



Two Weak Solutions for Fully Nonlinear Kirchhoff-Type Problem

Abdolrahman Razani^a

^aDepartment of Pure Mathematics, Faculty of Science,
Imam Khomeini International University,
Qazvin, Iran, P.O. Box 34194-168181.

Abstract. In this article, the existence of solutions for fully nonlinear Kirchhoff-type problem

$$-M \left(\int (\Phi(|\nabla u|) + \Phi(|u|)) dx \right) [\operatorname{div}(a(|\nabla u|)\nabla u) + a(|u|)u] = \lambda \sum_{i=1}^k (t^{q_i(x)-1} - t^{r_i(x)-1})$$

is proved via variational method. Finally, some new problems are introduced.

1. Introduction

Partial differential equation is an interdisciplinary area that one may study many physical phenomena (see [2, 8–11, 25–36]). One of the main problems in this area is wave equation. The wave equation is an important second-order linear partial differential equation for the description of waves. Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond D’Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange. In 1746, D’Alembert discovered the one-dimensional wave equation, and within ten years Euler discovered the three-dimensional wave equation. Later on, Kirchhoff [14] proposed the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0. \quad (1)$$

This equation is an extension of the classical D’Alembert’s wave equation by considering the effects of the length changes of the string produced by transverse vibrations. The parameters in (1) have the following meanings: h is the cross-section area, E is the Young modulus, ρ is the mass density, L is the length of the string, and P_0 is the initial tension. In recent years, p -Kirchhoff type problems have been studied by many researchers, we refer to [4, 12–21, 38].

Here we consider the problem

$$\begin{cases} -M \left(\int (\Phi(|\nabla u|) + \Phi(|u|)) dx \right) [\operatorname{div}(a(|\nabla u|)\nabla u) + a(|u|)u] = \lambda f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

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Email address: razani@sci.ikiu.ac.ir (Abdolrahman Razani)

where $M : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function, λ is a positive real parameter, Ω is a bounded domain in \mathbb{R}^N ($N \geq 3$) with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$. Set

$$f(x, t) = \begin{cases} \sum_{i=1}^k (t^{q_i(x)-1} - t^{r_i(x)-1}) & t \geq 0, \\ 0 & t < 0, \end{cases} \tag{3}$$

for $x \in \Omega$, where $q_i(x), r_i(x)$ are given by (11). Assume that $a : (0, \infty) \rightarrow \mathbb{R}$ is a function such that

$$\varphi(t) := \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases} \tag{4}$$

is an odd, increasing homeomorphism from \mathbb{R} onto \mathbb{R} and

$$\Phi(t) = \int_0^t \varphi(s)ds \quad \text{for all } t \in \mathbb{R}. \tag{5}$$

Notice that if $\varphi(t) = p|t|^{p-2}t$, then problem (2) becomes the well-known p -Kirchhoff type equation

$$\begin{cases} -M(\int_{\Omega} (|\nabla u|^p + |u|^p)dx)(\Delta_p u + |u|^{p-2}u) = \lambda f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

which is related to the stationary version of the Kirchhoff equation (1). Since the first equation in (6) contains an integral over Ω , it is no longer a pointwise identity, and therefore it is often called a nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population identity, see [5].

The existence of infinitely many solutions for

$$\begin{cases} -M(\int_{\Omega} \Phi(|\nabla u|)dx)div(a(|\nabla u|)\nabla u) = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is proved where $M : [0, +\infty) \rightarrow \mathbb{R}$ is a continuous function such that there exists a positive number m with $M(t) \geq m$ for all $t \geq 0$, (see [20]). Also problem (2) studied in [3], where $M(t) \equiv 1$. Recently, in [21], problem (2) is studied where $f(x, t) = t^{q(x)-1} - t^{r(x)-1}$ and the existence of solutions is proved. Here we consider an extension of $f(x, t)$ by (3) and generalize the result of [21]. In fact, we study the existence of weak solutions for problem (2). Here is the main result.

