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Decay Estimates for a Degenerate Wave Equation with a Dynamic Fractional Feedback Acting on the Degenerate Boundary

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Abstract. In this paper, we consider a one-dimensional weakly degenerate wave equation with a dynamic nonlocal boundary feedback of fractional type acting at a degenerate point. First We show well-posedness by using the semigroup theory. Next, we show that our system is not uniformly stable by spectral analysis. Hence, we look for a polynomial decay rate for a smooth initial data by using a result due Borichev and Tomilov which reduces the problem of estimating the rate of energy decay to finding a growth bound for the resolvent of the generator associated with the semigroup. This analysis proves that the degeneracy affect the energy decay rates.

1. Introduction

In this paper, we are concerned with the dynamic boundary stabilization of fractional type for degenerate wave equation of the form

$$\begin{cases} u_{tt}(x,t) - (x^\alpha u_x(x,t))_x = 0 & \text{in } (0,1)\times(0,+\infty), \\ -mu_{tt}(0,t) + (x^\alpha u_x)(0,t) = \varrho\partial_t^{\tau,\eta}u(0,t) & \text{in } (0,+\infty), \\ u(1,t) = 0 & \text{in } (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases} \tag{P}$$

where $(x,t) \in (0,1)\times(0,+\infty)$, $\alpha \in [0,1)$, $m > 0$ and $\varrho > 0$. The notation $\partial_t^{\tau,\eta}$ stands for the generalized Caputo's fractional derivative of order τ, $(0 < \tau \leq 1)$, with respect to the time variable (see **[11]**). It is defined as follows

$$\partial_t^{\tau,\eta}w(t) = \begin{cases} w_t(t) & \text{for } \tau = 1, \eta \geq 0, \\ \dfrac{1}{\Gamma(1-\tau)}\displaystyle\int_0^t (t-s)^{-\tau}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds, & \text{for } 0 < \tau < 1, \eta \geq 0. \end{cases}$$

The problem (P) describes the motion of a pinched vibration cable with tip mass $m > 0$ (see **[21]** and **[15]**). The situations where the coefficients are variables arise in engineering problems that generally use non-homogeneous materials such as smart materials.

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The bibliography of works concerning the stabilization of nondegenerate non-homogeneous wave equation with different types of dampings is truly long (see e.g. **[9]**, **[12]** and the references therein). D'Andrea-Novel and al. in **[12]** studied the wave equation with one feedback depending only on the boundary velocities and the boundary displacement i.e, they considered the following problem

$$\begin{cases} u_{tt}(x,t) - (d(x)u_x)_x = 0, & 0 < x < 1,\ t > 0, \\ (du_x)(0,t) = 0, & t > 0 \\ (du_x)(1,t) = -ku(1,t) - u_t(1,t), & t > 0\ k > 0, \end{cases}$$

where $d(x) = d_1 x + d_0$. They have established aymptotics stabilization.

Let us mention here that the case $\alpha = 0$ and $\tau = 1$ in (P) corresponds to a classical boundary damping and it has been extensively studied by many authors (see, for instance, **[18]**, **[14]**, and references therein). It has been proved, in particular that solutions exist globally with an optimal decay rate that is $E(t) \sim c/t$ by using Riesz basis property of the generalized eigenvector of the system.

Recently in **[6]**, Benaissa and Benkhedda considered the stabilization for the following wave equation with dynamic boundary feedback of fractional derivative type (CF):

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{in } (0,1) \times (0,+\infty) \\ u(0,t) = 0 & \text{in } (0,+\infty) \\ mu_{tt}(1,t) + u_x(1,t) = -\varrho \partial_t^{\tau,\eta} u(1,t) & \text{in } (0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } (0,1) \end{cases} \tag{PF}$$

They proved that the decay of the energy is not exponential but polynomial that is $E(t) \leq C1/t^{(2-\tau)}$.

