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How Many Are Projectable Classical Linear Connections with a Prescribed Ricci Tensor

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Abstract. How many are projectable classical linear connections with a prescribed Ricci tensor and a prescribed trace of torsion tensor on the total space of a fibered manifold? The questions are answered in the analytic case by using the Cauchy-Kowalevski theorem. In the C^∞ case, we answer how many are classical linear connections with a prescribed Ricci tensor on a 2-dimensional manifold. In the C^∞ case, we also deduce that any 2-form on the total space of a fibered manifold with at least 2-dimensional fibres can be realized locally as the Ricci tensor of a projectable classical linear connection.

Introduction

Except the last section, all manifolds considered in this paper are assumed to be finite dimensional and analytical. Maps between manifolds are assumed to be analytical.

A classical linear connection on a manifold N is a usual linear connection $\nabla : \mathcal{X}(N) \times \mathcal{X}(N) \to \mathcal{X}(N)$ as in Section III in the book [4].

Let $p : Y \to M$ be a fibered manifold. We recall that a classical linear connection ∇ on Y is projectable if there exists a (unique) classical linear connection $\underline{\nabla}$ on M being p-related with ∇. Let r be a (analytical) tensor field of type $(0, 2)$ on the total space Y.

In this note, using the Cauchy-Kowalevski theorem, we describe all local solutions ∇ of the equation

$$Ric(\nabla) = r \tag{1}$$

with unknown projectable classical linear connection ∇ on Y, where $Ric(\nabla)$ is the Ricci tensor field of ∇.

We also describe all local solutions ∇ of the equation (1) with unknown projectable classical linear connection ∇ with a prescribed trace of torsion tensor.

If M is a one point manifold, we get similar results to the ones from [8] (in particular, similar to the ones from [2, 3]). If M is a one point manifold and $r = 0$, we get similar results to the ones from [1].

In the last section, in the C^∞ case, we answer how many are classical linear connections with a prescribed Ricci tensor on a 2-dimensional manifold. In the C^∞ case, we also prove the existence of a local solution of equation (1) under the assumption that r is a 2-form (i.e. a skew-symmetric tensor field of type $(0, 2)$).

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1. Preliminaries

We adopt the notation $(f)_i = \frac{\partial f}{\partial x^i}$ for a function on a domain endowed with a (analytical) coordinate system $x^1, ..., x^n$. The Cauchy-Kowalevski theorem is the fundamental existence theorem for systems of partial differential equations with initial conditions, [6]. Recall the theorem of Cauchy-Kowalevski in the version we need for our considerations.

Theorem 1.1. *Consider a system of differential equations for unknown functions $U^1, ..., U^N$ in a neighborhood of $0 \in \mathbf{R}^n$ and of the form*

$$\begin{aligned}
(U^1)_n &= H^1(x^1, ..., x^n, U^1, ..., U^N, (U^1)_1, ..., (U^1)_{n-1}, ..., (U^N)_1, ..., (U^N)_{n-1}),\\
(U^2)_n &= H^2(x^1, ..., x^n, U^1, ..., U^N, (U^1)_1, ..., (U^1)_{n-1}, ..., (U^N)_1, ..., (U^N)_{n-1}),\\
&\ \,...\\
(U^N)_n &= H^N(x^1, ..., x^n, U^1, ..., U^N, (U^1)_1, ..., (U^1)_{n-1}, ..., (U^N)_1, ..., (U^N)_{n-1}),
\end{aligned}$$

where H^i, $i = 1, ..., N$, are analytic functions of all variables in a neighborhood of $(0, ..., 0, \varphi^1(0), ..., \varphi^N(0), (\varphi^1)_1(0), ..., (\varphi^1)_{n-1}(0), ..., (\varphi^N)_1(0), ..., (\varphi^N)_{n-1}(0)) \in \mathbf{R}^{(N+1)n}$ for analytic functions $\varphi^1, ..., \varphi^N$ given in a neighborhood of $0 \in \mathbf{R}^{n-1}$.