Theorem 1.1. *Let conditions (12) and (13) be satisfied. Then there exists a constant $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$, problem (14) admits at least two non-negative nontrivial weak solutions.*

In order to prove the main theorem, we apply the minimum principle [37] to prove that the problem (12) admits a non-negative, non-trivial weak solution (call u_1) as the global minimizer of J . The existence of this solution (as a first solution) is proved in Section 2.1. Also we apply the Mountain Pass Theorem to prove the existence of second nontrivial weak solution (call u_2) of (14) (see Section 2.2). The proof shows that the weak solutions u_2 and u_1 are distinct, since $J(u_2) = \bar{c} > 0 > J(u_1)$. Thus we need to prove the existence of u_1 and u_2 , separately. Due to do this, we introduce the Orlicz–Sobolev space as a suitable function space.

Notice that Φ defined in (5) is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{t \rightarrow \infty} \Phi(t) = +\infty$. Furthermore since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0$, and $\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = +\infty$, the function Φ is then called an N -function. The function Φ^* defined by

$$\Phi^*(t) = \int_0^t \varphi^{-1}(s)ds \quad \text{for all } t \in \mathbb{R},$$

is called the Complementary function of Φ and it satisfies

$$\Phi^*(t) = \sup\{st - \Phi(s) : s \geq 0\} \quad \text{for all } t \geq 0.$$

Notice that Φ^* is also an N -function and satisfies the following Young inequality

$$st \leq \Phi(s) + \Phi^*(t) \quad \text{for all } s, t \geq 0.$$

Assume that

$$1 < \liminf_{t \rightarrow \infty} \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 < \infty, \tag{7}$$

$$N < \varphi_0 \leq \liminf_{t \rightarrow \infty} \frac{\log(\Phi(t))}{\log(t)}, \tag{8}$$

where

$$\varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \quad \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}.$$

The set $K_\Phi(\Omega)$ which is defined by

$$K_\Phi(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \int_\Omega \Phi(|u(x)|) dx < \infty \right\}$$

for the N -function Φ is called the Orlicz class. The Orlicz space $L_\Phi(\Omega)$ is defined by the linear hull of the set $K_\Phi(\Omega)$. Considering Luxemburg norm

$$\|u\|_\Phi := \inf \left\{ k > 0 : \int_\Omega \Phi\left(\frac{u(x)}{k}\right) dx \leq 1 \right\},$$

the Orlicz space $L_\Phi(\Omega)$ is a Banach space. Also $\|u\|_\Phi$ is equivalent to the Orlicz norm

$$\|u\|_{L_\Phi} := \sup \left\{ \left| \int_\Omega u(x)v(x) dx \right| : v \in K_{\Phi^*}(\Omega), \int_\Omega \Phi^*(|v(x)|) dx \leq 1 \right\}.$$

The Hölder inequality holds for Orlicz spaces as follows (see [28])

$$\int_\Omega uv dx \leq 2\|u\|_{L_\Phi} \|v\|_{L_{\Phi^*}} \quad \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\Phi^*}(\Omega).$$

The Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is defined by

$$W^{1,\Phi}(\Omega) := \left\{ u \in L_\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), i = 1, 2, \dots, N \right\},$$

and it is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \|\nabla u\|_\Phi.$$

This norm is equivalent to the following Luxemburg-norm

$$\|u\| := \inf \left\{ \mu > 0 : \int_\Omega \Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right) dx \leq 1 \right\}.$$

Condition (7) assures that Φ satisfies the Δ_2 -condition, i.e.

$$\Phi(2t) \leq l\Phi(t) \quad \text{for all } t \geq 0, \tag{9}$$

where l is a positive constant, see [23, proposition 2.3]. In the sequel we also need the condition

$$\text{the function } t \mapsto \Phi(\sqrt{t}) \text{ is convex for all } t \geq 0. \tag{10}$$

Set $X := W^{1,\Phi}(\Omega)$. Notice that the spaces $L_\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable, reflexive Banach spaces, see [1, p. 241 and p. 247]. Notice that for $1 \leq q < \varphi_0^* := \frac{N\varphi_0}{N-\varphi_0}$, $W^{1,\Phi}$ is continuously and compactly embedded in the classical Lebesgue space $L^q(\Omega)$ (see [3, 6]).

Finally, we remind [27, Mountain Pass Theorem], which is the main tool in our problem. Also we recall the Palais-Smale condition (see [27]).

