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On solvability of Riemann problems in Banach Hardy classes

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Abstract. This work deals with the rearrangement invariant Banach function space X and Banach Hardy classes generated by this space, which consist of analytic functions inside and outside the unit circle. In these Hardy classes we consider homogeneous and nonhomogeneous Riemann problems with piecewise continuous coefficient. We define new characteristic of the space X related to the power functions in X. Canonical solution is defined depending on the jumps of the argument of the coefficient of the problem. In terms of the above characteristic, we find a condition on the jumps of the argument depending on the Boyd indices of space X, which is sufficient for the solvability of these problems, and, in case of solvability, we construct a general solution. We also give an orthogonality condition for the solvability of nonhomogeneous problem. As X, considering specific spaces, we obtain previously known results.

1. Introduction

Lately, there had been great interest in various non-standard function spaces in the context of problems of theoretical and applied mathematics. Among those spaces, we can mention Lebesgue spaces with variable summability index, Morrey spaces, grand-Lebesgue spaces, etc. Various problems of harmonic analysis, theory of partial differential equations, approximation theory, theory of conjugate boundary value problems for analytic functions, etc are studied in these spaces. Numerous research articles and review articles have been dedicated to these problems. Many results related to the problems of harmonic analysis have been obtained by Diening L., Harjuleto P., Hasto P., Ruzicka M. [30], D.V. Gruz-Uribe, A. Fiorenza [1], D.R. Adams [2], V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko [3, 4], R.E. Castillo, H. Rafeiro [5], M.M. Reo, Z.D. Ren [6], R. Lecniewicz [7], J. Musielak [8], W.M. Kozlowski [9], etc. In [10–23] (to name just a few), the problems of approximation theory, basis theory and theory of boundary value problems for analytic functions have been considered. Each of the above-mentioned spaces presents specific difficulties to treat this or that problem depending on the geometry of the space. The solutions of considered problems depend on the parameters of the space (including the norm it is supplied with) and the problem data, and you have to find the relationships between them to solve your problem. Despite these circumstances, it should be noted that these spaces are basically (unlike, for example, Lebesgue spaces with variable summability index) so called rearrangement invariant Banach function spaces (r.i.s. for short). For the theory of these spaces we refer the readers to the monographs [24, 25, 43]. The question naturally arises: is it possible

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to consider the above problems in general r.i.s.? Some problems of harmonic analysis have already been studied in r.i.s. (see [24, 25, 43] and [26–29]).

Note that the study of basis properties (completeness, minimality, basicity) of perturbed trigonometric systems of exponential form

$$\left\{e^{i\left(nt-\alpha(t)\,sign\,n\right)}\right\}_{n\in Z}, \tag{1}$$

(Z are integers) and those of sine and cosine systems

$$\{\cos\,(nt-\alpha\,(t))\}_{n\in Z_+}\ ;\ \{\sin\,(nt-\alpha\,(t))\}_{n\in N}, \tag{2}$$

(N is the set of natural numbers; $Z_+ = \{0\}\bigcup N; Z = \{-N\}\bigcup Z_+$) in different Banach function spaces is of special scientific interest for the spectral theory of differential operators. One of the known methods for treating basis properties of the systems (1) and (2) in function spaces (for example, in Lebesgue spaces L_p) is the method of Riemann boundary value problems in corresponding Hardy spaces. For the applications of this method in L_p and Morrey spaces we refer the readers to [31–39, 45–48]. This method requires definition of corresponding Hardy spaces, establishment of some properties of them, basicity of the parts of an exponential system in these spaces and solvability of Riemann problems. In [12, 13, 18, 19, 39], the method of Riemann boundary value problems to study basis properties of the systems (1), (2) has been developed for Lebesgue spaces with variable summability index and Morrey spaces. Similar matters have been treated in [45] for Orlicz-Hardy spaces.

In this work, we deal with the rearrangement invariant Banach function space X and Banach Hardy classes, generated by this space, which consist of analytic functions inside and outside the unit circle. In these Hardy classes, we consider homogeneous and nonhomogeneous Riemann problems with piecewise continuous coefficient. We define new characteristic of the space X related to the power functions in X. Canonical solution is defined with regard to the jumps of the argument of the coefficient of the problem. In terms of the above characteristic, we find a condition on the jumps of the argument which is sufficient for the solvability of these problems, and, in case of solvability, we construct a general solution. We also give an orthogonality condition for the solvability of nonhomogeneous problem. As X, considering specific spaces, we obtain previously known results.

2. Needful information and auxiliary facts

Further, we will use the following standard notations and concepts. $R_+ = (0,+\infty)$; $\chi_M(\cdot)$ is the characteristic function of the set M; R is the set of real numbers; C is the complex plane; $\omega = \{z \in C : |z| < 1\}$ is a unit disk in C; $\gamma = \partial\omega$ is a unit circle; $\bar{M}$ is the closure of the set M with respect to appropriate norm; $(\bar{\cdot})$ is the complex conjugate. By $[X]$ we denote the algebra of linear bounded operators acting in a Banach space X.

We will need some concepts and facts from the theory of Banach function spaces (see, e.g., [24, 25]). Let $(R;\mu)$ be a measure space. Let $\mathcal{M}^+$ be the cone of μ-measurable functions on R whose values lie in $[0,+\infty]$. The characteristic function of a μ-measurable subset E of R denote by χ_E.

Definition 2.1. *A mapping $\rho : \mathcal{M}^+ \to [0,+\infty]$ is called a Banach function norm (or simply a function norm) if, for all $f, g, f_n, n \in N$ in $\mathcal{M}^+$, for all constants $a \geq 0$ and for all μ-measurable subsets $E \subset R$, the following properties hold:*
(P1)$\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\rho(af) = a\rho(f)$; $\rho(f+g) \leq \rho(f) + \rho(g)$;
(P2) $0 \leq g \leq f$ μ-a.e. $\Rightarrow \rho(g) \leq \rho(f)$;
(P3) $0 \leq f_n \uparrow f$ μ-a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$;
(P4)$\mu(E) < +\infty \Rightarrow \rho(\chi_E) < +\infty$;
(P5) $\mu(E) < +\infty \Rightarrow \int_E f d\mu \leq C_E\rho(f)$, for some constant $C_E : 0 < C_E < +\infty$ depending on E and ρ, but independent of f.

Let $\mathcal{M}$ denote the collection of all extended scalar-valued (real or complex) μ-measurable functions and $\mathcal{M}_0 \subset \mathcal{M}$ the subclass of functions that are finite μ-a.e. .

Definition 2.2. *Let ρ be a function norm. The collection $X = X(\rho)$ of all functions f in $\mathcal{M}$ for which $\rho\left(\left|f\right|\right) < +\infty$ is called a Banach function space. For each $f \in X$, define $\left\|f\right\|_X = \rho\left(\left|f\right|\right)$.*

It is valid the following

Theorem 2.3. *Let ρ be a function norm and let $X = X(\rho)$ and $\|\cdot\|_X$ be as above. Then under the natural vector space operations, $(X; \|\cdot\|_X)$ is a normed linear space for which the inclusions*

$$\mathcal{M}_s \subset X \subset \mathcal{M}_0,$$

hold, where $\mathcal{M}_s$ is the set of μ-simple functions. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure, and hence some subsequence converges point wise μ-a.e. to f.

A space X equipped with the norm $\left\|f\right\|_X = \rho\left(\left|f\right|\right)$ is called a Banach function space.

In what follows we assume $R = \gamma$ and μ is linear Lebesgue measure on γ. We also identify the circle γ and the segment $(-\pi, \pi]$ by the mapping $e^{it} : (-\pi, \pi] \to \gamma$. Let

$$\rho'(g) = \sup\left\{\int_\gamma f(\tau)\, g(\tau)\, |dt| : f \in \mathcal{M}^+; \rho(f) \le 1\right\}, \forall g \in \mathcal{M}^+.$$

A space

$$X' = \left\{g \in \mathcal{M} : \rho'\left(\left|g\right|\right) < +\infty\right\},$$

is called an associate space (Kothe dual) of X.

Let $w : [-\pi, \pi] \to R_+$ ($R_+ = (0, +\infty)$) be some weight function. We define the weight space $X_w = \{f \in M_0 : fw \in X\}$, with the norm

$$\left\|f\right\|_{X_w} = \left\|fw\right\|_X, \forall f \in X_w.$$

The functions $f; g \in \mathcal{M}_0$ are called equimeasurable if

$$\left|\left\{\tau \in \gamma : \left|f(\tau)\right| > \lambda\right\}\right| = \left|\left\{\tau \in \gamma : \left|g(\tau)\right| > \lambda\right\}\right|, \forall \lambda \ge 0.$$

Banach function norm $\rho : \mathcal{M}^+ \to [0, \infty]$ is called rearrangement invariant if for arbitrary equimeasurable functions $f; g \in \mathcal{M}_0^+$ the relation $\rho(f) = \rho(g)$ holds. In this case, Banach function space X with the norm $\|\cdot\|_X = \rho(|\cdot|)$ is said to be rearrangement invariant function space (r.i.s. for short). Classical Lebesgue, Orlicz, Lorentz, Lorentcz–Orlicz spaces are r.i.s..

To obtain our main results, we will significantly use the following result of [41] (see also [25]). Let α_X and β_X be upper and lower Boyd indices for the space X (for Boyd indices the reader is referred to, e.g., [24–29]).

Theorem 2.4. *For every p and q such that*

$$1 \le q < \frac{1}{\beta_X} \le \frac{1}{\alpha_X} < p \le \infty,$$

we have

$$L_p \subset X \subset L_q,$$

with the inclusion maps being continuous.