Then the system has a unique solution $(U^1(x^1, ..., x^n), ..., U^N(x^1, ..., x^n))$ which is analytic around $0 \in \mathbf{R}^n$ and satisfies the initial conditions

$$U^i(x^1, ..., x^{n-1}, 0) = \varphi^i(x^1, ..., x^{n-1}) \ \ for\ \ i = 1, ..., N.$$

2. The projectable classical linear connections with prescribed Ricci tensor

Let $x^1, ..., x^n$ be an adapted local coordinate system on Y such that $x^1, ..., x^m$ are base coordinates and $x^{m+1}, ..., x^n$ are fiber ones.

A classical linear connection ∇ on Y is projectable if and only if the Christoffel symbols Γ^i_{jk} of ∇ in any adapted coordinate system $x^1, ..., x^n$ (as above) satisfy

$$\Gamma^a_{bc} = \Gamma^a_{bc}(x^1, ..., x^m)\ ,\ \ \Gamma^a_{\alpha j} = 0\ ,\ \ \Gamma^a_{i\beta} = 0\ ,\ \ \Gamma^\gamma_{ij} = \Gamma^\gamma_{ij}(x^1, ..., x^n)\ , \tag{2}$$

for $a, b, c = 1, ..., m$ and $\alpha, \beta, \gamma = m + 1, ..., n$ and $i, j = 1, ..., n$.

We are going to find all local projectable classical linear connections ∇ on Y satisfying (1), or (equivalently) we are going to find all local solutions of the system of differential equations

$$\sum_{k=1}^{n} [(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i] + \sum_{k,l=1}^{n} [\Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{kj}\Gamma^k_{il}] = r_{ij}\ ,\ i, j = 1, ..., n\ , \tag{3}$$

where unknown maps $\Gamma^i_{jk} = \Gamma^i_{jk}(x^1, ..., x^n)$ satisfy additional conditions (2).

We put

$$\Lambda_{ij} = -\sum_{k,l=1}^{n} [\Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{kj}\Gamma^k_{il}]\ ,\ i, j = 1, ..., n\ , \tag{4}$$

and rewrite the system (3) into the form

$$[(\Gamma^1_{ij})_1 + ... + (\Gamma^n_{ij})_n] - [(\Gamma^1_{1j})_i + ... + (\Gamma^n_{nj})_i] = r_{ij} + \Lambda_{ij}\ ,\ i, j = 1, ..., n\ . \tag{5}$$

Assume $n - m \geq 2$. For $i = n$ and $j = 1, ..., n$ we keep each derivative $(\Gamma^{m+1}_{m+1,j})_n$ on the left-hand side of the corresponding equation. (If $n - m = 1$, $(\Gamma^{m+1}_{m+1,j})_n$ is not presented in (5) because $(\Gamma^n_{nj})_n - (\Gamma^{m+1}_{m+1,j})_{m+1} = 0$.) We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{nj} and move it to the right-hand side, and multiply both sides of such obtained equalities by −1. For $i \neq n$ and $j = 1, ..., n$, we keep each derivative $(\Gamma^n_{ij})_n$ on the left-hand side of the corresponding equation. We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{ij} and move it to the right-hand side. Then we obtain the (equivalent) system of n^2 equalities

$$\begin{aligned} &(\Gamma^{m+1}_{m+1,j})_n = -r_{nj} - \Lambda_{nj} + \Lambda'_{nj}\ ,\ j = 1, ..., n\ , \\ &(\Gamma^n_{ij})_n = r_{ij} + \Lambda_{ij} - \Lambda'_{ij}\ ,\ i = 1, ..., n\ , i \neq n,\ j = 1, ..., n\ , \end{aligned} \tag{6}$$

where Λ_{ij} are as in (4) and (after small reduction)

$$\begin{aligned} \Lambda'_{nj} = {} & [(\Gamma^1_{nj})_1 + ... + (\Gamma^{n-1}_{nj})_{n-1}] + \\ & -[(\Gamma^1_{1j})_n + ... + \widehat{(\Gamma^{m+1}_{m+1,j})_n} + ... + (\Gamma^{n-1}_{n-1,j})_n]\ ,\ \ j = 1, ..., n\ , \\ \Lambda'_{ij} = {} & [(\Gamma^1_{ij})_1 + ... + (\Gamma^{n-1}_{ij})_{n-1}] + \\ & -[(\Gamma^1_{1j})_i + ... + (\Gamma^n_{nj})_i]\ ,\ i, j = 1, ..., n,\ i \neq n\ , \end{aligned} \tag{7}$$

where $\widehat{(...)}$ means that the term is dropped.

