\Delta y(n)] + F(p)\Delta[a(n-\tau)\Delta y(n-\tau)] + \lambda Q(n)F(y(n-\sigma)) \le 0. \tag{18}$$

By a similar argument, (7) reduces to

$$\underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + F(p)\underline{\Delta}[a(m_j-\tau-1)\Delta y(m_j-\tau-1)] + \lambda R(m_j-1)F(y(m_j-\sigma-1)) \le 0. \tag{19}$$

Summing (18) from n to $l-1$ ($l \ge n+1,\ n \ge n_2 > n_1+\sigma$) and then using (19), we get

$$\begin{aligned} &a(l)\Delta y(l) - a(n)\Delta z(n) + F(p)a(l-\tau)\Delta y(l-\tau) - F(p)a(n-\tau)\Delta y(n-\tau) \\ &\qquad - \sum_{n\le m_j-1\le l-1}\Big[\underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + F(p)\underline{\Delta}[a(m_j-\tau-1)\Delta y(m_j-\tau-1)]\Big] \\ &\qquad + \lambda\sum_{s=n}^{l-1} Q(s)F(y(s-\sigma)) \le 0, \end{aligned}$$

that is,

$$\begin{aligned} &\lambda\sum_{s=n}^{l-1} Q(s)F(y(s-\sigma)) + \lambda\sum_{n\le m_j-1\le l-1} R(m_j-1)F(y(m_j-\sigma-1)) \\ &\qquad\qquad \le a(n)\Delta y(n) + F(p)a(n-\tau)\Delta y(n-\tau), \\ &\qquad\qquad \le a(n-\tau)\Delta y(n-\tau) + F(p)a(n-\tau)\Delta y(n-\tau), \\ &\qquad\qquad = (1+F(p))a(n-\tau)\Delta y(n-\tau) \end{aligned}$$

for $n \ge n_2$. Consequently,

$$\begin{aligned} \Delta y(n-\tau) &\ge \frac{\lambda}{(1+F(p))a(n-\tau)}\left[\sum_{s=n}^{l-1} Q(s)F(y(s-\sigma)) + \sum_{n\le m_j-1\le l-1} R(m_j-1)F(y(m_j-\sigma-1))\right] \\ &\ge \frac{\lambda}{(1+F(p))a(n)}\left[\sum_{s=n}^{l-1} Q(s)F(y(s-\sigma)) + \sum_{n\le m_j-1\le l-1} R(m_j-1)F(y(m_j-\sigma-1))\right]. \end{aligned}$$

Again summing the preceding inequality from n_2 to $n-1$, we obtain

$$y(n-\tau) - y(n_2-\tau) \ge \frac{\lambda}{(1+F(p))}\sum_{s=n_2}^{n-1}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty} Q(t)F(y(t-\sigma)) + \sum_{j=1}^{\infty} R(m_j-1)F(y(m_j-\sigma-1))\right],$$

that is,

$$y(n-\tau) \ge \frac{\lambda A(n)}{(1+F(p))}\left[\sum_{t=s}^{\infty} Q(t)F(y(t-\sigma)) + \sum_{j=1}^{\infty} R(m_j-1)F(y(m_j-\sigma-1))\right].$$

As a result,

$$\frac{y(n-\tau)}{A(n-\sigma)} \ge \frac{\lambda}{(1+F(p))}\left[\sum_{t=s}^{\infty} Q(t)F(y(t-\sigma)) + \sum_{j=1}^{\infty} R(m_j-1)F(y(m_j-\sigma-1))\right]. \tag{20}$$