Definition 1.2. A continuously Fréchet differentiable functional $I \in C^1(H, \mathbb{R})$ from a Hilbert space H to the reals satisfies the Palais-Smale (PS) condition if every sequence $\{u_k\}_{k=1}^\infty \subset H$ such that:

(I) $\{I[u_k]\}_{k=1}^\infty$ is bounded, and

(II) $I'[u_k] \rightarrow 0$ in H

has a convergent subsequence in H .

Theorem 1.3. Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying PS condition. Suppose $I(0) = 0$ and

(I₁) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$,

(I₂) there is an $e \in E \setminus B_\rho$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0, 1], E); g(0) = 0, g(1) = e\}.$$

In the next section we study the existence of two weak solutions for the problem (2).

2. Two weak solutions

In this section we consider a general form of $f(x, t)$ as (3) and for $1 \leq i \leq k$

$$q_i, r_i \in C_+(\overline{\Omega}) := \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\} \tag{11}$$

are such that

$$\begin{aligned} 1 &< r_- := \inf_{x \in \Omega} \{r_i(x) : 1 \leq i \leq k\} \leq r_i(x) \leq r_+ := \sup_{x \in \Omega} \{r_i(x) : 1 \leq i \leq k\} \\ &< q_- := \inf_{x \in \Omega} \{q_i(x) : 1 \leq i \leq k\} \leq q_i(x) \leq q_+ := \sup_{x \in \Omega} \{q_i(x) : 1 \leq i \leq k\} \\ &< \rho\varphi_0, \end{aligned}$$

with

$$\rho \in \left(1, \min \left\{ \frac{N}{\varphi_0}, \frac{N\varphi_0}{\varphi_0(N - \varphi_0)} \right\} \right). \tag{12}$$

Also, we suppose that $M : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function satisfying

$$M(t) \geq k_0 t^{\rho-1} \quad \text{for all } t \in \mathbb{R}^+ \tag{13}$$

where k_0 is a positive constant. Here we assume the Kirchhoff function M is not degenerate i.e. set $\rho = 1$.

Consider the following nonlocal problem

$$\begin{cases} -M \left(\int_\Omega \Phi(|\nabla u|) + \Phi(|u|) dx \right) (\text{div}(a(|\nabla u|)\nabla u) + a(|u|)u) = \lambda f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Definition 2.1. A function $u : \Omega \rightarrow \mathbb{R}$ is said to be a weak solution of problem (14) if $u \in X$ ($u \geq 0$ a.e in Ω) and

$$M\left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|)dx\right) \int_{\Omega} (a(|\nabla u|)\nabla u \cdot \nabla v + a(|u|)uv)dx - \lambda \sum_{i=1}^k \int_{\Omega} (u^{q_i(x)-1} - u^{r_i(x)-1})v dx = 0,$$

for all $v \in X$.

We define the energy functional $J : X \rightarrow \mathbb{R}$ by

$$J(u) = \hat{M}\left(\int_{\Omega} [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx\right) \tag{15}$$

$$- \lambda \sum_{i=1}^k \int_{\Omega} \left(\frac{1}{q_i(x)} u_+^{q_i(x)} - \frac{1}{r_i(x)} u_+^{r_i(x)}\right) dx = \mathcal{M}(u) - \lambda \mathcal{F}(u), \quad u \in X, \tag{16}$$

where

$$\begin{aligned} \mathcal{M} &= \hat{M}\left(\int_{\Omega} \Phi(|\nabla u(x)|) + \Phi(|u(x)|)dx\right), \\ \hat{M} &:= \int_0^t M(s)ds, \\ \mathcal{F}(u) &= \sum_{i=1}^k \int_{\Omega} \left(\frac{1}{q_i(x)} u_+^{q_i(x)} - \frac{1}{r_i(x)} u_+^{r_i(x)}\right) dx, \\ u_{\pm}(x) &= \max\{\pm u(x), 0\}. \end{aligned} \tag{17}$$

We know that $u \in X$ implies $u_+, u_- \in X$ and

$$\nabla u_+ = \begin{cases} 0 & \text{for } u \leq 0, \\ \nabla u & \text{for } u > 0, \end{cases} \quad \text{and} \quad \nabla u_- = \begin{cases} 0 & \text{for } u \geq 0, \\ \nabla u & \text{for } u < 0. \end{cases}$$

The following proposition is from [9].

