Filomat 35:10 (2021), 3293–3302
https://doi.org/10.2298/FIL2110293D

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Spectral Mapping Theorem and Weyl's Theorem for (m,n)-Paranormal Operators

Preeti Dharmarha[a], Sonu Ram[a]

[a]*Department of Mathematics, University of Delhi, Delhi-110007, India*

Abstract. In the present paper, we prove spectral mapping theorem for (m,n)-paranormal operator T on a separable Hilbert space, that is, $f(\sigma_w(T)) = \sigma_w(f(T))$ when f is an analytic function on some open neighborhood of σ(T). We also show that for (m,n)-paranormal operator T, Weyl's theorem holds, that is, $\sigma(T) - \sigma_w(T) = \pi_{00}(T)$. Moreover, if T is algebraically (m,n)-paranormal, then spectral mapping theorem and Weyl's theorem hold.

1. Introduction

Let $\mathcal{H}$ denote an infinite dimensional separable complex Hilbert space with inner product $\langle , \rangle$ and $B(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. Throughout the paper, we denote the set of all complex numbers by $\mathbb{C}$. For $T \in B(\mathcal{H})$, we write $ker(T)$ and $ran(T)$ for the null space and the range space of T, respectively, $\sigma_a(T)$ for the approximate point spectrum of T, $\sigma(T)$ for the spectrum of T, $\sigma_p(T)$ for the point spectrum of T, $\sigma_w(T)$ for the Weyl spectrum of T and $\pi_{00}(T)$ denotes the set of all isolated points of spectrum of T, which are eigen values of finite multiplicity of T [5, 12]. An operator $T \in B(\mathcal{H})$ is called Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension. The index of a Fredholm operator T is given by $ind(T) = \alpha(T) - \beta(T)$, where $\alpha(T)$ is the dimension of $ker(T)$ and $\beta(T)$ is the dimension of $ker(T^*)$. Also, let $iso\sigma(T)$ be the set of isolated points of spectrum of T and an operator T is said to be isoloid if every $\lambda \in iso\sigma(T)$ is an eigen value.

An operator $T \in B(\mathcal{H})$ is called Weyl if it is Fredholm of index zero. The spectrum $\sigma(T)$, the point spectrum $\sigma_p(T)$, the Weyl spectrum $\sigma_w(T)$ and the isolated eigen values of finite multiplicity $\pi_{00}(T)$ of $T \in B(\mathcal{H})$ are defined by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\},$$

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : Tx_n = \lambda x_n \text{ for some sequence } (x_n) \text{ of unit vectors}\},$$

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ for some non-zero vector } x\},$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

$$\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

2020 *Mathematics Subject Classification*. Primary 47A10, 47B20
Keywords. (m,n)-paranormal operator, Riesz idempotent, Weyl spectrum, Weyl's theorem, Spectral mapping theorem
Received: 13 February 2020; Accepted: 20 September 2021
Communicated by Dragan S. Djordjević
Email addresses: drpreetidharmarha@hrc.du.ac.in (Preeti Dharmarha), ram.sonu02@gmail.com (Sonu Ram)

An operator $T \in B(\mathcal{H})$ is said to be (m,n)-paranormal if $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$, for all $x \in \mathcal{H}$, where m is a positive real number and n is a positive integer [6].

Let $p(T)$ and $q(T)$ denote, respectively, the ascent and descent of an operator T. We say that T has finite ascent and $p(T) = m_1$, if there exists a least non-negative integer m_1 such that $ker(T^{m_1}) = ker(T^{m_1+1})$. Also, if such integer does not exist, then $p(T) = \infty$. Analogous, we say that T has finite descent and $q(T) = m_2$, if there exits a least non-negative integer m_2 such that $ran(T^{m_2}) = ran(T^{m_2+1})$. Also, if such integer does not exist, then $q(T) = \infty$. If $p(T)$ and $q(T)$ are finite, then $p(T) = q(T)$ [13, Proposition 38.3]. Moreover, for a complex number λ, $0 < p(T-\lambda) = q(T-\lambda) < \infty$ if and only if λ is pole of resolvent.

The contents of this paper have been organized into four sections. In sect. 2, we discuss some spectral properties of (m,n)-paranormal operators and prove spectral mapping theorem on Weyl spectrum, that is, $f(\sigma_w(T)) = \sigma_w(f(T))$ for all $f \in \mathcal{H}(\sigma(T))$, where $\mathcal{H}(\sigma(T))$ is the set of all analytic functions on some open neighborhood of spectrum of T.

In sect. 3, we prove that if T is a (m,n)-paranormal operator and $\sigma(T) = \{\lambda\}$, then $T = \lambda$. Further, for the same class of operators and every Riesz idempotent E_λ, we prove that $ran(E_\lambda) = ker(T-\lambda)$, where λ is the isolated point of spectrum of T. Hence, Weyl's theorem holds.

In sect. 4, we prove that every part of algebraically (m,n)-paranormal is algebraically (m,n)-paranormal and algebraically (m,n)-paranormal operator satisfies property (H). Further, spectral mapping theorem and Weyl's theorem hold for algebraically (m,n)-paranormal operators.