We see that the first derivatives which are on the left-hand sides of system (6) are not presented in any term on the right-hand sides of system(6) (i.e. on the right-hand sides of (7)).

Theorem 2.1. *Let $p : Y \to M$ be an analytic fibered manifold with m-dimensional basis and $(n - m)$-dimensional fibers. Assume $n - m \geq 2$. Let r be an analytic tensor field of type $(0, 2)$ on some neighborhood of $0 \in Y$. The family of locally defined near 0 analytic projectable classical linear connections ∇ with the Ricci tensor equal to r depends bijectively on $(n - m)n^2 - n^2$ arbitrarily chosen analytic functions of n variables and m^3 arbitrarily chosen analytic functions of m variables and n^2 arbitrarily chosen analytic functions of $n - 1$ variables.*

In particular, if $n - m \geq 2$, any analytic tensor field of type $(0, 2)$ on Y can be realized locally as the Ricci tensor of an analytic projectable classical linear connection.

Proof. We can choose $(n-m)n^2 - n^2$ Christoffel symbols Γ^γ_{ij} with $\gamma = m+1, ..., n$ and $i, j = 1, ...n$ not presented in the left hand sides of (6) as arbitrary analytic functions of n variables $x^1, ..., x^n$ and we can choose m^3 Christoffel symbols Γ^c_{ab} with $a, b, c = 1, ..., m$ not presented in the left hand sides of (6) as arbitrary analytic functions of m variables $x^1, ..., x^m$ and we have to choose other Christoffel symbols not presented in the left hand sides of (6) equal to 0. Then n^2 analytic functions of $n - 1$ variables $x^1, ..., x^{n-1}$ appear by solving the system (6) by the Cauchy-Kowalevski theorem. □

Let $N = (N, \mathcal{F})$ be a n-dimensional manifold with a regular foliation $\mathcal{F}$ of $(n - m)$-dimensional leaves.

A vector field X on N is called a $\mathcal{F}$-infinitesimal automorphism if the flow of X is formed by local diffeomorphisms respecting leaves of $\mathcal{F}$.

Equivalently, a vector field X on N is an $\mathcal{F}$-infinitesimal automorphism if and only if $[X, V]$ is tangent to leaves of $\mathcal{F}$ for any vector field V on N tangent to leaves of $\mathcal{F}$.

A classical linear connection ∇ on N is called $\mathcal{F}$-adapted if $\nabla_X X_1$ is an $\mathcal{F}$-infinitesimal automorphism for any $\mathcal{F}$-infinitesimal automorphisms X and X_1 on N and $\nabla_X X_1$ is tangent to leaves of $\mathcal{F}$ for any $\mathcal{F}$-infinitesimal automorphisms X and X_1 on N such that X or X_1 is tangent to leaves of $\mathcal{F}$.

A foliated manifold $(N, \mathcal{F})$ (as above) is locally a fibered manifold with m-dimensional basis and $(n-m)$-dimensional fibres, and then a classical linear connection ∇ on N is $\mathcal{F}$-adapted if and only if it is locally projectable.

Thus we have the following foliation version of Theorem 2.1.

Corollary 2.2. *Let $N = (N, \mathcal{F})$ be an analytical n-dimensional manifold with a regular analytical foliation $\mathcal{F}$ of $(n - m)$-dimensional leaves. Assume $n - m \geq 2$. Let r be an analytical tensor field of type $(0, 2)$ on some neighborhood of $0 \in N$. The family of locally defined near 0 analytic $\mathcal{F}$-adapted classical linear connections ∇ on N with the Ricci tensor equal to r depends bijectively on $(n - m)n^2 - n^2$ arbitrarily chosen analytic functions of n-variables and m^3 arbitrarily chosen analytical functions of m variables and n^2 arbitrarily chosen analytic functions of $n - 1$ variables.*

In particular, if $n - m \geq 2$, any analytic tensor field of type $(0,2)$ on N can be realized locally as the Ricci tensor of an analytic $\mathcal{F}$-adapted classical linear connection.