Proposition 2.2. Let $u \in W^{1,\Phi}(\Omega)$, then

$$\begin{aligned} \int_{\Omega} \Phi(|u|) + \Phi(|\nabla u|)dx &\geq \|u\|^{\varphi_0}; & \text{if } \|u\| < 1 \\ \int_{\Omega} \Phi(|u|) + \Phi(|\nabla u|)dx &\geq \|u\|^{\varphi_0}; & \text{if } \|u\| > 1 \\ \int_{\Omega} \Phi(|u|) + \Phi(|\nabla u|)dx &\leq \|u\|^{\varphi_0}; & \text{if } \|u\| < 1 \\ \int_{\Omega} \Phi(|u|) + \Phi(|\nabla u|)dx &\leq \|u\|^{\varphi_0}; & \text{if } \|u\| > 1 \end{aligned}$$

Lemma 2.3. There exists $\lambda_0 > 0$ such that

$$\lambda_0 = \inf_{\|u\|>1} \frac{k_0 \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|)dx\right)^{\rho}}{\rho \int_{\Omega} |u|^{\rho\varphi_0} dx}.$$

Proof. Since $X = W^{1,\Phi}(\Omega)$ is continuously embedded in $W^{1,\varphi_0}(\Omega)$ and consequently it is continuously embedded in $L^{\rho\varphi_0}(\Omega)$, there exists $C_1 > 0$ such that

$$\|u\| \geq C_1 \|u\|_{L^{\rho\varphi_0}(\Omega)}, \text{ for all } u \in X. \tag{18}$$

On the other hand, Proposition 2.2 shows that

$$\int_{\Omega} [\Phi(|\nabla u|) + \Phi(|u|)] dx \geq \|u\|^{\varphi_0}, \text{ for all } u \in X. \text{ with } \|u\| > 1 \tag{19}$$

Now from (18) and (19), we have

$$\begin{aligned} k_0 \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right)^{\rho} &\geq k_0 \|u\|^{\rho\varphi_0} \\ &\geq k_0 C_1^{\rho\varphi_0} \int_{\Omega} |u|^{\rho\varphi_0} dx, \end{aligned} \tag{20}$$

for all $u \in X$ with $\|u\| > 1$, which implies that

$$\lambda_0 := \inf_{\|u\|>1} \frac{k_0 \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right)^{\rho}}{\rho \int_{\Omega} |u|^{\rho\varphi_0} dx}$$

exists and the proof is complete. \square

Also Proposition 2.2 and the continuous embedding (obtained from hypothesis (12)) imply that the functional J is well-defined on X , $J \in C^1(X)$ and

$$\begin{aligned} J'(u)(v) &= M \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right) \int_{\Omega} (a(|\nabla u|) \nabla u \cdot \nabla v + a(|u|) uv) dx \\ &\quad - \lambda \sum_{i=1}^k \left(\int_{\Omega} u_+^{q_i(x)-1} v dx - \int_{\Omega} u_+^{r_i(x)-1} v dx \right) \end{aligned}$$

for all $u, v \in X$. The critical points of the functional J are the weak solutions of problem (14). In fact,

$$\begin{aligned} 0 &= J'(u)(u_-) \\ &= M \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right) \int_{\Omega} (a(|\nabla u|) \nabla u \cdot \nabla u_- + a(|u|) uu_-) dx \\ &\quad - \lambda \sum_{i=1}^k \int_{\Omega} (u_+^{q_i(x)-1} u_- - u_+^{r_i(x)-1} u_-) dx \\ &\geq k_0 \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right)^{\rho-1} \int_{\Omega} a(|\nabla u|) \nabla u \cdot \nabla u_- + a(|u|) uu_- dx \\ &= k_0 \left(\int_{\Omega} \Phi(|\nabla u_-|) + \Phi(|u_-|) dx \right)^{\rho-1} \int_{\Omega} a(|\nabla u_-|) |\nabla u_-|^2 + a(|u_-|) |u_-|^2 dx \\ &\geq k_0 \varphi_0 \left(\int_{\Omega} \Phi(|\nabla u_-|) + \Phi(|u_-|) dx \right)^{\rho}. \end{aligned}$$

Proposition 2.2 and (13) show $u \geq 0$. This means that the nontrivial critical points of J are non-negative, nontrivial weak solutions of problem (14).

