



EP Matrices of Adjointable Operators on Hilbert C^* -Modules

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Abstract. In this paper, we investigate EP modular operator matrices on Hilbert C^* -modules setting. The necessary and sufficient conditions for some modular operator matrices to be EP are discussed, based on the generalized Schur complement. These enable us to generalize some results of Meenakshi (1958) [11] for block matrices.

1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$ the space of all bounded linear operators from H to H . An operator $T \in \mathcal{B}(H)$ is an EP operator if and only if $TT^\dagger = T^\dagger T$ [2, 8], where T^\dagger is the Moore-Penrose inverse of T . This originates from EP matrix introduced by Schwerdtfeger in [16], and a complex square matrix T is said to be an EP matrix if the ranges of T and T^* coincide. Note that, for an operator $T \in \mathcal{B}(H)$, T^\dagger exists if and only if its range, $\mathcal{R}(T)$, is closed. In fact, $T \in \mathcal{B}(H)$ is an EP operator if and only if $\mathcal{R}(T)$ is closed and $\mathcal{R}(T) = \mathcal{R}(T^*)$. The property of EP has also been studied by many other authors, see e.g. [1, 6, 12, 13, 15] and references therein. In this note, we study the generalized inverse of EP modular operators matrices on Hilbert C^* -modules, and then we formulate some results concerning this class of operators matrices.

A Hilbert C^* -module is a natural generalization of a Hilbert space equipped with inner product by an arbitrary C^* -algebra, instead of the complex field \mathbb{C} . Because the geometry of this module is provided by its special inner product, some basic properties of Hilbert spaces like Pythagoras's equality, self-duality, and decomposition into orthogonal complements do not necessarily hold true.

Throughout the present paper, let \mathbb{A} be a C^* -algebra. A (right) pre-Hilbert module over a C^* -algebra \mathbb{A} is a linear complex space \mathcal{H} which is an algebraic (right) \mathbb{A} -module equipped with an \mathbb{A} -valued inner product $\langle x, y \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{A}$ satisfying

- (i) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$,
- (ii) $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$,
- (iii) $\langle x, ya \rangle = \langle x, y \rangle a$,
- (iv) $\langle x, y \rangle = \langle y, x \rangle^*$,

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for each $x, y, z \in \mathcal{H}$, $\lambda \in \mathbb{C}$, $a \in \mathbb{A}$. A Hilbert \mathbb{A} -module is precisely a complete (right) pre-Hilbert module, with respect to the induced norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. Let \mathcal{H} and \mathcal{K} are Hilbert \mathbb{A} -modules. Denote by $\mathcal{L}(\mathcal{H}, \mathcal{K})$ the set of adjointable operators from \mathcal{H} to \mathcal{K} , i.e., the set of all maps $T : \mathcal{H} \rightarrow \mathcal{K}$ for which there is a map $T^* : \mathcal{K} \rightarrow \mathcal{H}$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for $x \in \mathcal{H}$ and $y \in \mathcal{K}$. It is well known that any element T of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ must be a bounded linear operator, which is also \mathbb{A} -linear in the sense that $T(xa) = (Tx)a$ for any $x \in \mathcal{H}$ and $a \in \mathbb{A}$. We use $\mathcal{L}(\mathcal{H})$ to denote the C^* -algebra $\mathcal{L}(\mathcal{H}, \mathcal{H})$. Let $\mathcal{L}(\mathcal{H})_{sa}$ be the set of hermitian elements of $\mathcal{L}(\mathcal{H})$. For any $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, the range and the kernel of T are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. We write $n(T)$ and $d(T)$ for the dimensions of the kernel $\mathcal{N}(T)$ and the quotient space $\mathcal{H}/\mathcal{R}(T)$. An operator T of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ is said to be *regular* if there is an operator $T^- \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $TT^-T = T$; T^- is called an *inner inverse* (or $\{1\}$ -inverse) of T . It is easy to prove that T is regular if and only if $\mathcal{R}(T)$ is closed. The inner inverse of T is not unique.

In this paper, we use the generalized inverse to the generalized Schur complement as defined in [7]. Suppose $M \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ is a modular operator matrix partitioned into the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

where $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $D \in \mathcal{L}(\mathcal{K})$, then the generalized Schur complement of A in M is

$$M/A = D - CA^-B, \tag{2}$$

where A^- is an inner inverse of A . Similarly, the generalized Schur complement of D in M is

$$M/D = A - BD^-C, \tag{3}$$

where D^- is an inner inverse of D . The formulas (2) and (3) have previously appeared in papers dealing with generalized inverses of partitioned matrices [4, 10, 11].