For $m = 0$ we reobtain the following result from [8].

Corollary 2.3. *Let N be an analytical n-dimensional manifold. Assume $n \geq 2$. Let r be an analytical tensor field of type $(0,2)$ on some neighborhood of $0 \in N$. The family of locally defined near 0 analytical classical linear connections ∇ on N with the Ricci tensor equal to r depends bijectively on $n^3 - n^2$ arbitrarily chosen analytic functions of n-variables and n^2 arbitrarily chosen analytic functions of $n-1$ variables.*

In particular, if $n \geq 2$, any analytic tensor field of type $(0,2)$ on N can be realized locally as the Ricci tensor of an analytic classical linear connection.

For $n = 2$ we obtain the following result.

Corollary 2.4. *Let N be an analytical 2-dimensional manifold. Let r be an analytical tensor field of type $(0,2)$ on some neighborhood of $0 \in N$. The family of locally defined near 0 analytical classical linear connections ∇ on N with the Ricci tensor equal to r depends bijectively on 4 arbitrarily chosen analytic functions of 2-variables and 4 arbitrarily chosen analytic functions of 1 variable.*

In particular, if $n = 2$, any analytic tensor field of type $(0,2)$ on N can be realized locally as the Ricci tensor of an analytic classical linear connection.

In the last section of the present paper we get the C^∞-version of Corollary 2.4.

3. The projectable classical linear connections with a prescribed Ricci tensor and a prescribed trace of torsion tensor

Now, we are going to find all (local) projectable classical linear connections ∇ satisfying (1) and with the trace of torsion tensor equal to τ, or (equivalently) we are going to find all (local) solutions Γ^i_{jk} of the system (3) satisfying conditions (2) and

$$\sum_{k=1}^{n}(\Gamma^k_{kj} - \Gamma^k_{jk}) = \tau_j \ , \ j = 1, ..., n \ . \tag{8}$$

If $n - m \geq 2$, it is equivalent to find all (local) solutions of the system (6) satisfying conditions (2) and conditions (8).

Theorem 3.1. *Let $p : Y \to M$ be an analytical fibered manifold with m-dimensional basis and $(n-m)$-dimensional fibers. Assume $n - m \geq 2$. Let r be an analytic tensor field of type $(0,2)$ on some neighborhood of $0 \in Y$ and τ be an analytical 1-form on some neighborhood of $0 \in Y$. The family of locally defined near 0 real analytic projectable classical linear connections ∇ with the trace of torsion tensor equal to τ and with the Ricci tensor equal to r depends bijectively on $(n-m)n^2 - n^2 - n$ arbitrarily chosen analytic functions of n variables and m^3 arbitrarily chosen analytical functions of m variables and n^2 arbitrarily chosen analytic functions of $n-1$ variables.*

In particular, if $n - m \geq 2$, any analytical tensor field r on Y of type $(0,2)$ can be realized locally as the Ricci tensor of an analytic projectable classical linear connection with the trace of torsion tensor equal to τ.

Proof. For $j = 1, ..., n-1$ from j-th equation of (8) we compute Γ^n_{nj}. For $j = n$ from n-th equation of (8) we compute $\Gamma^{n-1}_{n,n-1}$. We get

$$\begin{aligned} \Gamma^n_{nj} &= [\Gamma^1_{j1} + ... + \widehat{\Gamma^j_{jj}} + ... + \Gamma^n_{jn}] + \\ &\quad -[\Gamma^1_{1j} + ... + \Gamma^j_{jj} + ... + \Gamma^{n-1}_{n-1,j}] + \tau_j \ , \quad j = 1, ..., n-1 \ , \\ \Gamma^{n-1}_{n,n-1} &= [\Gamma^1_{1n} + ... + \Gamma^{n-1}_{n-1,n}] + \\ &\quad -[\Gamma^1_{n1} + ... + \Gamma^{n-2}_{n,n-2}] - \tau_n \ , \quad j = n \ . \end{aligned} \tag{9}$$