2.1. First solution

By a standard argument, if the functional J is coercive, bounded from below and weakly lower semicontinuous, then the problem (14) admits at least one weak solution u_1 which is the global minimizer of the functional J . Due to this, we study these properties for the functional J (see [37] for more details).

Lemma 2.4. *The functional J which is given by the formula (15) is coercive, bounded from below and weakly lower semicontinuous.*

Proof. Due to show that the functional J is coercive and bounded from blow, (12), implies

$$\lim_{t \rightarrow \infty} \frac{\sum_{i=1}^k \left(\frac{1}{q_i(x)} t^{q_i(x)} - \frac{1}{r_i(x)} t^{r_i(x)} \right)}{t^{\rho\varphi_0}} = 0$$

uniformly in $x \in \Omega$. Then for any $\lambda > 0$ there exists $C_\lambda > 0$ depending on λ which

$$\lambda \sum_{i=1}^k \left(\frac{1}{q_i(x)} t^{q_i(x)} - \frac{1}{r_i(x)} t^{r_i(x)} \right) \leq \frac{\lambda_0}{2} t^{\rho\varphi_0} + C_\lambda \tag{21}$$

for all $t \geq 0$ and all $x \in \Omega$, where λ_0 is defined in Lemma 2.3. Combining (21) and (13),

$$\begin{aligned} J(u) &= \hat{M} \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right) \\ &\quad - \lambda \sum_{i=1}^k \int_{\Omega} \left(\frac{1}{q_i(x)} u_+^{q_i(x)} - \frac{1}{r_i(x)} u_+^{r_i(x)} \right) V dx \\ &\geq \frac{k_0}{\rho} \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right)^\rho - \frac{\lambda_0}{2} \int_{\Omega} |u|^{\rho\varphi_0} dx - C_\lambda |\Omega| \\ &\geq \frac{k_0}{2\rho} \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right)^\rho - C_\lambda |\Omega| \\ &\geq \frac{k_0}{2\rho} \|u\|^{\rho\varphi_0} - C_\lambda |\Omega| \end{aligned} \tag{22}$$

for any $u \in X$ with $\|u\| > 1$. This shows that the functional J is coercive and bounded from blow.

Now we prove J is weakly lower semicontinuous. Suppose $\{u_m\}$ is a sequence such that $u_m \rightharpoonup u$ in X . First, by (12), one has compact embeddings $X \hookrightarrow L^{q_i(x)}(\Omega)$ and $X \hookrightarrow L^{r_i(x)}(\Omega)$, for $1 \leq i \leq k$. Therefore $\lim_{m \rightarrow \infty} \mathcal{F}(u_m) = \mathcal{F}(u)$. Secondly, (see the proof of [23, Lemma 4.3]) the functional

$$u \mapsto \int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx$$

is weakly lower semicontinuous, i.e.

$$\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \leq \liminf_{m \rightarrow \infty} \int_{\Omega} \Phi(|\nabla u_m|) + \Phi(|u_m|) dx. \tag{24}$$

From (24) and the continuity and monotonicity of the function $t \mapsto \psi(t) = \hat{M}(t)$, we have

$$\begin{aligned} \liminf_{m \rightarrow \infty} \mathcal{M}(u_m) &= \liminf_{m \rightarrow \infty} \hat{M} \left(\int_{\Omega} \Phi(|\nabla u_m|) + \Phi(|u_m|) dx \right) \\ &\geq \hat{M} \left(\liminf_{m \rightarrow \infty} \int_{\Omega} \Phi(|\nabla u_m|) + \Phi(|u_m|) dx \right) \\ &\geq \hat{M} \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right) \\ &\geq \mathcal{M}(u). \end{aligned} \tag{25}$$

Thus

$$J(u) = \mathcal{M}(u) - \lambda \mathcal{F}(u) \leq \liminf_{m \rightarrow \infty} \mathcal{M}(u_m) - \lambda \mathcal{F}(u_m) = \liminf_{m \rightarrow \infty} J(u_m)$$

and the proof is complete. \square

We can show u_1 is not trivial.