In finite dimensional spaces, Meenakshi [11] has, using generalized Schur complement, discussed when a block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is EP. Infinite dimensional EP operators have been studied by several literature [1, 2, 6, 8, 15]. In the present paper, we use generalized Schur complement to discuss EP modular operator matrices on Hilbert C^* -modules, and generalize some results of Meenakshi [11] to infinite dimensional spaces.

2. Preliminaries

In this section, we would like to recall some definitions and present a few simple facts about adjointable operators on Hilbert \mathbb{A} -modules.

Definition 2.1 ([9]). Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. The Moore-Penrose inverse T^\dagger of T (if it exists) is an element in $\mathcal{L}(\mathcal{K}, \mathcal{H})$ which satisfies

- (a) $TT^\dagger T = T$,
- (b) $T^\dagger TT^\dagger = T^\dagger$,
- (c) $(TT^\dagger)^* = TT^\dagger$,
- (d) $(T^\dagger T)^* = T^\dagger T$.

These properties imply that T^\dagger is unique and $T^\dagger T$ and TT^\dagger are orthogonal projections. Moreover, $\mathcal{R}(T^\dagger) = \mathcal{R}(T^\dagger T)$, $\mathcal{R}(T) = \mathcal{R}(TT^\dagger)$, $\mathcal{N}(T) = \mathcal{N}(T^\dagger T)$ and $\mathcal{N}(T^\dagger) = \mathcal{N}(TT^\dagger)$. Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$. The Moore-Penrose inverse T^\dagger of T exists if and only if T has closed range.

Similar to Lemma 2.2.4 of [14], we have the following conclusion on Hilbert C^* -modules.

Lemma 2.2. Let $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $C \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. If A has an inner inverse A^- , then

- (i) $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ if and only if $C = CA^-A$,
- (ii) $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ if and only if $B = AA^-B$.

Lemma 2.3. Let M be a modular operator matrix of the form (1) with $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$. If $\mathcal{R}(A)$ is closed, then M is regular if and only if M/A is regular, where $M/A = D - CA^-B$. In this case, an inner inverse of M is given by

$$M^- = \begin{pmatrix} A^- + A^-B(M/A)^-CA^- & -A^-B(M/A)^- \\ -(M/A)^-CA^- & (M/A)^- \end{pmatrix}.$$

Proof. See Lemma 2.2 in [3]. \square

From Lemma 2.3, we can obtain the following corollary.

Corollary 2.4. Let M be a modular operator matrix of the form (1) with $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$, $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$ and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$. If $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed, then the Moore-Penrose inverse M^\dagger of M can be expressed as

$$M^\dagger = \begin{pmatrix} A^\dagger + A^\dagger B(M/A)^\dagger CA^\dagger & -A^\dagger B(M/A)^\dagger \\ -(M/A)^\dagger CA^\dagger & (M/A)^\dagger \end{pmatrix}.$$

Definition 2.5 ([15]). Let \mathcal{H} be Hilbert \mathbb{A} -module. An operator $T \in \mathcal{L}(\mathcal{H})$ is called EP if $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^*)}$.

Obviously, the range of an EP operator on a Hilbert C^* -module is not necessarily closed, and we further have the following property.

Proposition 2.6 ([15]). Let \mathcal{H} be Hilbert \mathbb{A} -module and $T \in \mathcal{L}(\mathcal{H})$ with closed range. Then the following conditions are equivalent:

- (i) T is EP operator,
- (ii) $\mathcal{N}(T) = \mathcal{N}(T^*)$,
- (iii) T is Moore-Penrose invertible and $T^\dagger T = TT^\dagger$.

3. EP matrices of adjointable operators

In this section, using generalized Schur complements, we study the 2×2 EP matrices of adjointable operators acting on Hilbert C^* -modules.

Theorem 3.1. Let M be a modular operator matrix of the form (1) with $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed. Then the following conditions are equivalent:

- (i) M is an EP operator with closed range,
- (ii) A and M/A are EP operators, and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$.