In establishing the basicity of parts of the system of exponents in Banach Hardy classes we will use some results related to Fourier series in r.i.s.. Let's state some relevant concepts and notations.

Definition 2.5. *Let X be a Banach function space. The closure in X of the set of simple functions $\mathcal{M}_s$ is denoted by X_b.*

Recall the definition of resonant space.

Definition 2.6. *Suppose $f(\cdot)$ belongs to $\mathcal{M}_0$. The decreasing rearrangement of $f(\cdot)$ is the function f^* defined on $[0,\infty)$ by*

$$f^*(t) = \inf\left\{\lambda : \mu_f(\lambda) \le t\right\}, \ t \ge 0,$$

where $\mu_f(\lambda) = \mu\left\{t : \left|f(t)\right| > \lambda\right\}, \ \lambda \ge 0$, is a distribution function of $f(\cdot)$.

It is valid the following well known

Theorem 2.7. *(Hardy, Littlewood). If $f(\cdot)$ and $g(\cdot)$ belong to $\mathcal{M}_0$, then*

$$\int_R \left|f\, g\right| d\mu \le \int_0^\infty f^*(s)\, g^*(s)\, ds. \tag{3}$$

An immediate consequence of the Hardy-Littlewood inequality (3) *is that*

$$\int_R \left|f\, \tilde{g}\right| d\mu \le \int_0^\infty f^*(t)\, g^*(t)\, dt, \tag{4}$$

for every function $\tilde{g}$ on R equimeasurable with g.

Definition 2.8. *If the supremum on $\tilde{g}$ of the integrals on the left of* (4) *coincide with the value on the right, such measure spaces is called resonant. If the supremum is in fact attained, then the measure space will be called strongly resonant.*

In what follows, we assume that all the functions under consideration are defined on the interval $(-\pi, \pi]$, periodically continued on R with a period of 2π and the interval $(-\pi, \pi]$ will be identified with γ.

We denote by T_s the translation operator $(T_s f)(t) = f\left(e^{i(s+t)}\right), \ -\pi < s; t \le \pi$, and by $\omega_X(f, \cdot)$ the X -modulus of continuity of f:

$$\omega_X(f;\delta) = \sup_{|s|\le\delta} \left\|T_s f - f\right\|_X, \ 0 \le \delta \le \pi.$$

Definition 2.9. *Let X be a rearrangement-invariant Banach space (r.i.s.) over a resonant space $(R;\mu)$. For each finite value of t belonging to the range of μ, let E be a subset of R with $\mu(E) = t$ and let*

$$\varphi_X(t) = \|\chi_E\|_X.$$

The function φ_X is called the fundamental function of X.

If f belongs to $L_1(\gamma)$, then for each integer n the n-th Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{i\theta}\right) e^{-in\theta} d\theta, \ n \in Z.$$

So called the "multiplier" operator m is defined initially on trigonometric polynomials

$$P\left(e^{i\theta}\right) = \sum_{n=-r}^{r} a_n e^{in\theta} \text{ by } mP\left(e^{i\theta}\right) = \sum_{n=-r}^{r} -i\,sign\, n\, a_n e^{in\theta}.$$

It is evidently that

$$\left(\overset{\wedge}{mP}\right)(n) = \begin{cases} -i\,sign\,n\,a_n,\ \forall n = \overline{-r,r}, \\ 0,\quad n \neq \overline{-r,r}\,, \end{cases}$$

for arbitrary trigonometric polynomial $P\left(e^{i\theta}\right) = \sum_{n=-r}^{r} a_n e^{in\theta}$.

Let S_n 's be partial sums of the Fourier series of the function f :

$$S_n(f) = \sum_{|k|\leq n} \hat{f}(k)\, e^{ikt}.$$

In the sequel we also need the following

Theorem 2.10. *Suppose X is a r.i.s. on γ whose fundamental function satisfies $\varphi_X(+0) = 0$. Then the following conditions are equivalent:*

1. *Fourier series converge in norm in X_b;*
2. *the partial-sum operators S_n are uniformly bounded on X_b;*
3. *the multiplier operator m is bounded on X_b ;*
4. *the conjugate-function operator is bounded on X_b;*
5. *the Calderon operator*

$$S f^*(t) = \int_0^1 f^*(s) \min\left(1; \frac{s}{t}\right) \frac{ds}{s},$$

is bounded on $(X_b)^-$ – the Luxemburg representation of X_b on the interval $[0, 1]$.

More detail on the Luxemburg representation one can see the monograph [24].

The conjugate-function operator $\tilde{f}$ is defined by

$$\tilde{f}\left(e^{i\theta}\right) = \frac{1}{2\pi} \lim_{\varepsilon\to+0} \int_{\varepsilon<|s|\leq\pi} f\left(e^{i(\theta-s)}\right) \cot\frac{s}{2} ds, \forall\theta : -\pi < \theta \leq \pi.$$

If any one of these conditions holds, then $mf = \tilde{f}$ a.e. for $\forall f \in X_b$.

Corollary 2.11. *Let X be a separable r.i.s. on* $[-\pi, \pi]$. *Fourier series converge in norm in X if and only if the Boyd indices of X satisfy* $0 < \alpha_X; \beta_X < 1$.

We will need also the following lemma from the work [24].

Lemma 2.12. *[24] Let $X = X(\rho)$ be a Banach function space and suppose $f_n \in X$, $n \in N$.*

i) If $0 \leq f_n \uparrow f$ μ-a.e., then either $f \notin X$ and $\left\|f_n\right\|_X \uparrow +\infty$, or $f \in X$ and $\left\|f_n\right\|_X \uparrow \left\|f\right\|_X$.

ii) (Fatous lemma) If $f_n \to f$ μ-a.e., and if $\lim_{n\to\infty} \inf \left\|f_n\right\|_X < +\infty$, then $f \in X$ and $\left\|f\right\|_X \leq \lim_{n\to\infty} \inf \left\|f_n\right\|_X$.

In establishing the direct decomposition of the space X into Banach Hardy classes, the following easily proved lemma plays a key role.

Lemma 2.13. *Let the Banach space $\left(Y_1; \|\cdot\|_{Y_1}\right)$ be continuously embedded in the Banach space $\left(Y_2; \|\cdot\|_{Y_2}\right)$. Let $T \in [Y_2; Y_1]$ and $\overline{ImT} = Y_1$ (closure of the image of T). If the set $M \subset Y_2$ is everywhere dense in Y_2, i.e. $\bar{M} = Y_2$, then $\overline{TM} = Y_1$.*

In fact, let $y_1 \in Y_1$ be an arbitrary element and $\varepsilon > 0$ be an arbitrary number. It is clear that $\exists z_1 \in I_m T$:

$$\left\|z_1 - y_1\right\|_{Y_1} < \varepsilon.$$

Consequently, $\exists x_2 \in Y_2 : Tx_2 = z_1$. From $\overline{M} = Y_2$ it follows that $\exists m_2 \in M : \|m_2 - x_2\|_{Y_2} < \varepsilon$. We have

$$\left\|Tm_2 - y_1\right\|_{Y_1} \leq \|Tm_2 - z_1\|_{Y_1} + \varepsilon = \|Tm_2 - Tx_2\|_{Y_1} + \varepsilon \leq$$

$$\leq \|T\| \, \|m_2 - x_2\|_{Y_2} + \varepsilon = (1 + \|T\|)\, \varepsilon.$$

From arbitrariness of ε it directly follows that $\overline{TM} = Y_1$.

More details concerning the above results can be found in [24, 25, 43] and [26–29, 40–42].

In what follows, we will also use the concept of Nevanlinna class of analytic functions in ω. By $\mathcal{N}$ we denote the set of analytic functions $F(\cdot)$ in ω such that

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^+ \left|F\left(re^{it}\right)\right| dt < +\infty,$$

where

$$\log^+ u = \log \max \{1; u\}, u \geq 0.$$

It is known (see, e.g., [7, 44]) that the non-zero function $F(\cdot)$ belongs to the class $\mathcal{N}$ if and only if it can be represented as

$$F(z) = B(z) \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} dh(t)\right), \tag{5}$$

where $B(\cdot)$ is a Blaschke function, and $h(\cdot)$ is a function of bounded variation on $[-\pi, \pi]$. By $\mathcal{N}'$ (Nevanlinna class) we denote a class of functions $F \in \mathcal{N}$ such that the function $h(\cdot)$ in (5) is absolutely continuous on $[-\pi, \pi]$.

We need the following characteristic of the space X:

$$\gamma_X = \inf\{\alpha : |t|^\alpha \in X\}. \tag{6}$$

It is easy to see that $\gamma_X \geq -1$. This directly follows from the embedding $X \subset L_1$, since, for $\alpha \leq -1$, $|t|^\alpha \notin L_1$. It is clear that also $\gamma_X \leq 0$, since, for $\alpha \geq 0$, $|t|^\alpha$, is bounded and therefore belongs to X. Let us show that for $\forall \alpha > \gamma_X, |t|^\alpha \in X$. It suffices to show that if $|t|^{\alpha_1} \in X, \gamma_X \leq \alpha_1 < 0$, then for $\forall \alpha_2 \in (\alpha_1, 0)$ $|t|^{\alpha_2} \in X$ is true. And this, in turn, follows from the estimate

$$|t|^{\alpha_2} = |t|^{\alpha_2 - \alpha_1} |t|^{\alpha_1} \leq C |t|^{\alpha_1},$$

where $C > 0$ is some constant, as $\alpha_2 - \alpha_1 > 0$. So, the following lemma is true.