Lemma 2.5. *There exists $\lambda^* > 0$ such that $\inf_{u \in X} J(u) < 0$ for any $\lambda \geq \lambda^*$.*

Proof. We recall that q^-, r^+ are given by (11) and one can choose a number $t_0 > 1$ such that

$$t_0^{q^- - r^+} > 2 \frac{q^+}{r^-}, \tag{26}$$

since $1 < r^+ < q^-$. Let $\Omega_0 \subset \Omega$ be a compact subset and large enough. Also, choose a function $u_0 \in C_0^\infty(\Omega)$ with $u_0(x) = t_0$ in Ω_0 , and $0 \leq u_0(x) \leq t_0$ in $\Omega \setminus \Omega_0$. From (26) we get

$$t_0^{q(x) - r(x)} > \frac{q(x)}{r(x)} + \frac{q^+}{r^-} \quad \text{in } \Omega_0,$$

or

$$\frac{1}{q(x)} t_0^{q(x)} > \frac{1}{r(x)} t_0^{r(x)} + \frac{q^+ t_0^{r(x)}}{r^- q(x)} > \frac{1}{r(x)} t_0^{r(x)} + \frac{1}{r^-} \quad \text{in } \Omega_0. \tag{27}$$

This shows that for Ω_0 large enough,

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^k \left(\frac{1}{q_i(x)} u_0^{q_i(x)} - \frac{1}{r_i(x)} u_0^{r_i(x)} \right) dx \\ & \geq \sum_{i=1}^k \left(\int_{\Omega_0} \frac{1}{q_i(x)} u_0^{q_i(x)} dx - \int_{\Omega_0} \frac{1}{r_i(x)} u_0^{r_i(x)} dx - \int_{\Omega \setminus \Omega_0} \frac{1}{r_i(x)} u_0^{r_i(x)} dx \right) \\ & \geq \sum_{i=1}^k \left(\frac{1}{r^-} |\Omega_0| - \frac{t_0^{r^+}}{r^-} |\Omega \setminus \Omega_0| \right) \\ & = k \left(\frac{1}{r^-} |\Omega_0| - \frac{t_0^{r^+}}{r^-} |\Omega \setminus \Omega_0| \right) > 0. \end{aligned}$$

Therefore

$$\begin{aligned} J(u_0) &= \hat{M} \left(\int_{\Omega} \Phi(|\nabla u_0|) + \Phi(|u_0|) dx \right) - \lambda \sum_{i=1}^k \int_{\Omega} \left(\frac{1}{q_i(x)} u_+^{q_i(x)} - \frac{1}{r_i(x)} u_+^{r_i(x)} \right) dx \\ &\leq C_2 - \lambda k \left(\frac{1}{r^-} |\Omega_0| - \frac{t_0^{r^+}}{r^-} |\Omega \setminus \Omega_0| \right), \end{aligned}$$

where $C_2 > 0$ is a constant. Hence, there exists $\lambda^* > 0$ such that for any $\lambda \in [\lambda^*, \infty)$, $J(u_0) < 0$. It shows that $\inf_{u \in X} J(u) < 0$, and then $J(u_1) < 0$ for any $\lambda \geq \lambda^*$, therefore, u_1 is a non-negative and non-trivial weak solution for problem (14). \square

2.2. Second solution

We prove the existence of the second solution. In fact, by applying Theorem 1.3 we obtain the second weak solution $u_2 \in X$. To this aim, first we show that the functional J has the geometry of the Mountain Pass Theorem for all $\lambda \geq \lambda^*$ through the following lemmas.

Lemma 2.6. *There exist $\rho \in (0, \|u_1\|)$ and a positive constant R such that $J(u) \geq R$ for all $u \in X$ with $\|u\| = \rho$.*

Proof. For fixed $u \in X$ with $\|u\| < 1$, we have

$$\sum_{i=1}^k \frac{1}{q_i(x)} t^{q_i(x)} - \sum_{i=1}^k \frac{1}{r_i(x)} t^{r_i(x)} \leq 0 \tag{28}$$

for all $t \in [0, 1]$ and $x \in \Omega$, since $q^- > r^+$.

