Filomat 35:10 (2021), 3353–3364
https://doi.org/10.2298/FIL2110353B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Numerical Technique for Solving a Class of Nonlinear Singularly Perturbed Boundary Value Problems

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Abstract. In this work, we have studied a numerical scheme based on Sinc collocation method to solve a class of nonlinear singularly perturbed boundary value problems. The solution of the problems exhibit a boundary layer on the both sides or one side of the domain due to the presence of perturbation parameter ϵ. The Sinc method can control the oscillations in computed solutions at boundary layer regions naturally because the distribution of Sinc points is denser at near the boundaries. The convergence analysis is discussed and the method is shown to be an exponential convergent. The numerical results support the theoretical results and illustrate the efficiency and accuracy of the method compared with the results in the existing methods.

1. Introduction

A singularly perturbed differential equation belongs to the class of ordinary differential equations with highest derivative term multiplied by a small parameter. Many real life phenomena/applications arising in fluid mechanics, biosciences, control theory, economics and engineering can be modeled by these equations such as heat transfer problem with high P*é*clet numbers,evolutionary biological models, quantum mechanics, fluid dynamic elasticity, reactor diffusion process, oceanography, steady and unsteady viscous flow problem with large Reynolds numbers, signal transmission and magneto-hydrodynamics duct problems with high Hartman numbers.
Although the perturbation parameters are too small, the nature of the problem changes completely and boundary layers, which are rapid changes of the solution close to the boundary, are observed. It becomes harder to obtain stable solutions and more grid points are required to resolve the boundary layer [13].
We consider the following nonlinear singularly perturbed boundary value problem:

$$\epsilon y'' = F(x, y, y'), \qquad on \quad \Omega = (a, b) \tag{1}$$

subjected to boundary conditions :

$$y(a) = y_a, \quad y(b) = y_b, \tag{2}$$

2020 *Mathematics Subject Classification*. 65L04, 65L70, 65L11,65L60, 65L08
Keywords. Singularly perturbed equations; nonlinear boundary value problems; Sinc method; Convergence analysis.
Received: 10 May 2020; Revised: 23 March 2021; Accepted: 16 August 2021
Communicated by Marko Petković
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where ϵ is a small parameter perturbation satisfying $0 < \epsilon \ll 1$. In general, as ϵ tends to zero, the solution y of (1) may exhibit boundary or internal layers of various types (see [5] and [14]). Suppose F is smooth and satisfies the conditions:

- $\frac{\partial}{\partial y}F(x,y,z) \geq 0, \frac{\partial}{\partial z}F(x,y,z) \leq 0.$
- $\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)F(x,y,z) \geq \rho > 0, \rho$ is a positive constant.
- the growth condition $F(x,y,z) = O(|z|^2)$ as $z \to \infty$ for all $x \in (a,b)$ and all real y and z.

Under the conditions listed above the equations (1) and (2) has a unique solution [9].
The presence of the singular perturbation parameter, leads to occurrences of spurious oscillations in the computed solutions using finite difference schemes and finite element methods with piecewise polynomial basis functions. Therefore, in order to overcome such drawbacks associated with classical finite difference and finite element methods, we need to develop ϵ-uniformly convergent numerical methods; among them the fitted mesh method, which utilizes special layer-adapted mesh, is a satisfactory and popular technique to overcome the numerical difficulties.
In recent years, various numerical methods have been introduced and developed to solve the singularly perturbed differential equations such as least squares method [1], the finite difference scheme and Shishkin mesh [16], a homotopy technique [22], collocation method with Haar wavelets [21] and Bessel functions of the first kind of singularly perturbed differential equation[23]. A singularly perturbed problem having two parameters with discontinuous source term was considered in [7]. Authors [2] showed that Richardson extrapolation improves the order of convergence for interior turning point problem with two exponential boundary layers. Paper [8] presents an approximate solution for an interior layer problem by asymptotic expansion technique and reproducing kernel method. In [15] a numerical method using an approximate layer location was developed for a nonlinear singularly perturbed problems. Also, a neuro-evolutionary technique is developed for solving singularly perturbed linear and nonlinear boundary value problems in [24]. Moreover, we can point to other efficient methods for solving singularly perturbed boundary value problems such as [6], [10], [11].
Application of Sinc approach has been shown to be more suitable for handling singularities in boundary layers and semi-infinite domain than other methods [3], [4] and [17]. In the Sinc method the test functions are defined by the Sinc-function $s(x) = \sin(\pi x)/(\pi x)$, the Sinc method, which was developed by Stenger [20], is based on the Whittaker-Shannon-Kotel′ nikov sampling theorem for entire functions. In the presence of singularities, it gives a much better rate of convergence and accuracy than polynomial methods. The reference [12] provides excellent overviews of the Sinc method for several fields of mathematics.
The paper is organized as follows. In section 2, we review some basic facts about the Sinc approximation. In section 3, the Sinc collocation method is developed for solving of nonlinear singularly perturbed boundary value problems, In section 4, the convergence analysis of proposed method is given. Some numerical examples will be presented in section 5, and at the end we conclude implementation, application and efficiency of proposed scheme.