Lemma 2.14. *Let X be Banach function space and the quantity γ_X is defined by* (6). *Then $\gamma_X \in [-1, 0]$ and $|t|^\alpha \in X, \forall \alpha > \gamma_X$.*

Using this lemma, the following lemma is easily proved.

Lemma 2.15. *Let X be a Banach function space and γ_X is defined by* (6). *Then the finite product*

$$\mu(t) = \prod_{k=0}^{m} |t - t_k|^{\alpha_k},$$

belongs to X if $\alpha_k > \gamma_X$, $\forall k = \overline{1, m}$, where $-\pi \leq t_0 < t_1 < ... < t_m < \pi$ are some points.

In particular, it is true

Corollary 2.16. *Let X be a Banach space and γ_X is defined by* (6)*. Then the finite production*

$$\mu(t)=\prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\alpha_k}\ ,\ \ t\in(-\pi,\pi)\ ,$$

belongs to X , if $\alpha_k>\gamma_X$, $\forall k=\overline{1,r}$; where $-\pi\le s_0<s_1<...<s_m<\pi$ are some points.

Define the class of Muckenhoupt weights A_p. Let $p\in(1,\infty)$ and $\frac{1}{p}+\frac{1}{p'}=1$. We will say that the weight $\nu:\gamma\to R_+$ belongs to the class A_p if $\|\nu\chi_I\|_p\left\|\nu^{-1}\chi_I\right\|_{p'}\le C\,|I|$, where $|I|$ is a linear measure $I\subset\gamma$, and $C>0$ is a constant independent of I.

Consider the following singular integral with the Cauchy kernel

$$(Sf)(\tau)=\frac{1}{2\pi i}\int_\gamma\frac{f(\xi)\,d\xi}{\xi-\tau}\ ,\ \tau\in\gamma,$$

where $f\in X$ is some function. Denote by A_X a class of weights such that the singular operator S is bounded in the weighted space X_w, i.e.

$$A_X=\{w(\cdot):S\in[X_w]\}\,.$$

We need the following result of D.Boyd [41].

Theorem 2.17. *Suppose that $T\in\left[L_p\right]$ and $T\in\left[L_q\right]$, with $1<p<q<+\infty$. Let X be a r.i.s. with Boyd indices α_X and β_X which satisfy $\frac{1}{q}<\alpha_X\le\beta_X<\frac{1}{p}$. Then $T\in[X]$.*

This theorem immediately implies the following

Corollary 2.18. *Let X be a r.i.s. over $[-\pi,\pi]$ with Boyd indices $\alpha_X;\beta_X\in(0,1)$. Then singular operator S acts boundedly in X, i.e. $S\in[X]$.*

Let X be a r.i.s. with Boyd indices α_X and β_X. Let $\alpha_X;\beta_X\in(0,1)$. Then $\exists p;q\in(1,+\infty)$

$$1<q<\frac{1}{\beta_X}\le\frac{1}{\alpha_X}<p<+\infty\,. \tag{7}$$

Then, by Theorem 2.4 (see, e.g., [25, p. 134]), the following continuous inclusions hold

$$L_p\subset X\subset L_q\,.$$

Combined with the arbitrariness of the numbers p and q satisfying (7), the last statement implies $\gamma_X\in[-\beta_X,-\alpha_X]$. So the following lemma is true.

Lemma 2.19. *Let X be a r.i.s. with Boyd indices $\alpha_X;\beta_X\in(0,1)$. Then $\gamma_X\in[-\beta_X,-\alpha_X]$. In particular, if $\alpha_X=\beta_X$, then $\gamma_X=-\alpha_X$.*

Let's find out under which conditions $|t|^\alpha\in X_b$. Let $\alpha>-\alpha_X$ be an arbitrary number. For $\alpha\ge0$ it is clear that $|t|^\alpha\in X_b$. Therefore, it suffices to consider the case $\alpha<0$. Then $\exists p,q:$

$$-\frac{1}{q}<-\beta_X\le-\alpha_X<-\frac{1}{p}<\alpha<0\Leftrightarrow q<\frac{1}{\beta_X}\le\frac{1}{\alpha_X}<p<-\frac{1}{\alpha}\,.$$

Obviously, $|t|^\alpha\in L_p(-\pi,\pi)$. Then it follows from the continuous embedding $L_p\subset X$ that $|t|^\alpha$ has an absolutely continuous norm $\|\cdot\|_X$ and, consequently, $|t|^\alpha\in X_b$. So, it is valid

Lemma 2.20. *Let X be a r.i.s. with Boyd indices $\alpha_X;\beta_X\in(0,1)$. Then for $\forall\alpha>-\alpha_X:|t|^\alpha\in X_b$.*

We also need the following lemma from the monograph [24] (see page 10).

Lemma 2.21. *In order that a measurable function g belong to the associate space X', it is necessary and sufficient that $f(\cdot)\,g(\cdot)$ be integrable for every f in X.*

3. Banach Hardy classes and some facts about them

Let X be a Banach function space over $[-\pi, \pi]$. By H_X^+ we denote a Hardy class of functions $F(\cdot)$ analytic inside ω equipped with the norm

$$\|F\|_{H_X^{\pm}} = \overline{\lim_{r \to 1-0}} \|F_r(\cdot)\|_X, \quad \text{where } F_r(t) = F\left(re^{it}\right).$$

We also define its subclass

$$H_{X_b}^+ \equiv \left\{F \in H_X^+ : F^+ \in X_b\right\},$$

where $F^+(\cdot)$ are the non-tangential boundary values of F on γ.

Similar to classical case, we define the Banach Hardy class ${}_mH_X^-$ of analytic functions outside the unit circle which have a finite order at infinity. Let the function $f(\cdot)$, analytic outside ω, have a Laurent decomposition of the form

$$f(z) = \sum_{n=-\infty}^{m} a_n z^n, \ z \to \infty, \ a_m \neq 0,$$

in the vicinity of the infinitely remote point. So, for $m > 0$ the point $z = \infty$ is a pole of order m, and for $m \leq 0$ the point $z = \infty$ is a zero of order $(-m)$. Let $f(\cdot) = f_0(\cdot) + f_1(\cdot)$, where $f_0(\cdot)$ is the principal part, and $f_1(\cdot)$ is the regular part of Laurent decomposition in the vicinity of $z = \infty$. If the function $g(z) = \overline{f_0\left(\frac{1}{\bar{z}}\right)}$, $|z| < 1$, belongs to the class H_X^+, then we will say that the function $f(\cdot)$ belongs to the class ${}_mH_X^-$.

It is valid the following

Theorem 3.1. *Let X be a r.i.s. on γ. Then the system of exponent $\left\{e^{int}\right\}_{n\in Z}$ forms a basis for X_b if and only if the Boyd indices of X satisfy $0 < \alpha_X; \beta_X < 1$.*

Using this theorem, the validity of the following theorem is easily established.

Theorem 3.2. *Let X be a r.i.s. on γ with Boyd indices $\alpha_X;\ \beta_X \in (0,1)$. Let $X_b^+ = \left(\frac{1}{2}I + S\right)X_b$ and ${}_{-1}X_b^- = \left(\frac{1}{2}I - S\right)X_b$, where $S-$ is the singular integral Cauchy. Then the system $\left\{e^{int}\right\}_{n\in z_+}$ $\left(\left\{e^{-int}\right\}_{n\in N}\right)$ forms a basis for X_b^+ $\left({}_{-1}X_b^-\right)$.*

Theorem below has been proved in [49].

Theorem 3.3. *Let X be a r.i.s. with Boyd indices $\alpha_X;\ \beta_X \in (0,1)$. Then the following assertions are valid:*

i) Analytic function $F \in \mathcal{N}$ in ω belongs to the class $H_{X_b}^+$ $\left(H_X^+\right)$ if and only if its boundary values F^+ belong to X_b (X) and the Cauchy formula

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{F^+(\tau)}{\tau - z} d\tau, \ z \in \omega,$$

holds;

ii) The spaces $H_{X_b}^+$ and X_b^+; ${}_{-1}H_{X_b}^-$ and ${}_{-1}X_b^-$ (H_X^+ and X^+; ${}_{-1}H_X^-$ and ${}_{-1}X^-$) can be equated to each other from an isometric point of view: $\|F^+\|_{H_X^+} = \|F^+\|_X$; $\|F\|_{{}_{-1}H_X^-} = \|F^-\|_X$ and direct decompositions

$$X_b = H_{X_b}^+ \dot{+} {}_{-1}H_{X_b}^-; X = H_X^+ \dot{+} {}_{-1}H_X^-,$$

hold;

iii) The system $\{z^n\}_{n\in z_+}$ $(\{z^{-n}\}_{n\in N})$ forms a basis for $H_{X_b}^+$ (for ${}_{-1}H_{X_b}^-$).

The following analog of classical Riesz theorem is also true.

Theorem 3.4. *Let X be a r.i.s. with Boyd indices α_X; $\beta_X \in (0,1)$. Then the following assertions are valid:*
i) If $F \in H_X^+$, then $\lim_{r\to 1-0} \|F_r\|_X = \|F^+\|_X$;
ii) The relation $\lim_{r\to 1-0} \|F_r - F^+\| = 0$ is true if and only if $F \in H_{X_b}^+$.

Similar results are true for the classes ${}_mH_X^-$.