Since $a(n)\Delta y(n)$ is nonincreasing, then there exists a constant $C > 0$ and $n_3 > n_2$ such that $a(n)\Delta y(n) \le C$ for $n \ge n_3$ and thus $y(n) \le CA(n)$ for $n \ge n_3$. F is strongly sublinear implies that

$$F(y(n-\sigma)) = \frac{F(y(n-\sigma))}{y^\alpha(n-\sigma)}y^\alpha(n-\sigma) \ge \frac{F(CA(n-\sigma))}{C^\alpha A^\alpha(n-\sigma)}y^\alpha(n-\sigma)$$

and therefore, (20) can be written as

$$\frac{y(n-\tau)}{A(n-\sigma)} \geq \frac{\lambda C^{-\alpha}}{(1+F(p))}\left[\sum_{t=s}^{\infty} Q(t)\frac{F(CA(t-\sigma))}{A^{\alpha}(t-\sigma)}y^{\alpha}(t-\sigma)+\sum_{j=1}^{\infty} R(m_j-1)\frac{F(CA(m_j-\sigma-1))}{A^{\alpha}(m_j-\sigma-1)}y^{\alpha}(m_j-\sigma-1)\right].$$

The rest of the proof follows from Theorem 2.1. Hence, the theorem is proved. □

Theorem 2.10. *Let $1 \leq p(n) \leq p < \infty$, $\sigma \geq \tau$ and $a(n) > a(n-\sigma)$ for all $n \in \mathbb{N}_0$. Assume that F is strongly superlinear. In additiion to (H_0), (H_1), (H_4) and (H_5), assume that*
(H_7) $\sum_{s=n^*}^{\infty} \frac{1}{a(s)}\left[\sum_{t=s}^{\infty} Q(t)+\sum_{j=1}^{\infty} R(m_j-1)\right]=\infty$
hold, where $Q(n)$ and $R(m_j-1)$ are defined in Theorem 2.9. Then every solution of (E) is oscillatory.

Proof. The proof of the theorem follows from Theorem 2.9. Hence, details are omitted. □

Theorem 2.11. *Let $-\infty < p_2 \leq p(n) \leq p_3 < -1$, $p_2, p_3 > 0$. Assume that (H_0) and (H_1) hold. Then every bounded solution of (E) is oscillates or converges to zero if and only if (H_3) holds.*

Proof. Assume that $x(n)$ is a nonoscillatory solution of (E) which is bounded for $n \geq n_0 > \max\{\tau, \sigma\}$. Proceeding as in the proof of Theorem 2.1, we have that $a(n)\Delta y(n)$ is monotonically nonincreasing and $y(n)$ is monotonic. Here, we have following four cases:

1. $a(n)\Delta y(n) < 0, y(n) > 0$; 2. $a(n)\Delta y(n) < 0, y(n) < 0$,
3. $a(n)\Delta y(n) > 0, y(n) > 0$; 4. $a(n)\Delta y(n) > 0, y(n) < 0$.

The proofs for Case 1 and Case 2 follow from Theorem 2.4.
Case 3. We can find a $n_3 > n_2 + 1$ and a constant $L > 0$ such that $y(n-\sigma) \geq L$ and hence $y(m_j-\sigma-1) \geq L$ for $n \geq n_3$. $y(n) > 0$ implies that $x(n) \geq y(n)$. Therefore, for $n \geq n_4 > n_3 + \sigma$, the impulsive system (E) can be written as

$$\Delta[a(n)\Delta y(n)] + F(L)q(n) \leq 0,\ n \neq m_j,$$
$$\underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + F(L)r(m_j-1) \leq 0,\ j \in \mathbb{N}.$$

Summing the last impulsive system from n to $l-1$, we get

$$F(L)\left[\sum_{s=n}^{l-1} q(s)+\sum_{n\leq m_j-1<l-1} r(m_j-1)\right] \leq a(n)\Delta y(n) - a(l)\Delta y(l) \leq a(n)\Delta y(n),$$

that is,

$$\frac{F(L)}{a(n)}\left[\sum_{s=n}^{\infty} q(s)+\sum_{j=1}^{\infty} r(m_j-1)\right] \leq \Delta y(n)$$