2. Notation and background

In this section, we state preliminaries of the Sinc interpolation together with some essential definitions and theorems.
The Sinc function is defined on $-\infty < x < \infty$ by

$$\text{Sinc}(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For $h > 0$, we will denote the Sinc basis functions by $S(j,h)(x) = \text{Sinc}(\frac{x-jh}{h}), \quad j = 0, \pm 1, \pm 2, \ldots.$
Let f be a function defined on $\mathbb{R}$ then for $h > 0$ the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x),$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of Whittaker cardinal expansions have been studied and are thoroughly surveyed in Stenger [20]. These properties are derived in the infinite strip D_s of the complex plane where $d > 0$

$$D_s = \{\zeta = \xi + i\eta : |\eta| < d \leq \frac{\pi}{2}\}.$$

Approximations can be constructed for infinite, semi-finite, and finite intervals. But in this paper we construct approximation on the interval (a, b), we consider the conformal map

$$\phi(z) = \ln(\frac{z-a}{b-z}), \tag{3}$$

which maps the eye-shaped region

$$D_E = \{z = x + iy; |\arg(\frac{z-a}{b-z})| < d \leq \frac{\pi}{2}\},$$

onto the infinite strip D_s. For the Sinc method, the basis functions on the interval (a, b) for $z \in D_E$ are derived from the composite translated Sinc function:

$$S_j(z) = S(j, h) \circ \phi(z) = Sinc(\frac{\phi(z) - jh}{h}). \tag{4}$$

The function

$$z = \phi^{-1}(\sigma) = \frac{be^{\sigma} + a}{1 + e^{\sigma}},$$

is an inverse mapping of $\sigma = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\psi(u) = \phi^{-1}(u) \in D_E : -\infty < u < \infty\} = (a, b).$$

The Sinc grid points $z_k \in (a, b)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{be^{kh} + a}{1 + e^{kh}}, \qquad k = 0, \pm 1, \pm 2, ... \tag{5}$$

Definition 1. Let $B(D_E)$ be the class of functions f which are analytic in D_E such that

$$\int_{\psi(u+\Sigma)} |f(z)|dz \to 0, \qquad as \quad u \to \pm\infty \tag{6}$$

where $\Sigma = \{i\eta : |\eta| < d \leq \frac{\pi}{2}\}$ and satisfy

$$\aleph(f) \equiv \int_{\partial D_E} |f(z)|dz < \infty, \tag{7}$$

where ∂D_E represents the boundary of D_E.

Definition 2. Let $L_\alpha(D_E)$ be the set of all analytic function u in D_E, for which there exists a constant C such that

$$|u(z)| \leq C\frac{|\rho(z)|^{\alpha}}{[1 + |\rho(z)|]^{2\alpha}}, \qquad z \in D_E, \quad 0 < \alpha \leq 1, \tag{8}$$

where $\rho(z) = e^{\phi(z)}$.
Theorem 1. (Stenger [20]) If $\phi' u \in B(D_E)$, and let

$$\sup_{\frac{-\pi}{h} \leq t \leq \frac{\pi}{h}} \left| \left(\frac{d}{dx}\right)^l e^{it\phi(x)} \right| \leq C_1 h^{-l}, \qquad x \in \Gamma,$$

for $l = 0, 1, \ldots, m$ with C_1 a constant depending only on m and ϕ. If $u \in L_\alpha(D_E)$ then taking $h = \sqrt{\pi d/\alpha N}$ it follows that