Theorem 3.5. *Let X be a r.i.s. with Boyd indices α_X; $\beta_X \in (0,1)$. Then the following assertions are valid:*
i) If $F \in_m H_X^-$, then $\lim_{r\to 1+0} \|F_r\|_X = \|F^-\|_X$;
ii) The relation $\lim_{r\to 1+0} \|F_r - F^-\| = 0$ is true if and only if $F \in_m H_{X_b}^-$.
iii) Analytic function $F(\cdot)$ outside ω belongs to the class ${}_{-1}H_X^-({}_{-1}H_{X_b}^-)$ if and only if its boundary values $F^-(\cdot)$ belong to $X\,(X_b)$ and the Cauchy formula

$$F(z) = -\frac{1}{2\pi i}\int_\gamma \frac{F^-(\tau)}{\tau - z}d\tau\,,\ |z| > 1,$$

holds.

In the end of this section we will give the following proposition, which we will use in obtaining the main results.

Proposition 3.6. *If $f(\cdot)$ is a real function with $\|f\|_\infty < +\infty$, then Cauchy type integral ($z = \rho e^{i\sigma}$)*

$$\Phi(z) = \exp\left(\pm\frac{i}{2\pi}\int_{-\pi}^{\pi} f(s)\,\frac{e^{is} + \rho e^{i\sigma}}{e^{is} - \rho e^{i\sigma}}ds\right),$$

belongs to the Hardy class H_δ^+ for sufficiently small $\delta > 0$.

About this proposition one can see the monograph [43].

4. General solution of homogeneous Riemann problem

Consider homogeneous Riemann problem

$$F^+(\tau) - G(\tau)F^-(\tau) = 0, \tau \in \gamma, \quad F^+(\cdot) \in H_X^+; F^-(\cdot) \in {}_mH_X^-, \tag{8}$$

with complex-valued coefficient $G\left(e^{it}\right) = \left|G\left(e^{it}\right)\right| e^{i\theta(t)}, t \in [-\pi,\pi]$. By the solution of the problem (8) we mean a pair of analytic functions $(F^+;F^-) \in H_X^+ \times {}_mH_X^-$, whose non-tangential boundary values satisfy the equation (8) a.e. on γ. We assume that the coefficient $G(\cdot)$ satisfies the following conditions:
i) $G^{\pm 1}(\cdot) \in L_\infty(-\pi,\pi)$;
ii) $\theta(t) = \arg G\left(e^{it}\right)$ is a piecewise Hölder function on $[-\pi,\pi]$ with the jumps $h_k = \theta(s_k+0) - \theta(s_k-0)$, $k = \overline{1,r}$, at the points of discontinuity $\{s_k\}_1^r : -\pi < s_1 < \ldots < s_r < \pi$.
To solve the problem (8), we will follow [39]. Let

$$H(s;z) = \frac{e^{is} + z}{e^{is} - z}$$

be a Schwarz kernel. Consider the following analytic functions in $C\backslash\gamma$:

$$Z_1(z) \equiv \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left|G\left(e^{it}\right)\right| H(t;z)\,dt\right\},$$

$$Z_2(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta(t)\,H(t;z)\,dt, z \notin \gamma\right\}.$$

It is absolutely clear that the function $Z_2(\cdot)$ depends on the choice of the argument $\theta(\cdot)$. Let

$$Z_\theta(z) = Z_1(z) Z_2(z), z \notin \gamma.$$

Integral of the form

$$\Phi(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) H(s; z) \, ds, z \notin \gamma, \tag{9}$$

is called a Schwarz integral with the density $f \in L_1(-\pi, \pi)$. The following Sokhotski-Plemelj formulas are true for Schwarz integral

$$\Phi^{\pm}\left(e^{i\sigma}\right) \equiv \pm f(\sigma) + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) H\left(s; e^{i\sigma}\right) ds,$$

where $\Phi^+(\cdot)(\Phi^-(\cdot))$ are non-tangential boundary values of the function $\Phi(\cdot)$ inside (outside) ω on γ. From these formulas it immediately follows that

$$\left|G\left(e^{it}\right)\right| = \frac{Z_1^+\left(e^{it}\right)}{Z_1^-\left(e^{it}\right)}, e^{i\theta(t)} = \frac{Z_2^+\left(e^{it}\right)}{Z_2^-\left(e^{it}\right)}, \text{ a.e. } t \in [-\pi, \pi].$$

Consequently

$$Z_\theta^+(\tau) - G(\tau) Z_\theta^-(\tau) = 0, \text{ a.e. } \tau \in \gamma. \tag{10}$$

$Z_\theta(\cdot)$ *will be called a canonical solution of homogeneous problem* (8), *corresponding to the argument* $\theta(\cdot)$.

Considering (10) in (8), we have

$$\frac{F^+(\tau)}{Z_\theta^+(\tau)} = \frac{F^-(\tau)}{Z_\theta^-(\tau)}, \text{ a.e. } \tau \in \gamma.$$

Introduce the following piecewise analytic function

$$\Phi(z) = \frac{F(z)}{Z_\theta(z)}, z \notin \gamma.$$

We have

$$\Phi^+(\tau) = \Phi^-(\tau), \text{ a.e. } \tau \in \gamma.$$

Let's show that the function $\Phi(\cdot)$ satisfies all conditions of the uniqueness theorem. It is absolutely clear that the function $Z_\theta(\cdot)$ has no zeros and poles when $z \notin \gamma$. Therefore, the functions $\Phi(\cdot)$ and $F(\cdot)$ have the same order at infinity. Let's find the conditions which guarantee that the piecewise analytic function $\Phi(\cdot) = (\Phi^+(\cdot); \Phi^-(\cdot))$ belongs to the class $H_1^+ \times {}_mH_1^-$. The conditions *i*), *ii*) and the results of [8] imply the existence of sufficiently small number $\delta > 0$ such that the function $\Phi^+(z)$ belongs to the space H_δ^+(also, $\Phi^-(z)$ belongs to ${}_{\tilde{m}}H_\delta^-$ for some $\tilde{m} \in Z_+$). In fact, the assertion that the function $X_2^{\pm 1}(z)$ belongs to H_σ^+ for sufficiently small $\delta > 0$ follows from Proposition 3.6. As for the function $X_1^{\pm 1}(z)$, we apply Jensen's integral inequality of the form

$$\exp\left\{\frac{1}{\int_a^b |p(s)| \, ds} \int_a^b |p(s) f(s)| \, ds\right\} \leq \frac{1}{\int_a^b |p(s)| \, ds} \int_a^b |p(s)| \exp|f(s)| \, ds,$$

and obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X_1^{\pm 1}\left(\rho e^{i\sigma}\right)\right|^p d\sigma = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{\exp \frac{\pm p}{4\pi} \int_{-\pi}^{\pi} \ln\left|G\left(e^{is}\right)\right| P_\rho(\sigma - s) \, ds\right\} d\sigma \leq$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|G\left(e^{is}\right)\right|^{\pm\frac{p}{2}}P_{\rho}\left(\sigma-s\right)ds\right\}d\sigma \leq \|G\|_{\infty}^{\pm\frac{p}{2}},$$

where $P_r(\cdot)$ is a Poisson kernel. It follows that if the condition *i*) holds, then the function $X_1^{\pm}(z)$ belongs to all classes H_p^+, $\forall p > 0$. Applying Hölder's inequality, we conclude that the function $X^{\pm 1}(z)$ belongs to the Hardy class H_δ^+ for sufficiently small $\delta > 0$. From the representation $\Phi(z) = F(z)[X(z)]^{-1}$ it follows that the same is true also about the function $\Phi(z)$.

Let's find out under which conditions the function $\Phi(\cdot)$ belongs to the class H_1^+. To do so, it suffices to find out if the boundary values $\Phi^+(\tau)$ belong to $L_1(-\pi,\pi)$ (the rest will follow from the Smirnov theorem).

Let us represent the function $\theta(\cdot)$ in the form

$$\theta(t) = \theta_0(t) + \theta_1(t),$$

where $\theta_0(\cdot)$ −its continuous (Hölderian) part, and $\theta_1(\cdot)$ is a jump function, which is determined by the expression

$$\theta_1(-\pi) = 0,\ \theta_1(s) = \sum_{k:-\pi<s_k<s} h_k,\ \forall s \in (-\pi,\pi].$$

Let

$$h_0 = \theta(-\pi) - \theta(\pi),\ h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi).$$

Assume

$$u_0(t) = \left|\sin\frac{t+\pi}{2}\right|^{-\frac{h_0^{(0)}}{2\pi}} \exp\left(-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta_0(\tau)\, ctg\frac{t-\tau}{2}dt\right).$$

We also use the notation

$$u(t) = \prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}},$$

where $s_0 = -\pi$. Applying the Sokhotsky-Plemelj formulas to $Z_1(z)$ we have

$$Z_1^{\pm}\left(e^{i\sigma}\right) = \exp\left\{\pm\frac{1}{2}\ln\left|G\left(e^{i\sigma}\right)\right| + \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|G\left(e^{is}\right)\right|\frac{e^{is}+e^{i\sigma}}{e^{is}-e^{i\sigma}}ds\right\}.$$

Hence it follows directly that the following relation holds

$$\sup_{(-\pi,\pi)} vrai\left\{\left|Z_1^-\left(e^{it}\right)\right|^{\pm 1}\right\} < +\infty.$$

According to the results of the monograph [43], the boundary values of $\left|Z_2^-(\tau)\right|$ are expressed by the formula

$$\left|Z_2^-\left(e^{it}\right)\right| = u_0(t)\,u^{-1}(t) = u_0(t)\prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}.$$