which implies that

$$\sum_{s=n_3}^{n-1}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty} q(t)+\sum_{j=1}^{\infty} r(m_j-1)\right] \leq y(n)-y(n_3) \leq y(n) < \infty,$$

a contradiction to (H_3).
Case 4. Since $y(n)$ is nondecreasing and negative, then $\lim_{n\to\infty} y(n) = \delta$, $-\infty < \delta \leq 0$ exists. We claim that $\delta = 0$. If not, then there exists a $n_4 > n_3$ and a $C > 0$ such that $y(n+\tau-\sigma) < -C$ and $y(m_j+\tau-\sigma-1) < -C$, $j \in \mathbb{N}$.

Clearly, $y(n+\tau-\sigma) > p(n+\tau-\sigma)x(n-\sigma)$ implies that $y(n+\tau-\sigma) > p_2 x(n-\sigma)$, that is, $\frac{-C}{p_2} < x(n-\sigma)$. Also, $\frac{-C}{p_2} < x(m_j-\sigma-1)$ holds true. Therefore, (E) can be written as

$$\begin{aligned}
&\Delta[a(n)\Delta y(n)] + F\left(\frac{-C}{p_2}\right)q(n) \leq 0,\ n \neq m_j,\\
&\underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + F\left(\frac{-C}{p_2}\right)r(m_j-1) \leq 0,\ j \in \mathbb{N}.
\end{aligned}$$

Summing the above impulsive system from n to $l-1$ and then from n_4 to $n-1$, we get a contradiction to (H_3). Thus, our claim holds. As a result,

$$\begin{aligned}
0 &= \lim_{n\to\infty} y(n) = \liminf_{n\to\infty}(x(n) + p(n)x(n-\tau)) \leq \liminf_{n\to\infty}(x(n) + p_3 x(n-\tau))\\
&\leq \limsup_{n\to\infty} x(n) + \liminf_{n\to\infty}(p_3 x(n-\tau))\\
&= (1+p_3)\limsup_{n\to\infty} x(n).
\end{aligned}$$

Since $(1+p_3) < 0$, then $\lim_{n\to\infty} x(n) = 0$. Analogously, $\lim_{j\to\infty} x(m_j-1) = 0$ due to the nonimpulsive points $m_j-1, m_j-\tau-1, m_j-2\tau-1, \cdots$. Noting that $m_j-1 < m_j < n$ and an application of Sandwich theorem shows that $\lim_{j\to\infty} x(m_j) = 0$. Therefore, $\lim_{n\to\infty} x(n) = 0$ for all n.

Conversely, let us assume that (H_3) do not hold. Let $X = l_\infty^{n^*}$ be the Banach space of all real valued bounded functions $x(n)$ for $n \geq n^* > \rho + 1$ with the norm defined by $\|x\| = \sup\{|x(n)| : n \geq n^*\}$. Consider a closed subset Ω of X such that

$$\Omega = \{x \in X : \beta_5 \leq x(n) \leq \beta_6, n \geq n^*\},$$

where $\beta_5 > 0$ and $\beta_6 > 0$ are so chosen such that $-p_2\beta_5 < (-1-p_3)\beta_6$ and let $-p_2\beta_5 < \gamma \leq (-1-p_3)\beta_6$ be such that

$$\sum_{s=n^*}^{\infty} \frac{1}{a(s)}\left[\sum_{t=s}^{\infty} q(t) + \sum_{j=1}^{\infty} r(m_j-1)\right] < \frac{\gamma + p_2\beta_5}{M}, \tag{21}$$

where $M = \max\{F(x) : \beta_5 \leq x \leq \beta_6\}$. For $x \in \Omega$, we define two maps

$$(\mathcal{T}_1 x)(n) = \begin{cases} \mathcal{T}_1 x(n^*),\ n^*-\rho \leq n \leq n^*,\\ -\frac{\gamma}{p(n+\tau)} - \frac{1}{p(n+\tau)}x(n+\tau),\ n > n^*. \end{cases}$$

and

$$(\mathcal{T}_2 x)(n) = \begin{cases} \mathcal{T}_2 x(n^*),\ n^*-\rho \leq n \leq n^*,\\ \frac{1}{p(n+\tau)}\sum_{s=n^*}^{n+\tau}\frac{1}{a(s)}\Big[\sum_{t=s}^{\infty} q(t)F(x(t-\sigma)) + \sum_{j=1}^{\infty} r(m_j-1)F(x(m_j-\sigma-1))\Big],\ n > n^*. \end{cases}$$