$$\sup_{x \in \Gamma} \left| u^{(l)}(x) - \left(\frac{d}{dx}\right)^l \sum_{j=-N}^{N} u(x_j) S_j(x) \right| \leq C N^{(l+1)/2} \exp(-(\pi d \alpha N)^{1/2}),$$

where C is a constant depending only on u, d, m, ϕ and α.
The Sinc-collocation method requires that the derivatives of composite Sinc function be evaluated at the nodes. We need to recall the following lemma.
Lemma 1. (Lund and Bowers [12]) Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (4). Then

$$\delta_{jk}^{(0)} = [S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{9}$$

$$\delta_{jk}^{(1)} = h \frac{d}{d\phi} [S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{(k-j)}}{k-j}, & j \neq k, \end{cases} \tag{10}$$

$$\delta_{jk}^{(2)} = h^2 \frac{d^2}{d\phi^2} [S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{(k-j)}}{(k-j)^2}, & j \neq k, \end{cases} \tag{11}$$

in relations (9)-(11) h is step size and x_k is Sinc grid given by (5).

3. The Sinc-collocation method

We apply the Sinc-collocation method for solution of equation (1) with given boundary conditions (2). The approximate solution for $y(x)$ is represented by formula

$$y(x) \approx y_N(x) = \sum_{j=-N}^{N} c_j S_j(x), \qquad m = 2N + 1, \tag{12}$$

where $S_j(x)$ is function $S(j,h) \circ \phi(x)$ for some fixed step size h. The unknown coefficients c_j in relation (12) are determined by collocation method. Also the l-th derivative of the function $y(x)$ can be approximated as:

$$y^{(l)}(x) \approx \sum_{j=-N}^{N} c_j \frac{d^l}{dx^l} S_j(x). \tag{13}$$

Setting

$$\frac{d^i}{d\phi^i}[S_j(x)] = S_j^{(i)}(x), \qquad 0 \leq i \leq 2, \tag{14}$$

and noting that

$$\frac{d}{dx}[S_j(x)] = S_j^{(1)}(x)\phi^{'}(x), \tag{15}$$

$$\frac{d^2}{dx^2}[S_j(x)] = S_j^{(2)}(x)[\phi^{'}(x)]^2 + S_j^{(1)}(x)\phi^{''}(x), \tag{16}$$

and

$$\delta_{jk}^{(l)} = h^l \frac{d^l}{d\phi^l}[S_j(x)]_{x=x_k}. \tag{17}$$

Now by substituting each terms of (1) with given approximations in (12) and (13) and evaluating the result at the Sinc points x_k also using relations (14-17), we can obtain the discrete Sinc-collocation system of nonlinear equations to determining the unknown coefficients $\{c_j\}_{j=-N}^{N}$ as

$$\epsilon\left\{\sum_{j=-N}^{N} c_j\left(\frac{\delta_{jk}^{(2)}}{h^2}[\phi^{'}(x_k)]^2 + \frac{\delta_{jk}^{(1)}}{h}\phi^{''}(x_k)\right)\right\} = F\left(x_k, c_k, \sum_{j=-N}^{N} c_j\phi^{'}(x_k)\frac{\delta_{jk}^{(1)}}{h}\right), \qquad k = -N, -N+1, ..., N. \tag{18}$$

To obtain a matrix representation of the equations (18), let $\mathbf{I}^{(i)}, i = 0, 1, 2$ be the $m \times m$ matrices whose jk-th entry is given by (9)-(11). Note that the matrix $\mathbf{I}^{(2)}$ and $\mathbf{I}^{(1)}$ are symmetric and skew-symmetric matrices respectively, also $\mathbf{I}^{(0)}$ is identity matrix. We define the $m \times m$ diagonal matrix as follow:

$$\mathbf{D}(g(x))_{ij} = \begin{cases} g(x_i), & i = j, \\ 0, & i \neq j. \end{cases}$$

Therefore, by using the above definitions the system (18) can be represented by the following matrix form:

$$A\mathbf{C} + F(\mathbf{C}) = 0, \tag{19}$$

where $\mathbf{C}$ is m-vectors and A is $m \times m$ matrix as:

$$\mathbf{C} = (c_{-N}, c_{-N+1}, \ldots, c_N)^t,$$

$$A = \epsilon\Big\{\frac{1}{h^2}I^{(2)}D(\phi^{'})^2 + \frac{1}{h}I^{(1)}D(\phi^{''})\Big\}.$$

The system (19) is a nonlinear system of equations which consists of m equations and m unknowns. By solving this system by means of Newton's method, we can obtain an approximate solution $y_N(x)$ of (1) from (12).