Therefore, for the boundary values of the canonical solution $Z_\theta(\cdot)$we obtain

$$\left|Z_\theta{}^-\left(e^{it}\right)\right| = \left|Z_1^-\left(e^{it}\right)\right|\,|u_0(t)|\prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}.$$

Taking into account the expression

$$\frac{e^{is}+e^{i\sigma}}{e^{is}-e^{i\sigma}}=\frac{e^{i\frac{s-\sigma}{2}}+e^{i\frac{\sigma-s}{2}}}{e^{i\frac{s-\sigma}{2}}-e^{i\frac{\sigma-s}{2}}}=\frac{\cos\frac{s-\sigma}{2}}{i\sin\frac{s-\sigma}{2}}=ictg\frac{\sigma-s}{2},$$

for $Z_1^{\pm}(\cdot)$ we have

$$Z_1^{\pm}\left(e^{i\sigma}\right)=\exp\left\{\pm\frac{1}{2}\ln\left|G\left(e^{i\sigma}\right)\right|+\frac{i}{4\pi}\int_{-\pi}^{\pi}\ln\left|G\left(e^{is}\right)\right|ctg\frac{\sigma-s}{2}ds\right\}.$$

Consequently

$$\left|Z_1^{\pm}\left(e^{i\sigma}\right)\right|=\left|G\left(e^{i\sigma}\right)\right|^{\pm\frac{1}{2}},$$

and, as a result

$$\left|Z_\theta{}^{-}\left(e^{it}\right)\right|=\left|G\left(e^{it}\right)\right|^{-\frac{1}{2}}|u_0(t)|\prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}. \tag{11}$$

It is clear that $\theta_0(\cdot)$ is Holder function on $[-\pi,\pi]$. Then again, as follows from the results of the monograph [43], the relation

$$\sup_{[-\pi,\pi]} vrai\, |u_0(t)|^{\pm 1}<+\infty,$$

holds. We have

$$\Phi^{-}\left(e^{it}\right)=F^{-}\left(e^{it}\right)\left[Z_\theta^{-}\left(e^{it}\right)\right]^{-1}. \tag{12}$$

So, by definition of solution $F^{-}(\cdot)\in X$ is true. Paying attention to Lemma 2.21 and representation (12), we get that for the validity of $\Phi^{-}(\cdot)\in L_1(-\pi,\pi)$, it is sufficient that the inclusion $\left|Z_\theta^{-}(\cdot)\right|^{-1}\in X'$ is true. As follows from the representation (11), for this it suffices to show that $u(\cdot)\in X'$. We will suppose that X is a r.i.s. with Boyd indices $\alpha_X;\beta_X\in(0,1)$. The Boyd indices of space X' denote by $\alpha_{X'}$ and $\beta_{X'}$ (upper and lower, respectively). The following relations between the Boyd indices of the spaces X and X' are well known (see, e.g. [24, p.149]. Proposition 5.13).

$$\alpha_{X'}=1-\beta_X;\beta_{X'}=1-\alpha_X.$$

Also put

$$\gamma_{X'}=\inf\left\{\alpha:|t|^{\alpha}\in X'\right\}.$$

By Lemma 2.19

$$-\beta_{X'}\leq\gamma_{X'}\leq-\alpha_{X'}\Leftrightarrow -1+\alpha_X\leq\gamma_{X'}\leq-1+\beta_{X'},$$

holds. Applying Corollary 2.16 to this case, we obtain that if the inequalities

$$\frac{h_k}{2\pi}>\gamma_{X'},k=\overline{0,r};$$

are fulfilled, then $u(\cdot)\in X'\Leftrightarrow\left|Z_\theta^{-}(\cdot)\right|^{-1}\in X'$, and, as a result, $\Phi^{-}(\cdot)\in L_1(-\pi,\pi)$. Then it follows from Smirnov's theorem that $\Phi\in H_1^{+}$. From the same considerations, we obtain $\Phi\in{}_mH_1^{-}$. Since, $\Phi^{+}(\tau)=\Phi^{-}(\tau)$, a.e. $\tau \in \gamma$, then from the uniqueness theorem it follows that $\Phi(\cdot)$ is a polynomial $P_k(\cdot)$ of degree $k \le m$ (for $m < 0$ assume $P_k(z) \equiv 0$). As a result, for the function $F(\cdot)$ we have a presentation

$$F(z) \equiv Z_\theta(z) P_k(z), k \le m. \tag{13}$$

Let's find out under which conditions the function (13) belongs to the Hardy classes $H_X^+ \times_m H_X^-$. It is clear that for sufficiently small $\delta > 0$ the inclusion $\Phi(\cdot) \in H_\delta^+$ is true (this follows directly from the representation (13) and from the known facts regarding $Z_\theta(\cdot)$). Similar considerations hold for the outside ω. If $F^+ \in X$, then it is clear that $F^+ \in L_1$ and, as a result, $F \in H_1^+$ and the Cauchy formula is valid for it. Then by Theorem 3.3 we get that $F \in H_X^+$. Thus, it suffices to prove that $F^+ \in X$, and for this it is enough to prove that $F^- \in X$. It is clear that if $Z_\theta^- \in X$, then $F^- \in X$. Applying again Corollary 2.16 to the representation (11) we obtain that if the inequalities

$$-\frac{h_k}{2\pi} > \gamma_X, k = \overline{0,r};$$

are fulfilled, then $Z_\theta^- \in X \Rightarrow F^- \in X$, and, as a result, it is clear that $(F^+; F^-) \in H_X^+ \times {}_m H_X^-$. So, the following theorem is proved.

Theorem 4.1. *Let X be a r.i.s. with Boyd indices $0 < \alpha_X \le \beta_X < 1$. Suppose that the coefficient $G(\cdot)$ of problem (8) satisfies the conditions i), ii) and $Z_\theta(\cdot)$ is a canonical solution corresponding to argument $\theta(\cdot)$. Let jumps $\{h_k\}_0^r$ of the function $\theta(\cdot)$ ($h_0 = \theta(-\pi) - \theta(\pi)$), satisfy the inequalities*

$$\gamma_{X'} < \frac{h_k}{2\pi} < -\gamma_X, k = \overline{0,r}; \tag{14}$$

Then :

α) for $m \ge 0$ the homogeneous problem (8) has a general solution of the form

$$F(z) \equiv Z_\theta(z) P_k(z), \tag{15}$$

in the Hardy classes $H_X^+ \times {}_m H_X^-$, where $P_k(z)$ is an arbitrary polynomial of degree $k \le m$;

β) for $m < 0$ this problem has only a trivial solution, i.e. zero solution in the Hardy classes $H_X^+ \times {}_m H_X^-$.

This theorem directly implies

Corollary 4.2. *Let all the conditions of Theorem 4.1 be satisfied. Then under the condition $F(\infty) = 0$ the homogeneous problem (8) has only a trivial solution in the Hardy classes $H_X^+ \times {}_m H_X^-$.*

Consider the most general case. Define the argument $\theta(\cdot)$ as follows

$$\tilde{\theta}(t) \equiv \begin{cases} \theta(t), -\pi < t < s_1, \\ \theta(t) - 2\pi n_1, s_1 < t < s_2, \\ \vdots \\ \theta(t) - 2\pi n_r, s_r < t < \pi, \end{cases} \tag{16}$$

where $\{n_k\}_1^r \subset Z$ is some integer. Assume

$$\tilde{G}(t) \equiv |G(t)| e^{i\tilde{\theta}(t)}, t \in (-\pi, \pi).$$

It is obvious that $G(t) \equiv \tilde{G}(t)$. Therefore, in (8) instead of the coefficient $G(\cdot)$ we can take $\tilde{G}(\cdot)$. Denoting the jumps of the function $\tilde{\theta}(\cdot)$ at the points $\{s_k\}_1^r$, by $\left\{\tilde{h}_k\right\}_1^r$ we have

$$\tilde{h}_1 = h_1 - 2\pi n_1; \tilde{h}_k = h_k - 2\pi(n_k - n_{k-1}), k = \overline{2,r}; \tilde{h}_0 = h_0 + 2\pi n_r.$$

Applying Theorem 4.1 to the problem (8) with the coefficient $\tilde{G}(\cdot)$, we get the following theorem.

Theorem 4.3. *Let X be a r.i.s. with Boyd indices $\alpha_X; \beta_X \in (0,1)$, the coefficient $G(\cdot)$ of the problem (8) satisfy the conditions i),ii) and $\exists \{n_k\}_1^r \subset Z$ such that*

$$\gamma_{X'} < \frac{h_1}{2\pi} - n_1 < -\gamma_X; \gamma_{X'} < \frac{h_k}{2\pi} - n_k + n_{k-1} < -\gamma_X; k = \overline{2,r};$$
$$\gamma_{X'} < \frac{h_0}{2\pi} + 2\pi n_r < -\gamma_X . \tag{17}$$

Then:

α) for $m \geq 0$ the problem (8) has a general solution of the form

$$F(z) = Z_{\tilde{\theta}}(z) P_k(z),$$

in the Hardy classes $H_X^+ \times {}_m H_X^-$, where $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq m$, and $Z_{\tilde{\theta}}(\cdot)$ is a canonical solution corresponding to the argument $\tilde{\theta}(\cdot)$;

β) for $m < 0$ this problem has only a trivial solution.