2. Spectral Mapping Theorem for (m,n)-Paranormal Operators

In this section, we study a matrix representation of a (m,n)-paranormal operator with respect to the direct sum of an eigen-space and its orthogonal complement. By some authors, it was shown that the spectral mapping theorem on Weyl spectrum holds for classes of operators [12, 14, 20]. To start with we establish a few important results for (m,n)-paranormal operators.

Theorem 2.1. *Let $T \in B(\mathcal{H})$ be a (m,n)-paranormal operator on $\mathcal{H} = ker(T-\lambda) \oplus ker(T-\lambda)^\perp$ defined by 2×2 matrix representation $T = \begin{bmatrix} \lambda & T_1 \\ 0 & T_2 \end{bmatrix}$, where $0 \neq \lambda \in \sigma_p(T)$. Then*

$$\|T_2^{n+1}x\|^{\frac{2}{n+1}}\|x\|^{\frac{2n}{n+1}} \geq \|T_1x\|^2 + \|T_2x\|^2 \tag{1}$$

for $x \in ker(T-\lambda)^\perp$, provided $T_1\sum_{i=0}^{n}\lambda^{n-i}T_2^i = 0$ for each $a > 0$.

This result is a generalization of [21, Theorem 2.1].

Proof. Since T is (m,n)-paranormal, so

$$m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)a^nT^*T + m^{\frac{2}{n+1}}na^{n+1}I = \begin{bmatrix} Q(a) & R(a) \\ R^*(a) & S(a) \end{bmatrix}$$

is positive for each $a > 0$, where

$$Q(a) = m^{\frac{2}{n+1}}\bar{\lambda}^{n+1}\lambda^{n+1} - (n+1)a^n\bar{\lambda}\lambda + m^{\frac{2}{n+1}}na^{n+1}I,$$

$$R(a) = m^{\frac{2}{n+1}}\bar{\lambda}^{n+1}P - (n+1)a^n\bar{\lambda}T_1,$$

$$R^*(a) = m^{\frac{2}{n+1}}\lambda^{n+1}P^* - (n+1)a^n\lambda T_1^*,$$

and

$$S(a) = m^{\frac{2}{n+1}}(P^*P + T_2^{*n+1}T_2^{n+1}) - (n+1)a^n(T_1^*T_1 + T_2^*T_2) + m^{\frac{2}{n+1}}na^{n+1}I,$$

where

$$P = T_1 \sum_{i=0}^{n} \lambda^{n-i} T_2^i.$$

By (m,n)-paranormality of T, $S(a) \geq 0$ for each $a > 0$, thereby implying

$$m^{\frac{2}{n+1}} (P^*P + T_2^{*n+1} T_2^{n+1}) - (n+1)a^n (T_1^* T_1 + T_2^* T_2) + m^{\frac{2}{n+1}} n a^{n+1} I \geq 0.$$

As $P = 0$,

$$m^{\frac{2}{n+1}} T_2^{*n+1} T_2^{n+1} - (n+1)a^n (T_1^* T_1 + T_2^* T_2) + m^{\frac{2}{n+1}} n a^{n+1} I \geq 0, \tag{2}$$

for each $a > 0$. Now, by [6, Theorem 2.1], the result holds. □

Remark 2.2. *It is easy to conclude from* (2)*,* T_2 *is* (m,n)*-paranormal.*

Corollary 2.3. *Let* $T = \begin{bmatrix} 0 & T_1 \\ 0 & T_2 \end{bmatrix}$ *be a* (m,n)*-paranormal operator on* $\mathcal{H} = ker(T) \oplus ker(T)^{\perp}$*. Then* $ker(T_2) = \{0\}$*.*

Proof. First, assume that $x = 0 \oplus x_2 \in ker(T_2)$. Then $T_2 x_2 = 0$ and $Tx = T_1 x_2$. Consequently, $Tx \in ker(T)$, this implies that $x \in ker(T^{1+n}) = ker(T)$ by (m,n)-paranormality of T. So, $x \in ker(T) \cap ker(T)^{\perp} = 0$. Hence, $ker(T_2) = \{0\}$. □

Corollary 2.4. *If* $T = \begin{bmatrix} \lambda & T_1 \\ 0 & T_2 \end{bmatrix}$ *is a* (m,n)*-paranormal operator on* $\mathcal{H} = ker(T-\lambda) \oplus ker(T-\lambda)^{\perp}$ *with* $T_1 \sum_{i=0}^{n} \lambda^{n-i} T_2^i = 0$*, then* $ker(T_2 - \lambda) = \{0\}$*, where* $\lambda \neq 0$*.*

Proof. Let $x = 0 \oplus x_2 \in ker(T_2 - \lambda)$. Then $T_2 x_2 = \lambda x_2$ and $(T-\lambda)x = T_1 x_2$. By (1), we have $(T-\lambda)x = 0$. Therefore, $x \in ker(T-\lambda)$. So, $x \in ker(T-\lambda) \cap ker(T-\lambda)^{\perp} = \{0\}$. Hence, $ker(T_2 - \lambda) = \{0\}$. □