Proof. Suppose M is EP with closed range. Since $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed, we can define the following operator matrices:

$$L = \begin{pmatrix} I & 0 \\ CA^- & I \end{pmatrix}, R = \begin{pmatrix} I & B(M/A)^- \\ 0 & I \end{pmatrix}, P = \begin{pmatrix} A & 0 \\ 0 & M/A \end{pmatrix},$$

where $M/A = D - CA^-B$. Since $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, by Lemma 2.2, M can be decomposed as $M = LRP$. Hence $\mathcal{N}(M) = \mathcal{N}(P)$. From the invertibility of L and R , it follows that $\mathcal{N}(M^*) = \mathcal{N}(M) = \mathcal{N}(P)$ since M is EP with closed range. By using Lemma 2.2 again it is immediate that

$$M^* = M^*P^-P, \tag{4}$$

and

$$P^* = MM^-P^*, \tag{5}$$

hold for an arbitrary inner inverse P^- of P and M^- of M . In particular, P^- can be given by

$$P^- = \begin{pmatrix} A^- & 0 \\ 0 & (M/A)^- \end{pmatrix};$$

from $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ and Lemma 2.3, we know that M has inner inverses, and one of its inner inverse is given by

$$M^- = \begin{pmatrix} A^- + A^-B(M/A)^-CA^- & -A^-B(M/A)^- \\ -(M/A)^-CA^- & (M/A)^- \end{pmatrix}.$$

Thus the equations (4) and (5) give

$$\begin{aligned} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} &= \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} A^-A & 0 \\ 0 & (M/A)^-(M/A) \end{pmatrix} \\ &= \begin{pmatrix} A^*A^-A & C^*(M/A)^-(M/A) \\ B^*A^-A & D^*(M/A)^-(M/A) \end{pmatrix}, \end{aligned} \tag{6}$$

$$\begin{aligned} \begin{pmatrix} A^* & 0 \\ 0 & (M/A)^* \end{pmatrix} &= \begin{pmatrix} AA^- & 0 \\ (I-(M/A)(M/A)^-)CA^- & (M/A)(M/A)^- \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & (M/A)^* \end{pmatrix} \\ &= \begin{pmatrix} AA^-A^* & 0 \\ (I-(M/A)(M/A)^-)CA^-A^* & (M/A)(M/A)^-(M/A)^* \end{pmatrix}, \end{aligned} \tag{7}$$

which imply $A^* = A^*A^-A$ and $A^* = AA^-A^*$, and hence $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$ and $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$ simultaneously, i.e., $\mathcal{N}(A) = \mathcal{N}(A^*)$. This shows that A is an EP operator.

From (6), we have

$$D^* = D^*(M/A)^-(M/A) \text{ and } C^* = C^*(M/A)^-(M/A),$$

which together with $D = M/A + CA^-B$ yield $(M/A)^* = (M/A)^*(M/A)^-(M/A)$. This implies $\mathcal{N}(M/A) \subseteq \mathcal{N}((M/A)^*)$. Since $(M/A)^* = (M/A)(M/A)^-(M/A)^*$ by (7), we conclude $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(M/A)$. Then $\mathcal{N}((M/A)^*) = \mathcal{N}(M/A)$ and hence M/A is an EP operator. Besides, the relation $C^* = C^*(M/A)^-(M/A)$ implies $\mathcal{N}(M/A) \subseteq \mathcal{N}(C^*)$. Therefore $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$.

Conversely, the assumptions $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$ ensures that M^\dagger exists and is given by

$$M^\dagger = \begin{pmatrix} A^\dagger + A^\dagger B(M/A)^\dagger CA^\dagger & -A^\dagger B(M/A)^\dagger \\ -(M/A)^\dagger CA^\dagger & (M/A)^\dagger \end{pmatrix}$$

by Corollary 2.4. Using $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$, by Lemma 2.2, MM^\dagger is described as the form

$$MM^\dagger = \begin{pmatrix} AA^\dagger & 0 \\ 0 & (M/A)(M/A)^\dagger \end{pmatrix}.$$

Similarly, using $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$ gives

$$M^\dagger M = \begin{pmatrix} A^\dagger A & 0 \\ 0 & (M/A)^\dagger(M/A) \end{pmatrix}.$$

Since A and M/A are EP operators with closed range, $AA^\dagger = A^\dagger A$ and $(M/A)(M/A)^\dagger = (M/A)^\dagger(M/A)$. Thus $MM^\dagger = M^\dagger M$. Therefore M is EP with closed range. \square

Remark 3.2. Under the conditions $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$, M/A is independent on the choice of A^- . However, we always assume that M/A is given in terms of some specific choice of A^- .

The dual result of Theorem 3.1 is as follows.

Theorem 3.3. Let M be a modular operator matrix of the form (1) with $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed. Then the following conditions are equivalent:

- (i) M is EP operator with closed range,
- (ii) A and M/A are EP operators, $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$.

Proof. The proof of Theorem 3.3 is straightforward from Theorem 3.1 and from the fact that M is EP if and only if M^* is EP. \square

For the special case when $C = B^*$, we can get the following results.