4. Convergence analysis

In this section, we show that the approximate solution $y_N(x)$ given in (12) converges exponentially to the exact solution $y(x)$ of (1). In order to establish a bound of $|y(x) - y_N(x)|$, for this purpose, we assume that the analytic solution of equation (1) at the Sinc points x_k denoted by $\hat{y}(x)$ and defined by

$$\hat{y}(x) = \sum_{j=-N}^{N} y(x_j)S_j(x). \tag{20}$$

First, we rewrite equation (1) as follows:

$$\epsilon y'' = H(x, y, y') + g(x), \qquad on \quad \Omega = (a, b) \tag{21}$$

also, we rewrite (19) as follows

$$A\mathbf{C} + H(\mathbf{C}) = \mathbf{P}, \tag{22}$$

where

$$\mathbf{P} = (g(x_{-N}), g(x_{-N+1}), \ldots, g(x_N))^t,$$

we first need to get a bound of $\|y_N(x) - \hat{y}(x)\|_\infty$, so that, we need to prove the following theorem:

Theorem 2. If $y_N(x)$ and $\hat{y}(x)$ are defined in (12) and (20), then there exists a constant K_1 independent of N such that

$$\sup_{x\in\Omega} |y_N(x) - \hat{y}(x)| \leq K_1 N^2 \exp(-(\pi\alpha d N)^{1/2}). \tag{23}$$

Proof. According to (20) and Cauchy-Schwarz inequality we have

$$|y_N(x) - \hat{y}(x)| = \left|\sum_{j=-N}^{N} c_j S_j(x) - \sum_{j=-N}^{N} y(x_j) S_j(x)\right| \leq$$

$$\left(\sum_{j=-N}^{N} |c_j - y(x_j)|^2\right)^{\frac{1}{2}} \left(\sum_{j=-N}^{N} |S_j(x)|^2\right)^{\frac{1}{2}} = E, \tag{24}$$

we know that $\left(\sum_{j=-N}^{N} |S_j(x)|^2\right)^{\frac{1}{2}} \leq k_1$ where k_1 is a constant, then

$$E \leq k_1 \|\mathbf{C} - \widetilde{\mathbf{C}}\|_2, \tag{25}$$

where $\widetilde{\mathbf{C}} = (y(x_{-N}), y(x_{-N+1}), \ldots, y(x_N))^t$ and $\mathbf{C} = (c_{-N}, c_{-N+1}, \ldots, c_N)^t$.
Now we must obtain a bound for $\|\mathbf{C} - \widetilde{\mathbf{C}}\|_2$, to this aim, following (22) we have

$$A\widetilde{\mathbf{C}} + H(\widetilde{\mathbf{C}}) = \widetilde{\mathbf{P}}, \tag{26}$$

where

$$\widetilde{\mathbf{P}} = (\widetilde{g}(x_{-N}), \widetilde{g}(x_{-N+1}), \ldots, \widetilde{g}(x_N))^t.$$

By subtracting (26) from (22) we have

$$A(\mathbf{C} - \widetilde{\mathbf{C}}) + \left(H(\mathbf{C}) - H(\widetilde{\mathbf{C}})\right) = (\mathbf{P} - \widetilde{\mathbf{P}}). \tag{27}$$