Corollary 2.5. *Let* $T = \begin{bmatrix} \lambda & T_1 \\ 0 & T_2 \end{bmatrix}$ *be* (m,n)*-paranormal on* $\mathcal{H} = ker(T-\lambda) \oplus ker(T-\lambda)^{\perp}$ *with* $T_1 \sum_{i=0}^{n} \lambda^{n-i} T_2^i = 0$*, where* $\lambda \neq 0$*. Then* $ker(T-\lambda) \perp ker(T-\mu)$*, where* λ *and* μ *are two distinct eigen values of* T*.*

Proof. Let $x = x_1 \oplus x_2 \in ker(T-\mu)$. Then $0 = (T-\mu)x = (T-\mu)(x_1 \oplus x_2) = [(\lambda-\mu)x_1 + T_1 x_2] \oplus (T_2 - \mu)x_2$. Thus, $(T_2 - \mu)x_2 = 0$. Now, by (1), we have $\|T_1 x_2\| = 0$ and $(\lambda - \mu)x_1 = 0$. So, $x_1 = 0$. Hence, $x \in ker(T-\lambda)^{\perp}$ and thus $ker(T-\lambda) \perp ker(T-\mu)$. □

As a consequence of the above corollary, we obtain the following proposition.

Proposition 2.6. *If an operator* $T = \begin{bmatrix} \lambda & 0 \\ 0 & T_2 \end{bmatrix}$ *on* $\mathcal{H} = ker(T-\lambda) \oplus ker(T-\lambda)^{\perp}$ *is* (m,n)*-paranormal and* γ *is any eigen value of* T *such that* $\lambda \neq \gamma$*, then* $ker(T-\lambda) \perp ker(T-\gamma)$*.*

The proof of the following proposition is straightforward, so we omit it here.

Proposition 2.7. *If* $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$ *is a* (m,n)*-paranormal operator on* $\mathcal{H} \oplus \mathcal{H}$*, then* T_1 *and* T_2 *are* (m,n)*-paranormal.*

In the following theorem, we state a useful result on polar decomposition of bounded linear operators on Hilbert space. An operator T can be decomposed as $T = U|T|$, where U is a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$ with $ker(U) = ker(|T|)$.

Theorem 2.8. *[2, Corollary 2.2] If T is an operator with polar decomposition $T = U|T|$, where α is any non-zero complex number such that $\alpha = |\alpha|e^{i\theta}$ and (x_n) be a sequence of vectors in $\mathcal{H}$, then the following claims are equivalent.*
(1) $(T-\alpha)x_n \to 0$ and $(T^*-\bar{\alpha})x_n \to 0$
(2) $(|T|-|\alpha|)x_n \to 0$ and $(U-e^{i\theta})x_n \to 0$
(3) $(|T^*|-|\alpha|)x_n \to 0$ and $(U^*-e^{-i\theta})x_n \to 0$

Theorem 2.9. *Suppose that T is an operator on $\mathcal{H}$. If $(T-\lambda)x = 0$, then $(T^*-\bar{\lambda})x = 0$, where $\lambda \neq 0$.*

Proof. By assumption, we have $(T-\lambda)x = 0$, so

$$\|Tx\| = \|\lambda x\| \text{ and } \|T^{n+1}x\| = |\lambda|^{n+1}\|x\| \tag{3}$$

Now, by (3), we have

$$\langle |T|^2 x, x\rangle = |\lambda|^2\langle x, x\rangle \implies \langle (|T|^2 - |\lambda|^2)x, x\rangle = 0.$$

Therefore, $(|T|^2 - |\lambda|^2) = 0$ on $ker(T-\lambda)$, so we have

$$(|T| - |\lambda|)x = (|T| + |\lambda|)^{-1}(|T|^2 - |\lambda|^2)x = 0.$$

Thus,

$$|\lambda|(U - e^{i\theta})x = U(|\lambda| - |T|)x + (U|T| - |\lambda|e^{i\theta})x = 0,$$

that is, $(U-e^{i\theta})x = 0$ for $\lambda \neq 0$. So, $(T^*-\bar{\lambda})$x=0, by Theorem 2.8.
□

The next theorem provides spectral mapping theorem on Weyl spectrum of (m,n)-paranormal operators.

Theorem 2.10. *If T is a (m,n)-paranormal operator on $\mathcal{H}$, then spectral mapping theorem holds on Weyl spectrum, that is, $f(\sigma_w(T)) = \sigma_w(f(T))$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. First, let T be a (m,n)-paranormal operator defined by a matrix representation as: $T = \begin{bmatrix} \lambda & T_1 \\ 0 & T_2 \end{bmatrix}$ on $\mathcal{H} = ker(T-\lambda) \oplus ker(T-\lambda)^{\perp}$, where $\lambda \neq 0$. It is always true that $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ [10, Theorem 2(b)]. To prove converse part, assume that $\gamma \notin \sigma_w(f(T))$. Then $f(T)-\gamma$ is Fredholm operator of index zero. We take

$$f(z) - \gamma = (z-\gamma_1)(z-\gamma_2)\cdots(z-\gamma_k)g(z),$$

where $g(z) \neq 0$ for any z in G and γ_i, $i = 1, 2, \ldots, k$, are zeros of $f(z)-\gamma$ in G, where G is the open neighborhood of spectrum of T. Thus, consider the operator equation