One can easily see that the Christoffel symbols on the left-hand sides of the n equalities of (9) are different. They are not presented on the right-hand sides of the n equalities of (9). (Indeed, if Γ^n_{nj} is presented in the right-hand side of j_1-th equality then $\Gamma^n_{nj} = \Gamma^k_{j_1,k}$ or $\Gamma^n_{nj} = \Gamma^k_{k,j_1}$ for some k. Then $k = n = j_1$ or ($k = n$ and $j = j_1$). In the first case $\Gamma^n_{nj} = \Gamma^n_{nn}$, and we get a contradiction because in the right-hand side of the n-th equality Γ^n_{nn} is not presented. In the second case we get a contradiction because in the right-hand side of j-th equality Γ^n_{nj} is not presented, too. Further, if $\Gamma^{n-1}_{n,n-1}$ is presented in the right-hand side of j_1-th equality, we obtain a contradiction in similar way.) They are not presented on the left-hand sides of the n^2 equalities of (6), too. We substitute the above n equalities (9) into the n^2 equalities of (6). Then we obtain

$$\begin{aligned} &(\Gamma^{m+1}_{m+1,j})_n = -\tilde{\Lambda}_{nj} + \tilde{\Lambda}'_{nj} - r_{nj},\ j = 1, ..., n \\ &(\Gamma^n_{ij})_n = \tilde{\Lambda}_{ij} - \tilde{\Lambda}'_{ij} + r_{ij}\ , i = 1, ..., n\ ,\ i \neq n\ ,\ j = 1, ..., n\ , \end{aligned} \tag{10}$$

where $\tilde{\Lambda}_{ij}$ and $\tilde{\Lambda}'_{ij}$ are the Λ_{ij} (as in (4)) and Λ'_{ij} (as in (7)) after this substitution. Further, we see that the terms $(\Gamma^n_{nj})_n$ for $j = 1, ..., n-1$ and $(\Gamma^{n-1}_{n,n-1})_n$ are not presented in the right hand-sides of (7). Then $\tilde{\Lambda}'_{ij}$ and Λ'_{ij} have the same terms $(-)_n$. Then the first derivatives which are of the left-hand sides of the system (10) are not presented in any term of the right-hand sides. Now we can choose $(n-m)n^2 - n^2 - n$ Christoffel symbols Γ^γ_{ij}, $\gamma = m+1, ..., n$, $i, j = 1, ..., n$ which are not presented in the left-hand sides of (10) and (9) as arbitrary analytic functions of n variables $x^1, \ldots, x^n$ and we can choose m^3 Christoffel symbols Γ^c_{ab}, $a, b, c = 1, ..., m^3$ (not presented in the left-hand sides of (10) and (9)) as arbitrary analytical functions of m variables $x^1, \ldots, x^m$ and we have to choose other Christoffel symbols which are not presented in the left-hand sides of (10) and (9) to be 0. Then n^2 analytic functions of $n-1$ variables $x^1, \ldots, x^{n-1}$ appear by solving (10) by means of the Cauchy-Kowalevski theorem. □

Corollary 3.2. *Let $(N, \mathcal{F})$ be an analytical n-dimensional manifold with regular analytical foliation $\mathcal{F}$ of $(n-m)$-dimensional leaves. Assume $n - m \geq 2$. Let r be an analytical tensor field of type $(0, 2)$ on some neighborhood of $0 \in N$ and τ be an analytical 1-form on some neighborhood of $0 \in N$. The family of locally defined near 0 $\mathcal{F}$-adapted analytical classical linear connections ∇ on N with the trace of torsion tensor equal to τ and the Ricci tensor equal to r depends bijectively on $(n-m)n^2 - n^2 - n$ arbitrarily chosen analytic functions of n-variables and m^3 arbitrarily chosen analytic functions of m variables and n^2 arbitrarily chosen analytic functions of $n-1$ variables.*

In particular, if $n - m \geq 2$, any analytical tensor field r on N of type $(0, 2)$ can be realized locally as the Ricci tensor of an analytic $\mathcal{F}$-adapted classical linear connection with the trace of torsion tensor equal to τ.

If $m = 0$ we get the following result.