Now, we obtain a bound for $\|\mathbf{P} - \widetilde{\mathbf{P}}\|$ in maximum norm. Applying (21) we find

$$\left|g(x_k) - \widetilde{g}(x_k)\right| \leq \epsilon\left|y''(x_k) - \widetilde{y}''(x_k)\right| + \left|H(x_k, y(x_k), y'(x_k)) - H(x_k, \widetilde{y}(x_k), \widetilde{y}'(x_k))\right|, \tag{28}$$

for $k = -N, -N+1, \ldots, N$.
From (28) and by Theorem 1 and Theorem 9.19 reference [[18] p.218], we obtain
$\|\mathbf{P} - \widetilde{\mathbf{P}}\|_\infty \leq \epsilon\|\mathbf{y}'' - \widetilde{\mathbf{y}}''\|_\infty + \|H'(z)\|_\infty\left(\|\mathbf{y} - \widetilde{\mathbf{y}}\|_\infty + \|\mathbf{y}' - \widetilde{\mathbf{y}}'\|_\infty\right) \leq$

$$+\epsilon M_1 N^{3/2} \exp(-(\pi\alpha d N)^{1/2}) + M_2\left(M_3 N^{1/2} \exp(-(\pi\alpha d N)^{1/2}) + M_4 N\right)$$

where M_1, M_3 and M_4 are constants independent on N and $\|H'(z)\|_\infty \leq M_2$.
Thus we have

$$\|\mathbf{P} - \widetilde{\mathbf{P}}\|_\infty \leq M_5 N^{3/2} \exp(-(\pi\alpha d N)^{1/2}), \tag{29}$$

where $M_5 = \epsilon M_1 + M_2M_3 + M_2M_4$.
By using mean value theorem we find

$$H(\mathbf{C}) - H(\widetilde{\mathbf{C}}) = \Big(\int_0^1 JH\Big(\mathbf{C} + t(\mathbf{C} - \widetilde{\mathbf{C}})dt\Big)\Big(\mathbf{C} - \widetilde{\mathbf{C}}\Big) \tag{30}$$

in which JH is Jacobin matrix of H. From (27) and (30) we have

$$R(\mathbf{C} - \widetilde{\mathbf{C}}) = \mathbf{P} - \widetilde{\mathbf{P}}, \tag{31}$$

where

$$R = A + \int_0^1 JH\Big(\mathbf{C} + t(\mathbf{C} - \widetilde{\mathbf{C}})dt, \tag{32}$$

therefore, by taking L^2-norm from (31) and using (29) we have
$||\mathbf{C} - \widetilde{\mathbf{C}}||_2 \leq ||R^{-1}||_2||\mathbf{P} - \widetilde{\mathbf{P}}||_2 \leq ||R^{-1}||_2 \sqrt{N}||\mathbf{P} - \widetilde{\mathbf{P}}||_\infty$

$$\leq M_5N^2||R^{-1}||_2 \exp(-(\pi\alpha dN)^{1/2}). \tag{33}$$

Finally, by applying (24), (25) and (33) we can find

$$\sup_{x\in\Omega} |y_N(x) - \hat{y}(x)| \leq K_1N^2 \exp(-(\pi\alpha dN)^{1/2}). \tag{34}$$

where $K_1 = k_1M_5||R^{-1}||_2$.■

Theorem 3. Let $y(x)$ be the exact solution of (1) and $y_N(x)$ be its Sinc approximation defined by (12), then, under the assumptions of Theorem 1 and 2, there exists a constant K, independent of N, such that

$$\sup_{x\in\Omega} |y_N(x) - y(x)| \leq KN^2 \exp(-(\pi\alpha dN)^{1/2}). \tag{35}$$

Proof. By making use of the triangular inequality we have

$$|y_N(x) - y(x)| \leq |y_N(x) - \hat{y}(x)| + |\hat{y}(x) - y(x)|, \tag{36}$$

By applying Theorem 1 for second term of right hand side of (36) there exists a constant K_2 independent of N such that

$$\sup_{x\in\Omega} |\hat{y}(x) - y(x)| \leq K_2N^{\frac{1}{2}} \exp(-(\pi\alpha dN)^{1/2}). \tag{37}$$

Also, by using Theorem 2 for first term in the right hand side of (36) we can obtain

$$\sup_{x\in\Omega} |y_N(x) - \hat{y}(x)| \leq K_1N^2 \exp(-(\pi\alpha dN)^{1/2}). \tag{38}$$

Finally, by applying relations (36-38) we have

$$\sup_{x\in\Omega} |y_N(x) - y(x)| \leq KN^2 \exp(-(\pi\alpha dN)^{1/2}), \tag{39}$$

where $K = \max\{K_1, K_2\}$.■

Remark. It is worth to mention here that, the value K in the bound of error in (39) depends on values of ϵ. Therefore, this should be considered in numerical results. However, due to the nature of the Sinc method and the numerical results obtained, this non-uniform convergence has little effect on the loss of method efficiency.