$$f(T) - \gamma = (T-\gamma_1)(T-\gamma_2)\cdots(T-\gamma_k)g(T). \tag{4}$$

Considering (4), it is clear that $\gamma \in f(\sigma_w(T))$ if and only if $\gamma_i \in \sigma_w(T)$ for some i. Now, to prove the reverse inequality, it is suffices to prove that $\gamma_i \notin \sigma_w(T)$ for all i. Since $f(T)-\gamma$ is Fredholm operator of index zero, so by [18, Lemma 5.19], $T-\gamma_i$ is also Fredholm for each i and so is $g(T)$. Observe that

$$0 = ind(f(T)-\gamma) = ind(T-\gamma_1) + \cdots + ind(T-\gamma_k) + ind(g(T)). \tag{5}$$

Now, by [11, Corollary 5], we have $ind(T-\gamma_i) = ind(T_2-\gamma_i)$ for each i.

If $\gamma_i = 0$, then by Corollary 2.3, $ind(T) \leq 0$ for each i. If $\gamma_i \neq 0$, then by Theorem 2.9, $ind(T-\gamma_i) \leq 0$ for each i. As $g(T)$ is invertible, so (5) implies that $ind(T-\gamma_i) = 0$ for each i. Thus $\gamma_i \notin \sigma_w(T)$ for each i. Therefore, $\gamma \notin f(\sigma_w(T))$. Hence, the result holds. □

3. Weyl's Theorem for (m, n)-Paranormal Operators

According to Coburn, for any operator T, Weyl's theorem holds if $\sigma(T) - \sigma_w(T) = \pi_{00}(T)$ [4]. In recent years, many authors have extended this theorem on various classes of operators [12, 14, 20, 21].

If λ is an isolated point of spectrum of T, then the Riesz idempotent of T denoted as E_λ with respect to λ, is defined as

$$E_\lambda = \frac{1}{2\pi i}\int_{\partial D}(z - T)^{-1}dz,$$

where D is a the closed disk with center λ and with radius small enough so that $D \cap \sigma(T) = \{\lambda\}$. Then $E_\lambda^2 = E_\lambda$, $E_\lambda T = TE_\lambda$, $\sigma(T_{|ran(E_\lambda)}) = \{\lambda\}$ and $ker(T - \lambda) \subseteq ran(E_\lambda)$.

Lemma 3.1. *Let T be (m, n)-paranormal for $m \leq 1$ and $\sigma(T) = \{\lambda\}$, where λ is a complex number. Then $T = \lambda I$.*

Proof. For $\lambda = 0$, $\sigma(T) = \{0\}$. By [7, Theorem 2.8], T is normaloid, so $T = 0$. Now, we assume that $\lambda \neq 0$ and $A_1 = \frac{1}{\lambda}T$. Thus, $\sigma(A_1) = \{1\}$. Therefore, by [16, Theorem 1.5.14], A_1 is identity. Hence, $T = \lambda I$. □

Theorem 3.2. *Let T be a (m, n)-paranormal operator for $m \leq 1$. If λ is an isolated point of $\sigma(T)$, then Riesz idempotent E_λ with respect to λ satisfies $ran(E_\lambda) = ker(T - \lambda)$.*

Proof. By the general conditions of Riesz idempotent, we have that $\sigma(T_{|ran(E_\lambda)}) = \{\lambda\}$ and $ker(T-\lambda) \subseteq ran(E_\lambda)$. By [6, Proposition 2.2] and Lemma 3.1, we obtain $T_{|ran(E_\lambda)} = \lambda I$. Thus, $ran(E_\lambda) = ker(T - \lambda)$. □

Theorem 3.3. *For $m \leq 1$, if T is (m, n)-paranormal, then Weyl's theorem holds, that is, $\sigma(T) - \sigma_w(T) = \pi_{00}(T)$.*

Proof. If $\lambda \in \sigma(T) - \sigma_w(T)$, then $T - \lambda$ is Fredholm of index zero and $ker(T - \lambda)$ is also non-zero finite dimensional space. We represent T as the following 2×2 operator matrix with respect to the decomposition $\mathcal{H} = ker(T - \lambda) \oplus ker(T - \lambda)^\perp$:

$$T = \begin{bmatrix} \lambda & 0 \\ 0 & T_2 \end{bmatrix}.$$

Now, by [11, Corollary 5], we have

$$ind(T - \lambda) = ind(T_2 - \lambda) = 0.$$

Now, $ker(T_2 - \lambda) = \{0\}$, so $T_2 - \lambda$ is one-one, so it is invertible, this implies that $\lambda \notin \sigma(T_2)$. By [11], it is easy to see that $\sigma(T) = \sigma(T_2) \cup \{\lambda\}$, so $\lambda \in \pi_{00}(T)$.