Corollary 3.3. *Let N be an analytical n-dimensional manifold. Assume $n \geq 2$. Let r be an analytical tensor field of type $(0, 2)$ on some neighborhood of $0 \in N$ and τ be an analytical 1-form on some neighborhood of $0 \in N$. The family of locally defined near 0 analytical classical linear connections ∇ on N with the trace of torsion tensor equal to τ and the Ricci tensor equal to r depends bijectively on $n^3 - n^2 - n$ arbitrarily chosen analytic functions of n-variables and n^2 arbitrarily chosen analytic functions of $n-1$ variables.*

In particular, if $n \geq 2$, any analytical tensor field r on N of type $(0, 2)$ can be realized locally as the Ricci tensor of an analytic classical linear connection with the trace of torsion tensor equal to τ.

If $m = 0$ and $\tau = 0$ we reobtain the following result from [8].

Corollary 3.4. *Let N be an analytical n-dimensional manifold. Assume $n \geq 2$. Let r be an analytical tensor field of type $(0, 2)$ on some neighborhood of $0 \in N$. The family of locally defined near 0 analytical classical linear connections ∇ on N with the vanishing trace of torsion tensor and the Ricci tensor equal to r depends bijectively on $n^3 - n^2 - n$ arbitrarily chosen analytic functions of n-variables and n^2 arbitrarily chosen analytic functions of $n-1$ variables.*

In particular, if $n \geq 2$, any analytical tensor field r on N of type $(0, 2)$ can be realized locally as the Ricci tensor of an analytic classical linear connection with the vanishing trace of torsion tensor.

In [8], Corollary 3.4 is proved under stronger assumption $n \geq 3$.

Any classical linear connection on a 2-dimensional manifold with vanishing trace of torsion tensor is torsion free. So, we get the following result.

Corollary 3.5. *Let N be an analytical 2-dimensional manifold. Let r be an analytical tensor field of type $(0,2)$ on some neighborhood of $0 \in N$. The family of locally defined near 0 analytical torsion free classical linear connections ∇ on N with the Ricci tensor equal to r depends bijectively on 2 arbitrarily chosen analytic functions of 2 variables and 4 arbitrarily chosen analytic functions of 1 variable.*

In particular, any analytical tensor field r of type $(0,2)$ on a 2-dimensional analytic manifold N can be realized locally as the Ricci tensor of an analytic torsion free classical linear connection on N.

The skew-symmetric part of the Ricci tensor of a torsion free classical linear connection is exact (and then closed), see [7]. In [8], the authors proved the following result.

Theorem 3.6. *([8]) Let N be an analytical n-dimensional manifold. Assume $n \geq 2$. Let r be an analytical tensor field of type $(0,2)$ on some neighborhood of $0 \in N$ such that the skew-symmetric part of r is closed. Then r can be realized locally as the Ricci tensor of an analytic torsion free classical linear connection on N. The family of locally defined near 0 analytical torsion free classical linear connections ∇ on N with the Ricci tensor equal to r depends bijectively on $\frac{n^3-3n}{2}+1$ arbitrarily chosen analytic functions of n variables and $\frac{n^2+n}{2}$ arbitrarily chosen analytic functions of $n-1$ variables.*

Any 2-form on 2-dimensional manifold is closed. Thus for $n = 2$ we obtain the following result.

Corollary 3.7. *Let N be an analytical 2-dimensional manifold. Let r be an analytical tensor field of type $(0,2)$ on some neighborhood of $0 \in N$. The family of locally defined near 0 analytical torsion free classical linear connections ∇ on N with the Ricci tensor equal to r depends bijectively on 2 arbitrarily chosen analytic functions of 2 variables and 3 arbitrarily chosen analytic functions of 1 variable.*

We see that the number of arbitrarily chosen analytical functions of one variable in Corollary 3.7 is different than the one in Corollary 3.5. The reason is the following one. These bijections are different.