Corollary 3.4. Let $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ with $\mathcal{N}(A) \subseteq \mathcal{N}(B^*)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(B)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, where $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $D \in \mathcal{L}(\mathcal{K})_{sa}$. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed. Then the following conditions are equivalent:

- (i) M is an EP operator with closed range.
- (ii) A and M/A are EP operators.

Corollary 3.5. Let $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ with $\mathcal{N}(A) \subseteq \mathcal{N}(B^*)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, where $A \in \mathcal{L}(\mathcal{H})_{sa}$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $D \in \mathcal{L}(\mathcal{K})_{sa}$. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed. Then the following conditions are equivalent:

- (i) M is EP operator with closed range.
- (ii) M/A is EP operator.

As the dimensions of the kernel is introduced, one can get the following results.

Theorem 3.6. Let M be a modular operator matrix of the form (1) with $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, where $M/A = D - CA^\dagger B$. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed, and that $n(A^*)$ and $n((M/A)^*)$ are zero. Then the following conditions are equivalent:

- (i) M is an EP operator with closed range.
- (ii) A and M/A are EP operators, $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$.

Proof. Suppose M is EP with closed range. Write

$$L = \begin{pmatrix} I & 0 \\ CA^\dagger & I \end{pmatrix}, R = \begin{pmatrix} I & B(M/A)^\dagger \\ 0 & I \end{pmatrix}, P = \begin{pmatrix} A & 0 \\ 0 & M/A \end{pmatrix}.$$

Obviously, L and R are invertible. By Lemma 2.2 and assumption $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, it is clear M can be decomposed as $M = LRP$. Hence $\mathcal{N}(M) = \mathcal{N}(P)$. Since M is EP with closed range, $\mathcal{N}(M^*) = \mathcal{N}(M) = \mathcal{N}(P)$. By Lemma 2.2, it is immediate that $M^* = M^*P^\dagger P$, where P^\dagger is given by

$$P^\dagger = \begin{pmatrix} A^\dagger & 0 \\ 0 & (M/A)^\dagger \end{pmatrix}.$$

Then

$$\begin{aligned} M^* &= \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} A^\dagger A & 0 \\ 0 & (M/A)^\dagger (M/A) \end{pmatrix} \\ &= \begin{pmatrix} A^* A^\dagger A & C^* (M/A)^\dagger (M/A) \\ B^* A^\dagger A & D^* (M/A)^\dagger (M/A) \end{pmatrix}. \end{aligned}$$

Hence $A^* = A^*A^+A$ implies $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$. Since $n(A^*)$ is zero, $\mathcal{N}(A) = \mathcal{N}(A^*)$. Thus A is an EP operator.

From $B^* = B^*A^+A$ it follows that $\mathcal{N}(A^*) = \mathcal{N}(A) \subseteq \mathcal{N}(B^*)$. In the similar way as in the proof of Theorem 3.1, from $D = M/A + CA^+B$, $C^* = C^*(M/A)^\dagger(M/A)$ and $D^* = D^*(M/A)^\dagger(M/A)$, we can conclude that

$$(M/A)^* = (M/A)^*(M/A)^\dagger(M/A).$$

This implies $\mathcal{N}(M/A) \subseteq \mathcal{N}((M/A)^*)$. Since $n((M/A)^*)$ is zero, $\mathcal{N}(M/A) = \mathcal{N}((M/A)^*)$. Thus M/A is an EP operator, and $\mathcal{N}((M/A)^*) = \mathcal{N}(M/A) \subseteq \mathcal{N}(C^*)$ is further valid.

Conversely, the proof is the same as that of (ii) \Rightarrow (i) in Theorem 3.1. \square

The dual result of Theorem 3.6 is as follows.

Theorem 3.7. *Let M be a modular operator matrix of the form (1) with $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$, where $M/A = D - CA^+B$. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed, and that $n(A)$ and $n(M/A)$ are zero. Then the following conditions are equivalent:*

- (i) M is an EP operator with closed range.
- (ii) A and M/A are EP operators, $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$.

For the special case when $C = B^*$, we have

Corollary 3.8. *Let $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ with $\mathcal{N}(A) \subseteq \mathcal{N}(B^*)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, where $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $D \in \mathcal{L}(\mathcal{K})$. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed, and that $n(A^*)$ and $n((M/A)^*)$ are zero. Then the following conditions are equivalent:*

- (i) M is an EP operator with closed range.
- (ii) A and M/A are EP operators.

Remark 3.9. *Using the generalized Schur complement $M/D = A - BD^-C$ of D in M , similar to Theorem 3.1 and Theorem 3.6, one can get the analogous results above.*

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