Conversely, let λ be any arbitrary point of $\pi_{00}(T)$. It is clear from Theorem 3.2 that $ran(E_\lambda) = ker(T - \lambda)$. We know that $\sigma(T_{|ran(E_\lambda)}) = \{\lambda\}$, so by the general theory of Riesz idempotent, $\lambda \notin \sigma(T_{|ran(I-E_\lambda)})$. It follows that $ran(T - \lambda) = (T - \lambda)ran(E_\lambda) + (T - \lambda)ran(I - E_\lambda)$. Thus, we have $ran(T - \lambda) = (T - \lambda)ran(I - E_\lambda) = ran(I - E_\lambda)$.

However, $\beta(T - \lambda) = dim(\mathcal{H}/ran(T - \lambda)) = dim(\mathcal{H}/ran(I - E_\lambda)) = dim(ran(E_\lambda)) = \alpha(T - \lambda)$. Hence, $\alpha(T - \lambda) = \beta(T - \lambda)$. So, $T - \lambda$ is Fredholm operator of index zero. Therefore, $\lambda \in \sigma(T) - \sigma_w(T)$. This completes the proof. □

Theorem 3.4. *Suppose that T is (m, n)-paranormal and $ran(T)$ is not dense. We define the representation of T as 2×2 matrix as $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$ on $\mathcal{H} = \overline{ran(T)} \oplus ker(T^*)$. Then T_1 is (m, n)-paranormal on $\overline{ran(T)}$ and $T_3 = 0$. Moreover, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. If P is projection on $\overline{ran(T)}$, then we write $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Consider $\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} = TP = PTP$. By [6, Theorem 2.1], we obtain

$$P(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)a^n T^* T + m^{\frac{2}{n+1}} n a^{n+1} I)P \geq 0,$$

for each $a > 0$. It follows that

$$m^{\frac{2}{n+1}} T_1^{*n+1} T_1^{n+1} - (n+1)a^n T_1^* T_1 + m^{\frac{2}{n+1}} n a^{n+1} I \geq 0,$$

for each $a > 0$. Hence, T_1 is (m, n)-paranormal on $\overline{ran(T)}$.

For $x = x_1 \oplus x_2$ in $\mathcal{H}$, we deduce that $\langle T_3 x_2, x_2 \rangle = \langle T(I-P)x, (I-P)x \rangle = \langle (I-P)x, T^*(I-P)x \rangle = 0$. Therefore, $T_3 = 0$ on $ker(T^*)$. It is clear that $\sigma(T_1) \cap \sigma(T_3)$ has no interior point, so by [11, Corollary 8], we obtain that $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$, i.e., $\sigma(T) = \sigma(T_1) \cup \{0\}$. □

Corollary 3.5. *If we define an operator* $T = \begin{bmatrix} \lambda & T_2 \\ 0 & T_3 \end{bmatrix}$ *on* $\mathcal{H}$ *in Theorem 3.4, where* λ *is any complex number, then* λI *is* (m, n)*-paranormal operator on* $\overline{ran(T)}$, $T_3 = 0$ *and* $\sigma(T) = \{\lambda, 0\}$.

Theorem 3.6. *Let* $T \in B(\mathcal{H} \oplus \mathcal{H})$ *be an operator represented by* 2×2 *matrix as* $\begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix}$. *Then* T *is* (m, n)*-paranormal if and only if*

$$m^{\frac{2}{n+1}} C^{*n+1} C^{n+1} - (n+1)a^n C^* C + m^{\frac{2}{n+1}} n a^{n+1} I \geq (n+1)a^n D^* D - m^{\frac{2}{n+1}} (C^*)^n D^* D C^n, \tag{6}$$

for each $a > 0$.

Proof. If T is (m, n)-paranormal, then by [6, Theorem 2.1]

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)a^n T^* T + m^{\frac{2}{n+1}} n a^{n+1} I = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \tag{7}$$

is positive for each $a > 0$, where $X = m^{\frac{2}{n+1}} (C^{*n+1} C^{n+1} + (C^*)^n D^* D C^n) - (n+1)a^n (C^* C + D^* D) + m^{\frac{2}{n+1}} n a^{n+1}$ and $Y = m^{\frac{2}{n+1}} n a^{n+1}$. Now, by (m, n)-paranormality T, $X \geq 0$. Thus, (6) holds.

On the other hand, when X is non-negative, the matrix on right side of (7) is hermitian as Y is also non-negative, implying thereby the non-negativity of the determinant of the matrix. □

Lemma 3.7. *[3] Let* $\mathcal{H}$ *be a complex Hilbert space. Then there exists a Hilbert space* $\mathcal{K}$ *such that* $\mathcal{H} \subseteq \mathcal{K}$ *and a map* $\phi : B(\mathcal{H}) \to B(\mathcal{K})$ *such that*

1. ϕ *is a faithful* $*$*-representation of the algebra* $B(\mathcal{H})$ *on* $\mathcal{K}$;
2. $\phi(A) \geq 0$ *for any* $A \geq 0$ *in* $B(\mathcal{H})$;
3. $\sigma_a(T) = \sigma_a(\phi(T)) = \sigma_p(\phi(T))$ *for any* T *in* $B(\mathcal{H})$.