5. Numerical results

To demonstrate the applicability of the above presented method we have applied it to three nonlinear singular perturbation problems with left-end boundary layer. The maximum nodal errors and the convergence rate is calculated. In all of the examples considered in this paper, we choose $\alpha = 1$ and $d = \frac{\pi}{2}$ which yield $h = \frac{\pi}{\sqrt{2N}}$, also the maximum pointwise errors are reported on uniform grids

$$U = \{z_0, z_1, \ldots, z_p\}, \quad z_r = \frac{r}{p}, \quad r = 0, 1, \ldots, p. \tag{40}$$

When we have an analytical solution then for each ϵ the maximum errors at all mesh points can be calculated as

$$E_\epsilon^N = \max_r |y(z_r) - y_N(z_r)|. \tag{41}$$

Also, when the exact solution of problems is not available , we estimate maximum errors using the double mesh principle as

$$E_\epsilon^N = \max_r |y_{2N}(z_r) - y_N(z_r)|. \tag{42}$$

For each values of ϵ, order of convergence are

$$\rho_\epsilon^N = \log_2\Big(\frac{E_\epsilon^N}{E_\epsilon^{2N}}\Big). \tag{43}$$

Also, the global errors and global rates of convergence are

$$E^N = \max_\epsilon\Big(E_\epsilon^N\Big), \qquad \rho^N = \log_2\Big(\frac{E^N}{E^{2N}}\Big). \tag{44}$$

Example 1. We consider equation (1) as

$$\epsilon y'' = (y - u)^n, \quad u(x) = 0.5 + x(x-1), \qquad y(0) = 1, y(1) = 1.5. \tag{45}$$

As the exact solution $y(x)$ of (45) is unknown, therefore for each ϵ pointwise errors are estimated as (42). The estimated global errors E^N and rate of convergence ρ^N using the method applied to above example are shown in table 1 for values $\epsilon = 2^0, 2^{-1}, ..., 2^{-60}$. In table 1, we compare our results with obtained results in [16], this comparison verify efficiency and accuracy of the proposed method especially with increasing N. Figure 1 displays the approximate solution for various values of $\epsilon = 2^{-6}, 2^{-10}, 2^{-15}$ and 2^{-25}, it can be seen that the boundary layers are located at both edges $x = 0$ and $x = 1$. Also, we observe that the propagation front is steeper in the neighborhood of edges, the boundary layer regions, for the small values of the parameter ϵ. It has been observed that the maximum pointwise errors arise near the coarse grids, which are in the boundary layer regions, for small values of the perturbation parameter. Therefore, the Sinc method can treat this issue naturally because the distribution of Sinc points is denser at near the boundaries.
Example 2. Consider the problem

$$-\epsilon^2 y'' + 1.4y = -1 + 0.4y - (1 + y)^3, \qquad y(0) = y(1) = 0. \tag{46}$$

Errors E_ϵ^N and rates ρ_ϵ^N based on (42-44) for this example are reported in Table 2 for various values of ϵ and N. These results verify efficiency and accuracy of the proposed method compared with obtained results in [19]. Also, it can be seen that the errors of proposed method are related to value of the parameter ϵ, of course, this dependency does not have a large impact on the performance of our method.
The graph of approximate solution is represented in figure 2 for values $\epsilon = 10^{-1}, 5\times10^{-2}, 10^{-2}$ and 10^{-5}, this figure shows that there are no any the boundary layers for large value of ϵ, but boundary layers are located at both sides of domain for small values of ϵ, which validates the physical behavior of the solution.