4. Two similar results in the C^∞ situation

All manifolds and maps considered in this section are assumed to be of class C^∞.
In the case $m = 0$ and $n = 2$ equations (6) are

$$\begin{aligned}
(\Gamma^1_{1,1})_2 &= (\Gamma^1_{2,1})_1 - r_{2,1} - \Lambda_{2,1}\,, \\
(\Gamma^1_{1,2})_2 &= (\Gamma^1_{2,2})_1 - r_{2,2} - \Lambda_{2,2}\,, \\
(\Gamma^2_{1,1})_2 &= (\Gamma^2_{2,1})_1 + r_{1,1} + \Lambda_{1,1}\,, \\
(\Gamma^2_{1,2})_2 &= (\Gamma^2_{2,2})_1 + r_{1,2} + \Lambda_{1,2}\,.
\end{aligned} \tag{11}$$

If we choose $\Gamma^1_{2,1}, \Gamma^1_{2,2}, \Gamma^2_{2,1}$ and $\Gamma^2_{2,2}$ as arbitrary functions of two variables x^1, x^2 of class C^∞ , then we obtain the system of ordinary differential equations of first order (of the form $y' = F(t, x, y)$ with unknown function $y : \mathbf{R} \to \mathbf{R}^4$ of one variable x) with parameter. Thus we obtain the following C^∞-version of Corollary 2.4.

Theorem 4.1. *Let N be a 2-dimensional manifold of class C^∞. Let r be a tensor field of class C^∞ of type $(0,2)$ on some neighborhood of $0 \in N$. The family of locally defined near 0 classical linear connections ∇ on N of class C^∞ with the Ricci tensor equal to r depends bijectively on 4 functions of 2 variables of class C^∞ and 4 functions of 1 variable of class C^∞.*

In particular, any tensor field of class C^∞ of type $(0,2)$ on a 2-dimensional manifold can be realized locally as the Ricci tensor of a classical linear connection.

We end this note by the following realization theorem (in the C^∞-situation).

Theorem 4.2. *Let $p : Y \to M$ be a fibered manifold with m-dimensional basis and $(n-m)$-dimensional fibers of class C^∞. If $n - m \geq 2$, any 2-form r (i.e. skew-symmetric tensor field r of type $(0, 2)$) on Y of class C^∞ can be realized locally as the Ricci tensor of a projectable classical linear connection of class C^∞.*

Proof. Let r be a 2-form on Y of class C^∞. It is sufficient to show that the system (3) has a local solution (Γ^k_{ij}) near 0 of class C^∞ satisfying conditions (2) and the following additional conditions

$$\begin{aligned} &\Gamma^k_{ij} = 0 \ , \ \ k = 1, ..., n-2 \ , \ i, j = 1, ..., n \ , \\ &\Gamma^{n-1}_{ij} = 0 \ , \ \ i, j = 1, ..., n-1 \ , \\ &\Gamma^{n-1}_{in} = 0 \ , \ \ i = 1, ..., n \ , \\ &\Gamma^{n}_{nj} = 0 \ , \ \ j = 1, ..., n \ . \end{aligned} \tag{12}$$

Under assumptions (12), system (3) is equivalent to the system of systems of differential equations

$$\begin{aligned} &0 = r_{nn} \ , \\ &(\Gamma^{n-1}_{nj})_{n-1} = r_{nj} \ , \ \ j = 1, ..., n-1 \ , \\ &(\Gamma^{n}_{in})_n = r_{in} \ , \ \ i = 1, ..., n-1 \ , \\ &(\Gamma^{n}_{ij})_n - \Gamma^{n-1}_{nj}\Gamma^{n}_{i,n-1} = r_{ij} \ , \ \ i, j = 1, ..., n-1 \ . \end{aligned} \tag{13}$$

It remains to observe that the system of systems (13) has a solution of class C^∞ on some neighborhood of 0. The first equation of (13) is satisfied if r is skew-symmetric. Integrating both sides of each equation of the second system of (13) with respect to the $(n-1)$-th variable we get a solution Γ^{n-1}_{nj} of class C^∞ of the second system of (13), where $j = 1, ..., n-1$. Similarly, integrating with respect to the n-th variable, we get a solution Γ^n_{in} of class C^∞ of the third system of (13), where $i = 1, ..., n-1$. Substituting the obtained Γ^{n-1}_{nj} into the fourth system of (13) we get the system of ordinary first order differential equations with parameters $x^1, ..., x^{n-1}$. Such obtained system has a local solution of class C^∞ according to the well-known theory of differential equations. □

For $m = 0$, we reobtain the following result of [5].

Corollary 4.3. *([5]) Let N be a n-dimensional manifold of class C^∞. If $n \geq 2$, any 2-form r on N of class C^∞ can be realized locally as the Ricci tensor of a classical linear connection on N of class C^∞.*

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