Let all the conditions of Theorem 4.3 be satisfied. It is clear that for $\forall \alpha > \gamma_X : |t|^\alpha \in X$ and for $\forall \alpha' > \gamma_{X'} : |t|^{\alpha'} \in X'$. Consequently, from Corollary 4.2 it follows that the integral $\int_{-\pi}^{\pi} |t|^{\alpha+\alpha'} dt$ exists, and hence $\alpha + \alpha' > -1$. It immediately follows that $\gamma_{X'} + \gamma_X \geq -1$. As a result, we get the validity of

Lemma 4.4. *Let X be a r.i.s. with Boyd indices $\alpha_X; \beta_X \in (0,1)$. Then it is valid*

$$\alpha_X + \alpha_{X'} \leq -\gamma_X - \gamma_{X'} \leq 1.$$

Remark 4.5. *From Lemma 4.4 it immediately follows that the integers $\{n_k\}_1^r$ in (17) (if they exist) are defined uniquely.*

Consider the special case when the Boyd indices α_X and β_X coincide: $\alpha_X = \beta_X$. Then from the relation $\alpha_X + \beta_{X'} = 1$ follows $\alpha_X + \alpha_{X'} = 1$ and it is clear that $\gamma_X = -\alpha_X$. In this case, the restrictions on the coefficient $G(\cdot)$ can be weakened, namely, let $\left\{\frac{h_k}{2\pi} - \alpha_X\right\}_0^r \bigcap Z = \emptyset$. Let us define integers $\{n_k\}_1^r$ from the following relations

$$\left.\begin{array}{l} -\alpha_{X'} < \frac{h_1}{2\pi} - n_1 < \alpha_X; \\ -\alpha_{X'} < \frac{h_k}{2\pi} - n_k + n_{k-1} < \alpha_X; k = \overline{2,r}; \end{array}\right\} \tag{18}$$

Let the function $\tilde{\theta}(\cdot)$ be defined by the relations (16). We have

$$\tilde{h}_0 = \tilde{\theta}(-\pi) - \tilde{\theta}(\pi) = \theta(-\pi) - \theta(\pi) + 2\pi n_r \Rightarrow \frac{\tilde{h}_0}{2\pi} = \frac{h_0}{2\pi} + n_r.$$

Suppose $æ_0 = \left[\frac{h_0}{2\pi}\right] + n_r([\cdot]$ is an integer part). Let us reformulate problem (8) in the following form

$$\begin{array}{l} F^+(\tau) - G(\tau) F^-(\tau) = 0 \Leftrightarrow \\ F^+(\tau) - \tilde{G}(\tau) \tau^{æ_0} F_1^-(\tau) = 0, \tau \in \gamma, \end{array} \tag{19}$$

where

$$F_1(z) = \begin{cases} z^{-æ_0} F(z), |z| > 1, \\ 0, |z| < 1. \end{cases}$$

Let $G_{æ_0}(\tau) = G(\tau)\tau^{æ_0} = \tilde{G}(\tau)\tau^{æ_0}, \tau \in \gamma$. It is clear that $\left|G_{æ_0}(\tau)\right| = |G(\cdot)|$ and $\theta_{æ_0}(t) = \arg G_{æ_0}\left(e^{it}\right) = \tilde{\theta}(t) + æ_0 t, t \in (-\pi, \pi)$. It is obvious that the discontinuity points and jumps of the function $\theta_{æ_0}(\cdot)$ on the interval $(-\pi, \pi)$ coincide with the discontinuity points and jumps of the function $\tilde{\theta}(\cdot)$. Moreover

$$\begin{array}{l} \theta_{æ_0}(-\pi) - \theta_{æ_0}(\pi) = \tilde{\theta}(-\pi) - \tilde{\theta}(\pi) - 2\pi æ_0 \Rightarrow \\ \Rightarrow \left[\frac{\theta_{æ_0}(-\pi) - \theta_{æ_0}(\pi)}{2\pi}\right] = \left[\frac{\tilde{\theta}(-\pi) - \tilde{\theta}(\pi)}{2\pi}\right] - æ_0 = 0 . \end{array}$$

It is obvious that $F \in {}_mH_X^- \Leftrightarrow F_1 \in {}_{m-æ_0}H_X^-$. Consequently, problem (8) is solvable in Hardy classes $H_X^+ \times {}_mH_X^-$ if and only if problem (19) is solvable in $H_X^+ \times {}_{m-æ_0}H_X^-$. Obviously, with respect to the coefficient $G_{æ_0}(\cdot)$ all the conditions of Theorem 4.1 are satisfied. We have

$$\begin{aligned}&n_1 = \left[\frac{h_1}{2\pi} + \alpha_{X'}\right],\ n_2 = n_1 + \left[\frac{h_2}{2\pi} + \alpha_{X'}\right] = \\ &= \left[\frac{h_1}{2\pi} + \alpha_{X'}\right] + \left[\frac{h_2}{2\pi} + \alpha_{X'}\right], \ldots,\quad n_r = \sum_{k=1}^{r}\left[\frac{h_k}{2\pi} + \alpha_{X'}\right] \Rightarrow \\ &æ_0 = \sum_{k=0}^{r}\left[\frac{h_k}{2\pi} + \alpha_{X'}\right].\end{aligned}$$

Applying Theorem 4.1 to problem (19) we obtain the following final result.

Theorem 4.6. *Let X be a r.i.s. with Boyd indices $\alpha_X = \beta_X \in (0,1)$; the coefficient $G(\cdot)$ satisfy the conditions i), ii) and $\left\{\frac{h_k}{2\pi} - \alpha_X\right\}_0^r \bigcap Z = \emptyset$. Suppose $æ = m - \sum_{k=0}^{r}\left[\frac{h_k}{2\pi} + \alpha_{X'}\right]$.*

Then:

α) for $æ \geq 0$ the homogeneous Riemann problem (8) has a general solution of the form

$$F(z) = \begin{cases} Z_{\theta_æ}(z) P_k(z), |z| < 1, \\ z^{æ-m} Z_{\theta_æ}(z) P_k(z), |z| > 1, \end{cases}$$

in classes $H_X^+ \times {}_mH_X^-$, where $Z_{\theta_æ}(\cdot)$ is a canonical solution corresponding to the argument $\theta_æ(\cdot)$, $P_k(\cdot)$is an arbitrary polynomial of degree $k \leq æ$;

β) for $æ < 0$ this problem has only a trivial solution in classes $H_X^+ \times {}_mH_X^-$.

Quantity $æ = m - \sum_{k=0}^{r}\left[\frac{h_k}{2\pi} + \alpha_{X'}\right] = m - r - 1 - \sum_{k=0}^{r}\left[\frac{h_k}{2\pi} - \alpha_X\right]$ is called an index of the problem (8) in classes $H_X^+ \times {}_mH_X^-$.

5. Nonhomogeneous Riemann problem

Consider the following nonhomogeneous Riemann problem

$$\left.\begin{array}{l} F^+(\tau) - G(\tau) F^-(\tau) = f(\tau),\ \text{a.e.}\quad \tau \in \gamma, \\ (F^+; F^-) \in H_X^+ \times {}_mH_X^-, \end{array}\right\}, \tag{20}$$

where $f(\cdot) \in X$ is some function. We will assume that the coefficient $G(\cdot)$ satisfies the conditions i), ii), and, as before, we will denote by $Z_\theta(\cdot)$ a canonical solution corresponding to the argument $\theta(\cdot)$. We first will seek for a partial solution of the problem (20), and then we will construct the general one.

5.1. *Partial solution of the problem* (20).

Consider the following Cauchy type integral

$$F_1(z) = \frac{Z_\theta(z)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t) e^{it}}{Z_\theta^+(e^{it})} \frac{dt}{e^{it} - z}. \tag{21}$$

Denote by $(Kf)(\cdot)$ a singular integral with a Cauchy kernel

$$(Kf)(\tau) = \frac{Z_\theta^+(\tau)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t) e^{it}}{Z_\theta^+(e^{it})} \frac{dt}{e^{it} - \tau}, \tau \in \gamma.$$

Applying Sokhotski-Plemelj formula to (21), we obtain

$$F_1^{\pm}(\tau) = Z_\theta^{\pm}(\tau) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} \frac{e^{it} dt}{e^{it} - z}\right]_\gamma^{\pm} =$$

$$= Z_\theta^{\pm}(\tau)\left(\pm\tfrac{1}{2}\left[Z_\theta^{+}(\tau)\right]^{-1} f(\tau) - \left[Z_\theta^{+}(\tau)\right]^{-1}(Kf)(\tau)\right), \text{a.e. } \tau\in\gamma,$$

where $[\,\cdot\,]_\gamma^{\pm}$ denotes the boundary values on γ from inside (with "+") and outside (with "-"), respectively. It immediately follows

$$\frac{F_1^{+}(\tau)}{Z_\theta^{+}(\tau)} - \frac{F_1^{-}(\tau)}{Z_\theta^{-}(\tau)} = \frac{f(\tau)}{Z_\theta^{+}(\tau)}, \quad \text{a.e. } \tau\in\gamma. \tag{22}$$

As the canonical solution $Z_\theta(\tau)$ satisfies

$$Z_\theta^{+}(\tau) - G(\tau) Z_\theta^{-}(\tau) = 0, \quad \text{a.e. } \tau\in\gamma,$$

from (22) it follows that

$$F_1^{+}(\tau) - G(\tau) F_1^{-}(\tau) = f(\tau), \quad \text{a.e. } \tau\in\gamma.$$