For $y \in \Omega$ and using (21), we have

$$\begin{aligned}
&\mathcal{T}_1 x_1(n) + \mathcal{T}_2 x_2(n)\\
&\leq -\frac{\gamma}{p(n+\tau)} - \frac{x_1(n+\tau)}{p(n+\tau)} + \frac{1}{p(n+\tau)}\sum_{s=n^*}^{n+\tau}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty} q(t)F(x_2(t-\sigma)) + \sum_{j=1}^{\infty} r(m_j-1)F(x_2(m_j-\sigma-1))\right]\\
&\leq -\frac{1}{p_3}\left[(-1-p_3)\beta_6 + \beta_6\right] = \beta_6
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{T}_1x_1(n)+\mathcal{T}_2x_2(n)\\
&\geq -\frac{\gamma}{p(n+\tau)}+\frac{1}{p(n+\tau)}\sum_{s=n^*}^{n+\tau}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty}q(t)F(x_2(t-\sigma))+\sum_{j=1}^{\infty}r(m_j-1)F(x_2(m_j-\sigma-1))\right]\\
&\geq -\frac{1}{p_2}\left[\gamma-M\left(\frac{\gamma+p_2\beta_5}{M}\right)\right]\geq -\frac{1}{p_2}\left[\gamma-\gamma-p_2\beta_5\right]=\beta_5.
\end{aligned}$$

Therefore, $\beta_5 \leq \mathcal{T}_1x_1+\mathcal{T}_2x_2 \leq \beta_6$ for every $n \geq n^*$. It is easy to verify that $\mathcal{T}_1$ is a contraction mapping with contraction constant $0 \leq \left(\frac{-1}{p_3}\right) < 1$. Proceeding as in the proof of Theorem 2.7, we can show that T_2 is completely continuous. By Krasnoselskii's fixed point theorem, T_1+T_2 has a unique fixed point $x \in \Omega$ such that $T_1x+T_2x = x$ and hence

$$x(n)=\begin{cases} x(n), \ n_2-\rho \leq n \leq n_2,\\ \frac{-\gamma}{p(n+\tau)}-\frac{x(n+\tau)}{p(n+\tau)}+\frac{1}{p(n+\tau)}\sum_{s=n^*}^{n+\tau}\frac{1}{a(s)}\Big[\sum_{t=s}^{\infty}q(t)F(x(t-\sigma))+\sum_{j=1}^{\infty}r(m_j-1)F(x(m_j-\sigma-1))\Big],\ n>n^*.\end{cases}$$

Indeed, $x(n)$ is a positive solution of the impulsive system (E). This completes the proof of the theorem. □

Theorem 2.12. *Let $1 < p_4 \leq p(n) \leq p_5 < \infty$. If*
(H_8) $\sum_{s=n^*}^{\infty}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty}q(t)+\sum_{j=1}^{\infty}r(m_j-1)\right]<\infty$,
then (E) admits a positive bounded nonoscillatory solution.