We define the following set for the above function u

$$\Omega_u := \{x \in \Omega : u(x) > 1\}.$$

For $x \in \Omega \setminus \Omega_u$ one has $0 \leq u_+ \leq 1$ then from (28) we get

$$F(x, u) = \sum_{i=1}^k \frac{1}{q_i(x)} u_+^{q_i(x)} - \sum_{i=1}^k \frac{1}{r_i(x)} u_+^{r_i(x)} \leq 0 \quad \text{for all } x \in \Omega. \tag{29}$$

On the other hand, there exists a constant $\gamma \in (\rho\varphi^0, \frac{N\varphi_0}{N-\varphi_0})$ such that

$X \hookrightarrow L^\gamma(\Omega)$ since $\rho\varphi^0 < \min\{N, \frac{N\varphi_0}{N-\varphi_0}\}$. Hence, there exists a constant $C_3 > 0$ such that

$$\|u\|_{L^\gamma(\Omega)} \leq C_3 \|u\| \quad \text{for all } u \in X. \tag{30}$$

Now, combining (13), (29), (30) and Proposition 2.2, we have

$$\begin{aligned} J(u) &= \hat{M} \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right) - \int_{\Omega} F(x, u) dx \\ &\geq \frac{k_0}{\rho} \left(\int_{\Omega} \Phi(|\nabla u|) + \Phi(|u|) dx \right) - \lambda \int_{\Omega_u} F(x, u) dx - \lambda \int_{\Omega \setminus \Omega_u} F(x, u) dx \\ &\geq \frac{k_0}{\rho} \|u\|^{\rho\varphi^0} - \lambda \int_{\Omega_u} F(x, u) dx \\ &= \frac{k_0}{\rho} \|u\|^{\rho\varphi^0} - \lambda \sum_{i=1}^k \int_{\Omega_u} \left(\frac{1}{q_i(x)} u_+^{q_i(x)} - \frac{1}{r_i(x)} u_+^{r_i(x)} \right) dx \\ &\geq \frac{k_0}{\rho} \|u\|^{\rho\varphi^0} - \frac{\lambda}{q^-} \sum_{i=1}^k \int_{\Omega_u} u_+^{q_i(x)} dx \\ &\geq \frac{k_0}{\rho} \|u\|^{\rho\varphi^0} - \frac{\lambda k}{q^-} \int_{\Omega_u} u_+^\gamma dx \\ &\geq \frac{k_0}{\rho} \|u\|^{\rho\varphi^0} - \frac{\lambda k C_3^\gamma}{q^-} \|u\|^\gamma \\ &= \left(\frac{k_0}{\rho} - \frac{\lambda k C_3^\gamma}{q^-} \|u\|^{|\gamma - \rho\varphi^0|} \right) \|u\|^{\rho\varphi^0} \end{aligned}$$

Since $\rho\varphi^0 < \gamma$, there exists positive constant ρ small enough with $\rho < \|u_1\|$, such that for any $u \in X$ with $\|u\| = \rho$, one has $J(u) \geq R > 0$. \square

If the functional J satisfies the PS condition, then Mountain Pass Theorem 1.3 shows there exists $u_2 \in X$ such that $J'(u_2)(v) = 0$ for all $v \in X$. Thus u_2 is the nontrivial second weak solution of (14). Thus it remains the check the PS condition.

Lemma 2.7. *The functional J satisfies the PS condition.*

Proof. Let $\{u_m\}$ be a sequence in X such that

$$J(u_m) \rightarrow \bar{c} > 0, \quad J'(u_m) \rightarrow 0 \quad \text{in } X^*. \tag{31}$$

It follows from (31) that $\{u_m\}$ is bounded in X since the functional J is coercive. On the other hand, since the Banach space X is reflexive, there exists $u \in X$ such that passing to a subsequence, denoted by $\{u_m\}$, $u_m \rightharpoonup u$

in X . Therefore, $\{u_m\}$ converges strongly to u in $L^{q_i(x)}(\Omega)$ and $L^{r_i(x)}(\Omega)$, for $1 \leq i \leq k$. Applying the Hölder inequality (see [22]) we have

$$\begin{aligned}
 |\mathcal{F}'(u_m)(u_m - u)| &\leq \sum_{i=1}^k \left| \int_{\Omega} \left((u_m)_+^{q_i(x)-1} - (u)_+^{q_i(x)-1} \right) (u_m - u) dx \right| \\
 &\leq \sum_{i=1}^k \left(\| |u_m|^{q_i(x)-1} \|_{L^{\frac{q_i(x)}{q_i(x)-1}}} \| u_m - u \|_{L^{q_i(x)}} \right. \\
 &\quad \left. + \| |u_m|^{r_i(x)-1} \|_{L^{\frac{r_i(x)}{r_i(x)-1}}} \| u_m - u \|_{L^{r_i(x)}} \right)
 \end{aligned}
 \tag{32}$$

which leads to 0 as $m \rightarrow \infty$.