Thus, $F_1(\cdot)$ satisfies (20). Let's find out under which conditions the function $F_1(\cdot)$ belongs to the Hardy classes $H_X^{+}\times {}_mH_X^{-}$. So, let's assume that the coefficient $G(\cdot)$ satisfies the conditions i),ii). In terms of the notations of previous section, we have

$$\left|Z_\theta^{-}\left(e^{it}\right)\right| = \left|Z_1^{-}\left(e^{it}\right)\right| u_0(t)\prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}.$$

Consequently

$$\left|Z_\theta^{+}\left(e^{it}\right)\right|^{-1} = \left|G\left(e^{it}\right)\right|^{-1}\left|Z_\theta^{-}\left(e^{it}\right)\right|^{-1} \sim \prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}, t\in(-\pi,\pi). \tag{23}$$

By the conditions of the problem, we have $f\in X$. Therefore it is clear that if $\left|Z_\theta^{+}(\cdot)\right|^{-1}\in X'$, then $f(\cdot)\left[Z_\theta^{+}(\cdot)\right]^{-1}\in L_1(-\pi,\pi)$. Applying Corollary 2.16 to (23), we conclude that if

$$\frac{h_k}{2\pi} > \gamma_{X'}, k=\overline{0,r}, \tag{24}$$

then $\left|Z_\theta^{+}(\cdot)\right|^{-1}\in X'$, and, as a result, we have $f(\cdot)\left|Z_\theta^{+}(\cdot)\right|^{-1}\in L_1(-\pi,\pi)$. Then, by the well-known Smirnov theorem (see, e.g. [7, 43]), Cauchy type integral

$$F_2(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{Z_\theta^{+}(e^{it})}\frac{e^{it}dt}{e^{it}-z}, z\in\omega,$$

belongs to the Hardy class $H_\delta^{+}, \forall\delta\in(0,1)$. As already established, $Z_\theta(\cdot)$ belongs to the Hardy class H_δ^{+} for sufficiently small $\delta>0$. Applying Hölder's inequality, we find that the product $F_1(\cdot) = Z_\theta(\cdot)F_2(\cdot)$ belongs to the Hardy class H_δ^{+} for sufficiently small $\delta>0$. Consequently, $F_1(\cdot)\in\mathcal{N}'$ (Nevanlinna class). Let's show that $F_1^{+}(\cdot)\in X$. From Smirnov theorem we have $F_1\in H_1^{+}$(because $X\subset L_1$), and hence, by Theorem 3.4, Cauchy formula holds for F_1, which implies $F_1\in H_X^{+}$. We have

$$F_1^{+}(\tau) = \frac{1}{2}f(\tau) - (Kf)(\tau), \tau\in\gamma. \tag{25}$$

The last relation implies that it only remains to prove the validity of the inclusion $(Kf)(\cdot)\in X$. Let

$$g(\tau) = f(\tau)\left(Z_\theta(\tau)\right)^{-1}, \tau\in\gamma,$$

and consider the following singular operator

$$(Sg)(\tau) = \frac{1}{2\pi i}\int_{\gamma} \frac{g(\xi)\,d\xi}{\xi - \tau}, \tau \in \gamma.$$

Consider the following weight function

$$\rho_0(t) = \left|t^2 - \pi^2\right|^{-\frac{h_0}{2\pi}} \prod_{k=1}^{r} |t - s_k|^{-\frac{h_k}{2\pi}}, t \in (-\pi, \pi). \tag{26}$$

It is not difficult to see that

$$\left|Z_\theta\left(e^{it}\right)\right| \sim \rho_0(t), t \in (-\pi, \pi).$$

Consequently, $\|g\|_{X_{\rho_0}} \sim \|f\|_X$. From these relations it immediately follows that the operator K acts boundedly in X if and only if the operator S acts boundedly in the weighted space X_{ρ_0} :

$$(Kf)(\cdot) \in X \Leftrightarrow (Sg)(\cdot) \in X_{\rho_0}.$$

Let $\rho_0 \in A_X \Rightarrow (Kf)(\cdot) \in X$. Then, from previous considerations we obtain $F_1(\cdot) \in H_X^+$. It is not difficult to see that there exists a finite limit $\lim_{z\to\infty} Z_\theta(z) \neq 0$. Then from the expression for $F_1(\cdot)$ we immediately obtain $\lim_{z\to\infty} F_1(z) = 0$. Based on this relation, it can be similarly proved that $F_1(\cdot) \in {}_{-1}H_X^-$. Thus, the following theorem is true.

Theorem 5.1. *Let X be a r.i.s. with Boyd indices α_X and β_X, and the coefficient $G(\cdot)$ satisfy the conditions i), ii). Let the following relations hold:*

$$\frac{h_k}{2\pi} > \gamma_{X'}, k = \overline{0, r};\ \rho_0(\cdot) \in A_X,$$

where the weight $\rho_0(\cdot)$ is defined by (26). *Then the function $F_1(\cdot)$ defined by* (21) *is a solution of the nonhomogeneous problem* (20) *in the Hardy classes $H_X^+ \times {}_{-1}H_X^-$.*

Let us find concrete conditions for the validity of the inclusion $\rho_0(\cdot) \in A_X$. Assume that $\alpha_X; \beta_X \in (0,1)$ and $\rho_0(\cdot) \in A_{\frac{1}{\alpha_X}} \bigcap A_{\frac{1}{\beta_X}}$. It follows that $\rho_0(\cdot) \in A_X$, i.e. the operator S is bounded in X (see. e.g. [40, 41]). It is known that

$$|t|^\alpha \in A_p \Leftrightarrow -\frac{1}{p} < \alpha < -\frac{1}{p} + 1, \quad 1 < p < +\infty.$$

Consequently

$$\rho_0(\cdot) \in A_{\frac{1}{\alpha_X}} \Leftrightarrow -\alpha_X < -\tfrac{h_k}{2\pi} < -\alpha_X + 1, k = \overline{0, r};$$
$$\rho_0(\cdot) \in A_{\frac{1}{\beta_X}} \Leftrightarrow -\beta_X < -\tfrac{h_k}{2\pi} < -\beta_X + 1, k = \overline{0, r}.$$

Taking into account the relation $\alpha_X \leq \beta_X$ we obtain

$$\rho_0 \in A_{\frac{1}{\alpha_X}} \bigcap A_{\frac{1}{\beta_X}} \Leftrightarrow \beta_X - 1 < \frac{h_k}{2\pi} < \alpha_X, k = \overline{0, r}.$$

Taking into account these relations from Theorem 5.1 we get the following

Corollary 5.2. *Let X be a r.i.s. with continuous norm and with Boyd indices $\alpha_X; \beta_X \in (0,1)$. Let the coefficient $G(\cdot)$ satisfy the conditions i), ii) and the following relations hold*

$$\max\{-\alpha_{X'}; \beta_X - 1\} < \frac{h_k}{2\pi} < \alpha_X, k = \overline{0,r}.$$

Then the function $F_1(\cdot)$, which is defined by (21)*, be the solution of nonhomogeneous problem* (20) *in the classes Hardy $H_X^+ \times {}_{-1}H_X^-$.*

In addition if $\alpha_X = \beta_X \in (0,1)$, then it is evident that $\alpha_{X'} = \beta_{X'} = 1 - \alpha_X \Rightarrow -\alpha_{X'} = \alpha_X - 1 = \beta_X - 1$. In the result we obtain the following

Corollary 5.3. *Let all conditions of Corollary 5.2 are satisfied and $\alpha_X = \beta_X$. If the following inequalities hold*

$$\alpha_X - 1 < \frac{h_k}{2\pi} < \alpha_X, k = \overline{0,r},$$

then the function (21) *is the solution of the problem* (20) *in the Hardy classes $H_X^+ \times {}_{-1}H_X^-$.*

5.2. General solution of nonhomogeneous problem.

Let's find a general solution of the nonhomogeneous problem (20) in the Hardy classes $H_X^+ \times {}_mH_X^-$. It is absolutely clear that the general solution of the problem (20) can be expressed in the form $F(\cdot) = F_0(\cdot) + F_1(\cdot)$, where $F_0(\cdot)$ is a general solution of corresponding homogeneous problem (8), and $F_1(\cdot)$ is some partial solution of the homogeneous problem (20). We first consider the case $m \geq -1$. In this case it is not difficult to see that the function $F_1(\cdot)$ defined by (21) belongs to the Hardy classes $H_X^+ \times {}_mH_X^-$ if $\rho_0(\cdot) \in A_X$ and the inequalities (24) hold. As already established, the general solution of the homogeneous problem in this case has the form $F_0(\cdot) = Z_\theta(\cdot) P_k(\cdot)$ if all the conditions of Theorem 4.1 hold, where $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq m$ (for $m = -1$ we assume $P_k(\cdot) \equiv 0$). Consider the case $m < -1$. In this case, with all the conditions of Theorem 4.1 fulfilled, the homogeneous problem (8) has only a trivial solution in the classes $H_X^+ \times {}_mH_X^-$. Let's show that if the problem (20) has a solution $\Phi_1(\cdot)$, then it coincides with$F_1(\cdot)$, i.e. $\Phi_1(\cdot) \equiv F_1(\cdot)$. But this follows directly from the trivial solvability of the homogeneous problem, because the function $F_0(\cdot) = \Phi_1(\cdot) - F_1(\cdot)$ is a solution of the homogeneous problem in the Hardy classes $H_X^+ \times {}_mH_X^-$. Let's show that the function $F_1(\cdot)$ belongs to the classes $H_X^+ \times {}_mH_X^-$. It is clear that $F_1(\cdot) \in H_X^+$. Moreover, $F_1^-(\cdot) \in X$. Then it is evident that $F_1(\cdot) \in {}_mH_X^+$ if and only if $F_1(\cdot)$ has a Laurent decomposition of the form

$$F_1(z) = \sum_{k=-\infty}^{m} a_k z^k, z \to \infty,$$

in the vicinity of the infinitely remote point. As $\exists \lim_{z\to\infty} \left|Z_\theta^-(z)\right|^{\pm 1} \neq 0$, from $F_1(\cdot) = Z_\theta(\cdot) K(\cdot)$ it immediately follows that the Cauchy type integral

$$K(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} K(z,t)\, dt,$$

has a decomposition of the form

$$K(z) = \sum_{k=-\infty}^{m} b_k z^k, z \to \infty, \tag{27}$$

as $z \to \infty$. We have

$$K(z) = -\frac{z^{-1}}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} \frac{e^{it} dt}{1 - e^{it} z^{-1}} = -\sum_{k=-\infty}^{-1} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} e^{-ikt} dt\, z^k. \tag{28}$$

Comparing the decompositions (27) and (28), we see that the function $F_1(\cdot)$ belongs to the class ${}_mH_X^-$ if and only if the orthogonality conditions

$$\int_{-\pi}^{\pi}\frac{f(t)}{Z_\theta^+(e^{it})}e^{ikt}dt=0, k=\overline{1,-m-1}, \tag{29}$$

hold.