Proof. The proof of the theorem can similarly be dealt with the necessary part of Theorem 2.11. □

3. Examples

Example 3.1. *Consider the impulsive difference equation of the form:*

$$(E')\begin{cases}\Delta[a(n)\Delta(x(n)+p(n)x(n-1))]+q(n)x^{1/3}(n-3)=0,\ n\neq m_j,\ n\geq 4,\ j\in\mathbb{N},\\ \underline{\Delta}[a(m_j-1)\Delta(x(m_j-1)+p(m_j-1)x(m_j-2))]+r(m_j-1)x^{1/3}(m_j-4)=0,\end{cases}$$

where $\tau=1$, $\sigma=3$, $p(n)=-1/2$, $q(n)=\frac{2e+e^{-1}+3}{e^{\frac{n-3}{3}}}$, $a(n)=\frac{1}{e^n}$ and $A(n)=\frac{e^n-e}{e-1}$. Here, $r(m_j-1)=\frac{2e^3+e^2+e^{-1}+2}{e^{\frac{m_j-4}{3}}}$ when $m_j=3j$ for $j\in\mathbb{N}$ and $F(u)=u^{1/3}$. Let $\alpha=\frac{1}{2}$. Then $\frac{F(u)}{u^{\frac{1}{2}}}\leq\frac{F(v)}{v^{\frac{1}{2}}}$ if and only if $u^{\frac{1}{6}}\geq v^{\frac{1}{6}}$ for $u\geq v>0$ and $F(-u)=(-u)^{\frac{1}{3}}=-u^{\frac{1}{3}}=-F(u)$. Clearly,

$$\begin{aligned}
\sum_{s=\sigma}^{\infty}q(s)F(CA(s-\sigma))+\sum_{j=1}^{\infty}r(m_j-1)F(CA(m_j-\sigma-1))\\
&\geq\sum_{s=\sigma}^{\infty}q(s)F(CA(s-\sigma))\\
&=\sum_{s=3}^{\infty}\frac{2e+e^{-1}+3}{e^{\frac{s-3}{3}}}\times\left(\frac{e^{s-3}-e}{e-1}\right)^{1/3}\to\infty \text{ as } n\to\infty.
\end{aligned}$$

Indeed, all conditions of Theorem 2.1 are satisfied. In particular, $x(n)=(-1)^ne^n$ is an unbounded oscillatory solution of the first equation of (E') while $(-1)^{m_j}e^{m_j}$ is an unbounded oscillatory solution of the second equation of (E').

Example 3.2. *Consider the impulsive difference equation of the form:*

$$(E'')\begin{cases} \Delta[a(n)\Delta(x(n)+p(n)x(n-2))]+q(n)x^3(n-1)=0,\ n\neq m_j,\ n>2,\\ \underline{\Delta}[a(m_j-1)(x(m_j-1)+p(m_j-1)x(m_j-3)]+r(m_j-1)x^3(m_j-2)=0,\ j\in\mathbb{N}, \end{cases}$$

where $\tau=2$, $\sigma=1$, $p(n)=\frac{1}{n}$, $q(n)=4+\frac{1}{n+2}+\frac{2}{n+1}+\frac{1}{n}$, $a(n)=1$ *and* $A(n)=n$. *Here,* $r(m_j-1)=4+\frac{1}{m_j+3}+\frac{1}{m_j+2}+\frac{1}{m_j}+\frac{1}{m_j-1}$ *when* $m_j=3j$ *for* $j\in\mathbb{N}$ *and* $F(u)=u^3$. *Let* $\beta=2$. *Then* $\frac{F(u)}{u^2}\geq\frac{F(v)}{v^2}$ *if and only if* $u\geq v$ *for* $u,v>0$ *and* $F(-u)=-F(u)$. *Clearly,*

$$\sum_{s=2}^{\infty}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty}q(t)+\sum_{j=1}^{\infty}r(m_j-1)\right]\geq\sum_{s=2}^{\infty}\frac{1}{a(s)}\left[\sum_{t=s}^{\infty}q(t)\right]\geq\sum_{s=2}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t}\to\infty.$$

Indeed, all conditions of Theorem 2.7 are satisfied. In particular, $x(n)=(-1)^n$ *is an oscillatory solution of the first equation of* (E'') *while* $(-1)^{m_j}$ *is an oscillatory solution of the second equation of* (E'').

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