Table 1: *Estimated global errors and rates of convergence defined in (44) for example 1 with $p = 2N$.*

	$n = 2$		$n = 3$	
$N \downarrow$	[16]	Our scheme	[16]	Our scheme
16	0.0296	0.0110	0.0284	0.0111
	0.45	0.81	0.27	0.79
32	0.0216	0.0061	0.0235	0.0064
	0.64	0.93	0.33	0.60
64	0.0139	0.0032	0.0187	0.0042
	0.83	1.83	0.40	1.58
128	0.0078	0.0009	0.0142	0.0014
	0.91	3.16	0.47	2.81

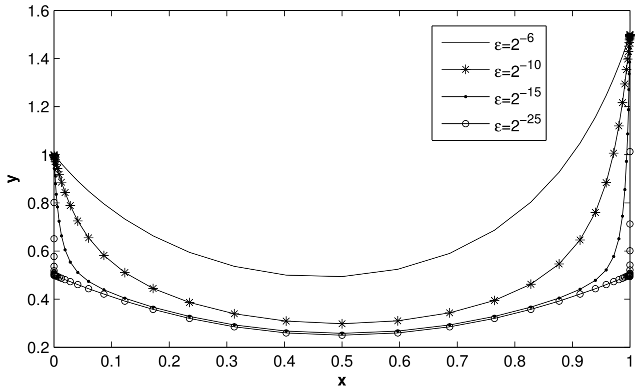

Figure 1: Approximate solutions for example 1 for various values of ϵ and for fix $N = 64$.

Table 2: *Comparison of estimated errors E_ϵ^N and rates ρ_ϵ^N for Example 2 with $p = 2N$.*

	$N = 32$		$N = 64$		$N = 128$	
ϵ	[19]	Our scheme	[19]	Our scheme	[19]	Our scheme
10^0	$7.25e-06$	$1.87e-06$	$1.81e-6$	$1.15e-7$	$4.53e-7$	$4.54e-9$
	2.00	4.02	1.99	4.66		
10^{-2}	$8.49e-4$	$4.68e-5$	$2.10e-4$	$6.52e-6$	$5.24e-5$	$9.98e-8$
	1.81	2.84	2.00	6.03		
10^{-5}	$1.16e-2$	$1.07e-4$	$3.39e-3$	$1.56e-5$	$1.06e-3$	$8.15e-7$
	1.77	2.77	1.67	4.25		
10^{-10}	$1.16e-2$	$3.20e-3$	$3.39e-3$	$5.17e-4$	$1.06e-3$	$3.04e-5$
	1.77	2.62	1.67	4.08		
10^{-15}	$1.16e-2$	$3.20e-3$	$3.39e-3$	$7.87e-4$	$1.06e-3$	$1.32e-4$
	1.77	2.02	1.67	2.57		
ρ^N	1.77	2.02	1.67	2.57		

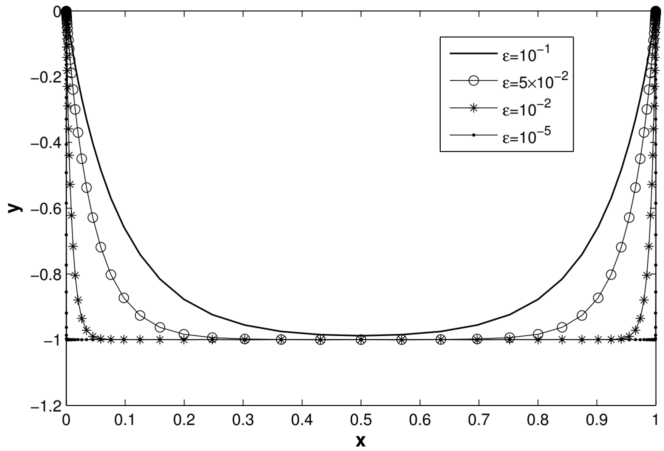

Figure 2: Approximate solutions for example 2 for various values of ϵ and $N = 64$.

Table 3: *Comparison of estimated maximum pointwise errors for Example 3 with uniform mesh for various values of N with* $\epsilon = 10^{-2}$ *and* $p = 2N$.