Theorem 3.8. *Let* T *be a* (m, n)*-paranormal operator. If* $\phi : B(\mathcal{H}) \to B(\mathcal{K})$ *is Berberian's faithful* $*$*-representation of Lemma 3.7, then* $\phi(T)$ *is also* (m, n)*-paranormal.*

Proof. Since T is (m, n)-paranormal, by [6, Theorem 2.1], we have

$$(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)a^n T^* T + m^{\frac{2}{n+1}} n a^{n+1} I) \geq 0,$$

for each $a > 0$. By using Lemma 3.7,

$$\phi(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)a^n T^* T + m^{\frac{2}{n+1}} n a^{n+1} I) \geq 0,$$

that is,

$$m^{\frac{2}{n+1}} \phi(T)^{*n+1} \phi(T)^{n+1} - (n+1)a^n \phi(T)^* \phi(T) + m^{\frac{2}{n+1}} n a^{n+1} I \geq 0,$$

for each $a > 0$. This completes the result. □

Theorem 3.9. *For $m \leq 1$, let T be (m, n)-paranormal for some positive integer n. Then Weyl's theorem holds for $f(T)$, i.e., $\sigma(f(T)) - \sigma_w(f(T)) = \pi_{00}(f(T))$, for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. By Theorem 3.2, the operator T is isoloid. Then by [17, Lemma],

$$\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T)), \tag{8}$$

for every $f \in \mathcal{H}(\sigma(T))$. Now, from Theorem 3.3 and Theorem 2.10, it follows that

$$\begin{aligned} f(\sigma(T) - \pi_{00}(T)) &= f(\sigma_w(T)) \\ &= \sigma_w(f(T)). \end{aligned} \tag{9}$$

From (8) and (9), Weyl's theorem holds for $f(T)$. □

4. Algebraically (m, n)-Paranormal Operators

In this section, we show that the class of algebraically (m, n)-paranormal operators is independent from the class of (m, n)-paranormal and also show that algebraically (m, n)-paranormal operators are Polaroid. Moreover, we prove spectral mapping theorem and Weyl's theorem for the class of algebraically (m, n)-paranormal operators.

Definition 4.1. *An operator T on $\mathcal{H}$ is said to be an algebraically (m, n)-paranormal operator if there exists a non-constant complex polynomial $p(t)$ such that $p(T)$ is (m, n)-paranormal.*

Now, we define quasi nilpotent part of T, i.e., is subspace of $\mathcal{H}$, that is,

$$H(T) = \{x \in \mathcal{H} : \lim_{n \to \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is easy to see that $ker(T^m) \subseteq H(T)$ for each $m \in \mathbb{N}$. Moreover, we say that an operator T is said to have property (H) if $ker(T - \lambda) = H(T - \lambda)$ for all complex number λ.

Proposition 4.2. *[6, Proposition 2.2] If an operator T is algebraically (m, n)-paranormal, then T is algebraically (m, n)-paranormal on every invariant subspace of $\mathcal{H}$.*

Lemma 4.3. *Let T be a (m, n)-paranormal operator for $m \leq 1$, then T has property (H).*

Proof. By Theorem 3.2 and [15, Theorem 3.1], T satisfies the required result. Consequently, by [1, Theorem 2.5], $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$. □

It is easy to understand every (m, n)-paranormal operator is algebraically (m, n)-paranormal but converse need not be true. The following example shows that an algebraically (m, n)-paranormal operator is not (m, n)-paranormal.

Example 4.4. *First, we take an operator T on $\mathcal{H}\oplus\mathcal{H}$ as matrix representation $T = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$. By [6, Theorem 2.1], T is $((\frac{1}{2})^{\frac{3}{2}}, 2)$-paranormal if and only if*

$$\frac{1}{2}T^{*3}T^3 - 3a^2T^*T + a^3I \geq 0, \tag{10}$$

for each $a > 0$. Observe that

$$\frac{1}{2}T^{*3}T^3 - 3a^2T^*T + a^3I = \begin{bmatrix} \frac{37}{2} - 15a^2 + a^3 & -3 + 6a^2 \\ -3 + 6a^2 & \frac{37}{2} - 3a^2 + a^3 \end{bmatrix}$$

is not positive at $a = 1.3$. Thus, T is not $((\frac{1}{2})^{\frac{3}{2}}, 2)$-paranormal.

Now, consider the polynomial $f(z) = (z + 1)^2$, this implies that, $f(T) = (T + I)^2$. From (10), we have the operator

$$\frac{1}{2}f(T)^{*3}f(T)^3 - 3a^2f(T)^*f(T) + a^3I = \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix}$$

is positive for each $a > 0$. Therefore, $f(T)$ is $((\frac{1}{2})^{\frac{3}{2}}, 2)$-paranormal operator.

In the following example, we show that an operator T is algebraically (m, n)-paranormal and (m, n)-paranormal both.

Example 4.5. *An operator $T \in B(\mathcal{H}\oplus\mathcal{H})$ defined by the matrix representation $T = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$. The operator T is $(2^{\frac{3}{2}}, 2)$-paranormal if and only if*

$$2T^{*3}T^3 - 3a^2T^*T + 4a^3I \geq 0, \tag{11}$$

for each $a > 0$. Consider the operator equation

$$2T^{*3}T^3 - 3a^2T^*T + 4a^3I = \begin{bmatrix} 416 - 15a^2 + 4a^3 & 192 - 6a^2 \\ 192 - 6a^2 & 128 - 12a^2 + 4a^3 \end{bmatrix}.$$

The above operator is positive for each $a > 0$. So, T is $(2^{\frac{3}{2}}, 2)$-paranormal.