So, the following theorem is true

Theorem 5.4. *Let X be a r.i.s. with Boyd indices $\alpha_X;\beta_X\in(0,1)$. Let the coefficient $G(\cdot)$ satisfies the conditions i), ii) and $Z_\theta(\cdot)$ is the canonical solution corresponding to the argument $\theta(\cdot)$. Suppose that the jumps $\{h_k\}_0^r$ of the argument $\theta(\cdot)$ satisfy the following inequalities*

$$\gamma_{X'}<\frac{h_k}{2\pi}<-\gamma_X, k=\overline{0,r},$$

and the weight function $\rho_0(\cdot)$ belongs to the class A_X. Then:

α) for $m\geq-1$ the non-homogeneous problem (20) *for $\forall f\in X$ has a general solution of the form*

$$F(z)\equiv Z_\theta(z)P_k(z)+\frac{Z_\theta(z)}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{Z_\theta^+(e^{it})}\frac{e^{it}dt}{e^{it}-z}, \tag{30}$$

in classes $H_X^+\times{}_mH_X^-$, where $P_k(\cdot)$ is an arbitrary polynomial of degree $k\leq m$ (for $m=-1$ we assume $P_k(\cdot)\equiv0$);

β) for $m<-1$ this problem is solvable in classes $H_X^+\times{}_mH_X^-$ if and only if right-hand side $f(\cdot)$ satisfies the orthogonality conditions (29) *and in this case the unique solution $F_1(\cdot)$ is representable in the form of the integral* (21).

This theorem immediately implies the following

Corollary 5.5. *Let all the conditions of Theorem 5.4 be satisfied. Then the nonhomogeneous problem* (20) *has a unique solution of the form* (21) *in classes $H_X^+\times{}_{-1}H_X^-$ for $\forall f\in X$.*

Consider the case where the Boyd indices of the space X coincide with each other. Let $\left\{\frac{h_k}{2\pi}-\alpha_X\right\}_0^r\bigcap Z=\emptyset$. Set $æ_0=\left[\frac{1}{2\pi}\sum_{k=0}^r h_k-\alpha_X\right]+1$. Absolutely similar to the homogeneous case, we establish that the nonhomogeneous problem (20) is solvable in the classes $H_X^+\times{}_mH_X^-$ with the right-hand side $f(\cdot)$ if and only if the nonhomogeneous problem

$$F^+(\tau)-G_{æ_0}(\tau)F^-(\tau)=f(\tau), \tau\in\gamma,$$

is solvable in the classes $H_X^+\times{}_{m-æ_0}H_X^-$, where the coefficient $G_{æ_0}(\cdot)$ is defined by the formula

$$G_{æ_0}(\tau)=G(\tau)\tau^{æ_0},\ \tau\in\gamma.$$

Taking this into account, we obtain the following result.

Theorem 5.6. *Let X be a r.i.s. with Boyd indices $\alpha_X=\beta_X\in(0,1)$, the coefficient $G(\cdot)$ satisfy the conditions i), ii) and $\left\{\frac{h_k}{2\pi}-\alpha_X\right\}_0^r\bigcap Z=\emptyset$ holds. Let*

$$æ=m-r-1-\sum_{k=0}^r\left[\frac{h_k}{2\pi}-\alpha_X\right].$$

Then:

α) for $æ\geq-1$ nonhomogeneous problem with arbitrary right-hand side $f\in X$ has a general solution of the form (30) *in classes $H_X^+\times{}_mH_X^-$, where $P_k(\cdot)$ is an arbitrary polynomial of degree $k\leq æ$ (for $æ=-1$ assume $P_k(\cdot)\equiv0$);*

β) for $æ < -1$ this problem is solvable in the classes $H_X^+ \times {}_mH_X^-$ if and only if the the right-hand side $f(\cdot)$ satisfies the orthogonality conditions

$$\int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} e^{ikt} dt = 0, k = \overline{1, -æ - 1}, \tag{31}$$

and unique solution is representable in the form of the integral (21)*:*

$$F(z) = \frac{Z_\theta(z)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z_\theta^+(e^{it})} \frac{e^{it} dt}{e^{it} - z}, z \notin \gamma. \tag{32}$$

From this theorem in particular we obtain the following

Corollary 5.7. *Let all the conditions of Theorem 5.6 be satisfied. If the index of the problem* (20) *is equal to (-1), i.e.* $æ = -1$*, then for* $\forall f \in X$ *this problem has a unique solution in classes* $H_X^+ \times {}_mH_X^-$ *representable as an integral* (32).

It should be noted that the Lebesgue, Orlicz, Lorentz, Lorentz-Orlicz, Morrey, grand-Lebesgue and other spaces are the examples of r.i.s. Consequently, the results of this work are applicable to any one of these spaces. Note, however, that in every single case the Boyd indices of these spaces must be calculated. Let us consider some special cases.

5.3. Lebesgue case.

Consider a particular case $X = L_p(-\pi, \pi), 1 < p < +\infty$. In this case we have $\alpha_X = \beta_X = \frac{1}{p}$. Then following Theorem 5.6 we have

Corollary 5.8. *Let the coefficient $G(\cdot)$ satisfy the conditions i), ii) and $\left\{\frac{h_k}{2\pi} - \frac{1}{p}\right\}_0^r \cap Z = \emptyset$ holds. Assume*

$$æ = m - r - 1 - \sum_{k=0}^{r} \left[\frac{h_k}{2\pi} - \frac{1}{p}\right].$$

Then:

α) for $æ \geq -1$ the nonhomogeneous problem (20) *for $\forall f \in L_p(-\pi, \pi), 1 < p < +\infty$, has a general solution of the form* (30)*, in the Hardy classes $H_p^+ \times {}_mH_p^-$, where $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq æ$ (for $æ = -1$ assume $P_k(\cdot) \equiv 0$);*

β) for $æ < -1$ this problem is solvable in the classes $H_p^+ \times {}_mH_p^-$ if and only if the orthogonality conditions (31) *hold.*

The results of this corollary are well-known, and the corresponding theory was developed by I.I.Daniliuk [43].

5.4. Symmetric Morrey space.

Consider the following Morrey type space (we will call this space as symmetric Morrey space) $SL^{p,\alpha}(-\pi, \pi)$ of Lebesgue measurable functions on $(-\pi, \pi)$ with norm $\|\cdot\|_{p,\alpha}$:

$$\|f\|_{p,\alpha} = \sup_{I \subset (-\pi,\pi)} \left(\frac{1}{|I|^{1-\alpha}} \int_I |f(t)|^p dt\right)^{\frac{1}{p}},$$

where $I \subset (-\pi, \pi)$ is an arbitrary Lebesgue measurable set, $|I|$ is Lebesgue measure of I and $\alpha \in (0, 1)$ is some parameter. This is not a separable space. In this case, the Boyd indices are equal $\alpha_X = \beta_X = \frac{\alpha}{p}$. Hardy Banach classes corresponding to Morrey space $SL^{p,\alpha}(-\pi, \pi)$, denote by $H_{p,\alpha}^+$ and ${}_mH_{p,\alpha}^-$. Again following Theorem 5.6 we obtain the following

Corollary 5.9. *Let the coefficient $G(\cdot)$ satisfy the conditions i), ii) and $\left\{\frac{h_k}{2\pi}-\frac{\alpha}{p}\right\}_0^r \cap Z=\emptyset$ holds. Denote*

$$æ = m - r - 1 - \sum_{k=0}^{r}\left[\frac{h_k}{2\pi}-\frac{\alpha}{p}\right],\ 1<p<+\infty,\ 0<\alpha<1.$$

Then:

α) for $æ \geq -1$ the nonhomogeneous problem (20) for $\forall f \in SL^{p,\alpha}(-\pi,\pi)$, has a general solution of the form (30), in the Hardy-Morrey classes $H^+_{p,\alpha}\times {}_mH^-_{p,\alpha}$, where $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq æ$(for $æ=-1$ assume $P_k(\cdot)\equiv 0$);

β) for $æ < -1$ this problem is solvable in the classes $H^+_{p,\alpha}\times {}_mH^-_{p,\alpha}$ if and only if the orthogonality conditions (31) hold.

It should be noted that the same result was obtained in the work [10] with respect to Morrey space $L^{p,\alpha}(-\pi,\pi)$.

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