N	[1]	Our scheme
4	1.12	0.20
8	$8.45e-2$	$2.12e-2$
16	$1.62e-2$	$7.18e-3$
32	$3.99e-3$	$2.35e-4$

Example 3. Consider the singularly perturbed boundary value problem

$$-\epsilon y'' + y^2 = f(x), \qquad y(0) = 1, y(1) = \exp(-\frac{1}{\epsilon}), \tag{47}$$

where $f(x) = \exp(-\frac{2x}{\epsilon}) - \dfrac{1}{\epsilon}\exp(-\frac{x}{\epsilon})$, with exact solution

$$y(x) = \exp(-\frac{x}{\epsilon}).$$

The maximum errors in (41) for this example are tabulated in table 3 for various values of N with fixed value $\epsilon = 10^{-2}$ and compared with results in [1]. Also, table 4 displays E_ϵ^N for various values of ϵ, this table shows that by increasing N the errors in our scheme decrease. Figure 3 represents the approximate solutions of example 3 for different values of ϵ, this graph exhibit existence of the boundary layers on the left side of the domain especially for small values of ϵ.
For these examples, the convergence curves are plotted for various values of ϵ in figure 4. This figure shows that the treatment of maximum errors is exponential with increasing N and verifies the theoretical results. Of course, with decreasing perturbation parameter ϵ this behavior is almost near to exponential.

6. Conclusions

In this article, a numerical method was employed successfully for solving a class of nonlinear singularly perturbed boundary value problems. This approach is based on the Sinc collocation method.
We know that the oscillations in the computed solutions arise near the coarse grids, which are in the

Table 4: *Errors E_ϵ^N for Example 3 for various values of ϵ and N with $p = 2N$.*

ϵ	$N = 32$	$N = 64$
10^{-2}	$2.35e-4$	$1.45e-6$
10^{-3}	$2.42e-3$	$1.48e-5$
10^{-4}	$2.02e-2$	$1.14e-4$
10^{-5}	$5.19e-2$	$1.40e-3$

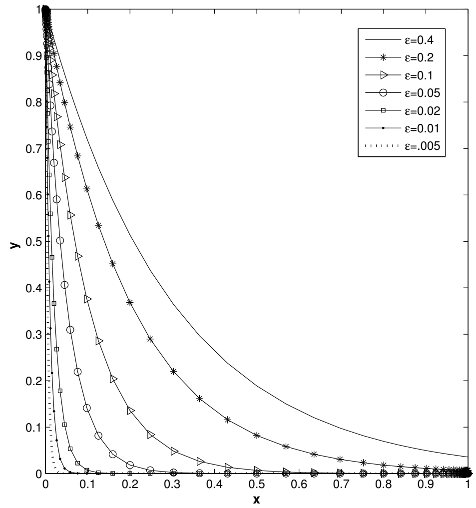

Figure 3: Approximate solutions for example 3 for various values of ϵ and for fix $N = 64$.

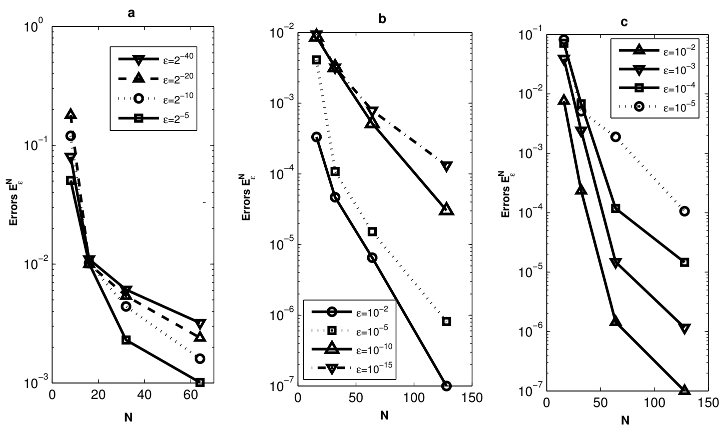

Figure 4: Convergence curves of the method for various values of ϵ: (a) for example 1, (b) for example 2 and (c) for example 3.

boundary layer regions, for small values of the perturbation parameter. On the other hand, the distribution of Sinc points is denser at near the boundaries. Therefore, the Sinc method can control the oscillations at edges naturally.

The convergence analysis of the proposed method is presented and an exponential convergence is achieved as well, the figure 4 verify this matter. Results from numerical experiments indicate the efficiency and accuracy of proposed method compared with other methods. To substantiate the suitability of the proposed method, graphs have been plotted for different values of the parameter ϵ in figures 1, 2 and 3. We can see that the boundary layers are located at both sides or one side of domain and the width of boundary layers decrease and become more and more stiff at $x = 0$ and $x = 1$ for small values of ϵ, which validates the physical behavior of the solution.

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