Now from the complex polynomial $f(z) = (z - 2)^2$ follows the polynomial in operator variable $f(T) = (T - 2)^2$. So, observe that the following operator equation

$$2f(T)^{*3}f(T)^3 - 3a^2f(T)^*f(T) + 4a^3I = \begin{bmatrix} 4a^3 & 0 \\ 0 & 4a^3 \end{bmatrix}.$$

The above operator is positive for each $a > 0$.

Definition 4.6. *An operator T is said to have single valued extension property (abbreviated as SVEP) at $\gamma_0 \in \mathbb{C}$, if for every open neighborhood G of γ_0, the only analytic function $f : G \to \mathcal{H}$ which satisfies the equation $(T - \gamma I)f(\gamma) = 0$ for all $\gamma \in G$ is the function $f = 0$.*

An operator T has SVEP if T has SVEP at every $\gamma \in \mathbb{C}$.

By using Proposition 2.6, the following theorem is proved.

Theorem 4.7. *[19, Theorem 8] If an operator T is (m, n)-paranormal, then T has SVEP.*

Immediate consequences, we got the following result.

Corollary 4.8. *Let T be an algebraically (m, n)-paranormal operator. Then $p(T)$ has SVEP for some polynomial $p(t)$.*

Theorem 4.9. *[16, Theorem 3.3.9] Let T be an algebraically (m, n)-paranormal operator. Then T has SVEP.*

We also provide spectral mapping theorem for algebraically (m, n)-paranormal operators for Weyl spectrum.

Theorem 4.10. *If T is an algebraically (m, n)-paranormal operator for some positive real number $m \leq 1$ and positive integer n, then $f(\sigma_w(T)) = \sigma_w(f(T))$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. It can be easily verified that $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$. To prove the converse part it suffices to prove that $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. Suppose $\lambda \notin \sigma_w(f(T))$. Then $f(T) - \lambda$ is Weyl operator, i.e., Fredholm operator of index zero. Observe that

$$f(T) - \lambda = (T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_k)g(T),$$

where $\lambda_1, \lambda_1, \cdots, \lambda_k$ are zeros of $f(T) - \lambda$ and $g(T)$ is an invertible operator. Thus, we get

$$0 = ind(f(T) - \lambda) = ind(T - \lambda_1) + ind(T - \lambda_2) + \cdots + ind(T - \lambda_k) + ind(g(T)).$$

Since $g(T)$ is invertible, so $ind(g(T)) = 0$. By Theorem 4.9 and [13, Proposition 38.5], $ind(T - \lambda_i) \leq 0$, for each $i = 1, 2, \cdots, k$. Therefore, $ind(T - \lambda_i) = 0$ for each i. Thus, $T - \lambda_i$ is Fredholm operator of index zero for each i, so $\lambda_i \notin \sigma_w(T)$. Hence, $\lambda \notin f(\sigma_w(T))$, which yields the required result. □

We say that an operator T is Polaroid if for every isolated point of spectrum of T is a pole of the resolvent. In the following theorem, we show that (m, n)-paranormal operator is Polaroid.

Theorem 4.11. *Let T be an algebraically (m, n)-paranormal operator for $m \leq 1$. Then $g(T)$ is Polaroid for some polynomial $g(t)$.*

Proof. Since T is algebraically (m, n)-paranormal operator then there exists a non-constant polynomial $g(t)$ such that $g(T)$ is (m, n)-paranormal. Let $\lambda \in iso\sigma(g(T))$ and corresponding spectral projection $E_\lambda = \frac{1}{2\pi i}\int_{\partial D}(z - g(T))^{-1}dz$, where D is closed disk of center λ with $\sigma(g(T)) \cap D = \{\lambda\}$. We can then represent $g(T)$ as:

$$g(T) = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

with $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(g(T)) - \{\lambda\}$. By Proposition 2.7, T_1 and T_2 are (m, n)-paranormal operators for $m \leq 1$. If $\sigma(T_1) = \{\lambda\}$, then by Lemma 3.1, $T_1 - \lambda = 0$. This implies that $ker(T_1 - \lambda) = \mathcal{H}$ and $ker(T_1 - \lambda)^2 = \mathcal{H}$. Thus, $p(T_1 - \lambda) = 1$. Analogously, $ran(T_1 - \lambda) = \{0\}$ and $ran(T_1 - \lambda)^2 = \{0\}$ and hence $q(T_1 - \lambda) = 1$. This implies that $p(T_1 - \lambda) = q(T_1 - \lambda)$. In the same way, $T_2 - \lambda$ is invertible, so $ker(T_2 - \lambda) = \{0\}$ and $ker(T_2 - \lambda)^0 = ker(I) = \{0\}$, it follows that $p(T_2 - \lambda) = 0$. Also, $ran(T_2 - \lambda) = \mathcal{H}$ and $ran(T_2 - \lambda)^0 = ran(I) = \mathcal{H}$, this implies that $q(T_2 - \lambda) = 0$. Consequently, $p(T_2 - \lambda) = q(T_2 - \lambda)$.