The relation (31) shows

$$\lim_{m \rightarrow \infty} J'(u_m)(u_m - u) = 0.
 \tag{33}$$

Also (31)-(33) imply

$$\lim_{m \rightarrow \infty} \mathcal{M}'(u_m)(u_m - u) = 0.
 \tag{34}$$

Since $\{u_m\}$ is bounded in X , by Proposition 2.2, passing to a subsequence, we have

$$\int_{\Omega} \Phi(|\nabla u_m|) + \Phi(|u_m|) dx \rightarrow t_1 \geq 0 \quad \text{as } m \rightarrow \infty$$

If $t_1 = 0$ then $u_m \rightarrow 0$ in X and the proof is complete.

If $t_1 > 0$ then

$$M \left(\int_{\Omega} \Phi(|\nabla u_m|) + \Phi(|u_m|) dx \right) \rightarrow M(t_1) \quad \text{as } m \rightarrow \infty.$$

For sufficiently large m and (13)

$$M \left(\int_{\Omega} \Phi(|\nabla u_m|) + \Phi(|u_m|) dx \right) \geq C_4 > 0.
 \tag{35}$$

Finally (34) and (35) imply

$$\lim_{m \rightarrow \infty} \int_{\Omega} (a(|\nabla u_m|) \nabla(\nabla u_m - \nabla u) + a(|u_m|) u_m (u_m - u)) dx = 0.$$

Thus $\{u_m\}$ converges strongly to u in X and J satisfies the PS condition (see [24, Lemma 5]). \square

3. Some problems

Here, we introduce some interesting problems. These problems are the existence of multiple solutions for the following model problems.

(I) One can study the existence of solutions for

$$\begin{cases} -M \left(\int \sum_{i=1}^n (|\nabla u|^{p_i} + |u|^{p_i}) dx \right) (\Delta_{p_i} u + |u|^{p_i-2} u) = \lambda f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \end{cases}
 \tag{36}$$

where $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function which may change sign, $M : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function, λ is a positive real parameter, Ω is a bounded domain in \mathbb{R}^N ($N \geq 3$) with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$.

(II) One can study the existence of solutions for

$$\begin{cases} -M \left(\int (\Phi(|\nabla u|) + \Phi(|u|)) dx \right) (\operatorname{div}(a(|\nabla u|)\nabla u) + a(|u|)u) \\ \quad \quad \quad = \lambda f(x, u, \nu) \quad \text{in } \Omega, \\ -M \left(\int (\Phi(|\nabla v|) + \Phi(|v|)) dx \right) (\operatorname{div}(a(|\nabla v|)\nabla v) + a(|v|)v) \\ \quad \quad \quad = \lambda g(x, u, \nu) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \quad \quad \text{on } \partial\Omega, \end{cases} \quad (37)$$

where $f, g : \Omega \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are two Carathéodory function which change sign, $M : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function, λ is a positive real parameter, Ω is a bounded domain in \mathbb{R}^N ($N \geq 3$) with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$.

(III) One can study the existence of solutions for

$$\begin{cases} -M \left(\int \sum_{i=1}^n (|\nabla u|^{p_i} + |u|^{p_i}) dx \right) (\Delta_{p_i} u + |u|^{p_i-2}u) = \lambda f(x, \nu) \quad \text{in } \Omega, \\ -M \left(\int \sum_{i=1}^n (|\nabla v|^{p_i} + |v|^{p_i}) dx \right) (\Delta_{p_i} v + |v|^{p_i-2}v) = \lambda f(x, u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \quad \quad \text{on } \partial\Omega, \end{cases} \quad (38)$$

where $M : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function, λ is a positive real parameter, Ω is a bounded domain in \mathbb{R}^N ($N \geq 3$) with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$. Suppose function $f(x, t)$ is given by

$$f(x, t) = \begin{cases} \sum_{i=1}^k (t^{q_i(x)-1} - t^{r_i(x)-1}) & t \geq 0, \\ 0 & t < 0, \end{cases}$$

for $x \in \Omega$.

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