Now, by above computation $ker(g(T) - \lambda) = ker(g(T) - \lambda)^2$ and hence $p(g(T) - \lambda) = 1$. In the sequel, $ran(g(T) - \lambda) = ran(g(T) - \lambda)^2$, it follows that $q(g(T) - \lambda) = 1$. Consequently, by [13, Proposition 38.3], we get $p(g(T) - \lambda) = q(g(T) - \lambda)$. Thus, $0 < p(g(T) - \lambda) = q(g(T) - \lambda) < \infty$. So, λ is a pole of resolvent of $g(T)$. This implies that $g(T)$ is Polaroid operator. □

Corollary 4.12. *[16, Theorem 3.3.9] If T is an algebraically (m, n)-paranormal operator for $m \leq 1$, then T is Polaroid. Consequently, by [9, Lemma 3.3], T is an isoloid operator.*

The proof of the following theorem is obvious by using Corollary 4.12 and Theorem 4.9.

Theorem 4.13. *[8, Theorem 2.2] If an operator T is algebraically (m, n)-paranormal for $m \leq 1$, then Weyl's theorem holds for T.*

Theorem 4.14. *If an operator T algebraically (m, n)-paranormal for $m \leq 1$, then Weyl's theorem holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. By Corollary 4.12, the operator T is isoloid, so by [17, Lemma], we get $\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T))$. So, theorem 4.13 implies that $f(\sigma(T) - \pi_{00}(T)) = f(\sigma_w(T))$.

Now, by Theorem 4.10, we get $f(\sigma_w(T)) = \sigma_w(f(T))$. Finally, we have $\sigma(f(T)) - \sigma_w(f(T)) = \pi_{00}(f(T))$. This completes the proof. □

References

[1] P. Aiena and F. Villafañe, Weyl's theorems for some classes of operators, Integral Equations Operator Theory **53** (2005), no. 4, 453–466.

[2] A. Aluthge and D. Wang, The joint approximate point spectrum of an operator, Hokkaido Math. J. **31** (2002), no. 1, 187–197.

[3] S. K. Berberian, Approximate proper vectors, Proc. Amer. Math. Soc. **13** (1962), 111–114.

[4] L. A. Coburn, Weyl's theorem for nonnormal operators, Michigan Math. J. **13** (1966), 285–288.

[5] J. B. Conway, A course in functional analysis, second edition, Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1990.

[6] P. Dharmarha and S. Ram, (m, n)-paranormal operators and $(m, n)^*$-paranormal operators, Commun. Korean Math. Soc. (2020) 35(1) 151–159.

[7] P. Dharmarha and S. Ram, *A note on* (m, n)*-paranormal operators*, Electronic Journal of Mathematical Analysis and Applications 9(2021), no. 1, 274-283.

[8] B. P. Duggal, Polaroid operators satisfying Weyl's theorem, Linear Algebra Appl. **414** (2006), no. 1, 271–277.

[9] B. P. Duggal, Polaroid operators, SVEP and perturbed Browder, Weyl theorems, Rend. Circ. Mat. Palermo (2) **56** (2007), no. 3, 317–330.

[10] B. Gramsch and D. Lay, Spectral mapping theorems for essential spectra, Math. Ann. **192** (1971), 17–32.

[11] J. K. Han, H. Y. Lee and W. Y. Lee, Invertible completions of 2×2 upper triangular operator matrices, Proc. Amer. Math. Soc. **128** (2000), no. 1, 119–123.

[12] Y. M. Han, J. I. Lee and D. Wang, Riesz idempotent and Weyl's theorem for w-hyponormal operators, Integral Equations Operator Theory **53** (2005), no. 1, 51–60.

[13] H. Heuser, Functional Analysis, Marcel Dekker, New York 1982.

[14] I. H. Kim, On (p, k)-quasihyponormal operators, Math. Inequal. Appl. **7** (2004), no. 4, 629–638.

[15] J. J. Koliha, Isolated spectral points, Proc. Amer. Math. Soc. **124** (1996), no. 11, 3417–3424.

[16] K. B. Laursen and M. M. Neumann, An introduction to local spectral theory, London Mathematical Society Monographs. New Series, 20, The Clarendon Press, Oxford University Press, New York, 2000.

[17] W. Y. Lee and S. H. Lee, A spectral mapping theorem for the Weyl spectrum, Glasgow Math. J. **38** (1996), no. 1, 61–64.

[18] M. Schechter, Principles of functional analysis, second edition, Graduate Studies in Mathematics, 36, American Mathematical Society, Providence, RI, 2002.

[19] K. Tanahashi and A. Uchiyama, A note on $*$-paranormal operators and related classes of operators, Bull. Korean Math. Soc. **51** (2014), no. 2, 357–371.

[20] A. Uchiyama, Weyl's theorem for class A operators, Math. Inequal. Appl. **4** (2001), no. 1, 143–150.

[21] A. Uchiyama, On the isolated points of the spectrum of paranormal operators, Integral Equations Operator Theory **55** (2006), no. 1, 145–151.