Very recently in **[10]**, Cheheb and al. considered the stabilization for the following wave equation with a general dynamic boundary feedback of diffusive type (CF):

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{in } (0,1) \times (0,+\infty), \\ u(0,t) = 0 & \text{in } (0,+\infty), \\ mu_{tt}(1,t) + u_x(1,t) = -\zeta \displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi,t)\, d\xi & \text{in } (0,+\infty), \\ \partial_t \varphi(\xi,t) + (\xi^2 + \eta)\varphi(\xi,t) - u_t(1,t)\nu(\xi) = 0 & \text{in } (-\infty,\infty) \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } (0,1), \\ \varphi(\xi,0) = \varphi_0 & \text{in } (-\infty,\infty). \end{cases} \tag{P}$$

They proved that the decay of the energy is not exponential. Moreover, they obtained a precise and optimal energy decay estimate for a general feedback of diffusive type, from which the usual feedback of fractional derivative type is a special case.

Very recently in **[5]**, Benaissa and Aichi studied the degenerate wave equation of the type

$$u_{tt}(x,t) - (d(x)u_x(x,t))_x = 0 \qquad \text{in } (0,1) \times (0,+\infty), \tag{1}$$

where the coefficient d is a positive function on $]0,1]$ but vanishes at zero. The degeneracy of (1) at $x = 0$ is measured by the parameter μ_d defined by

$$\mu_d = \sup_{0<x\leq 1} \frac{x|d'(x)|}{d(x)} \tag{2}$$

and the initial conditions are

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{3}$$

followed by the boundary conditions

$$\begin{cases} \begin{cases} u(0,t) = 0 & \text{if } 0 \leq \mu_d < 1 \\ (du_x)(0,t) = 0 & \text{if } 1 \leq \mu_d < 2 \end{cases} & \text{in } (0,+\infty), \\ u_x(1,t) + \varrho \partial_t^{\tau,\eta} u(1,t) + \beta u(1,t) = 0 & \text{in } (0,+\infty). \end{cases} \tag{P1}$$

They proved an optimal polynomial decay rate. It is proved that the presence of feedback of fractional time derivative type and located at a nondegenerate point $x = 1$ has no effect on the stabilisation results in **[5]**.

Very recently, Zerkouk and al. **[24]** extended the result of Mbodje **[20]** to the case of degenerate polynomial coefficient and boundary control of diffusive type acting on degenerate point as in this paper (with $m = 0$) and established a precise decay estimate by adopting the resolvent estimate method.

Here we want to focus on the following remarks:

- The method based on the theory of Riesz basis property of the generalized eigenvector of the system does not seem to be work in the presence of a fractional feedback.
- The frequency method based on multiplier techniques used in **[5]** and the enegy method based on multiplier techniques used in **[16]** do not seem to be work in the case of a feedback located at a degenerate point $x = 0$.

In this work, we explain the influence of the relation between the tip mass term, the degenerate coefficient and the fractional feedback on decay estimates. We prove a sharp polynomial decay rate depending on parameters α, τ. To the best of our knowledge, there is no result concerning the stabilization of a degenerate wave equation with the presence of a dynamic fractional feedback acting on the degenerate boundary.

This paper is organized as follows. In section 2, we give preliminaries results and we reformulate the system (P) into an augmented system by coupling the degenerate wave equation with a suitable diffusion equation and we show the well-posedness of our problem by semigroup theory. In section 3, we prove lack of exponential stability by spectral analysis and by using Bessel functions. In the last section, we prove an optimal decay rate. Our approach is based on a result due to Borichev and Tomilov, which reduces the problem of estimating the rate of energy decay to finding a growth bound for the resolvent of the semigroup generator using an explicit representation of the resolvent by the help of Bessel functions.

2. Preliminary results

Now, we introduce the following weighted Sobolev spaces:

$$H^1_{0,\alpha}(0,1) = \left\{u \text{ is locally absolutely continuous in } (0,1] : x^{\alpha/2}u_x \in L^2(0,1)/\ u(1) = 0\right\},$$

$$H^1_{\alpha}(0,1) = \left\{u \text{ is locally absolutely continuous in } (0,1] : x^{\alpha/2}u_x \in L^2(0,1)\right\}.$$

We remark that $H^1_\alpha(0,1)$ is a Hilbert space with the scalar product

$$(u,v)_{H^1_\alpha(0,1)} = \int_0^1 (u\overline{v} + x^\alpha u'(x)\overline{v'(x)})\,dx, \quad \forall u, v \in H^1_\alpha(0,1).$$

Let us also set

$$|u|_{H^1_{0,\alpha}(0,1)} = \left(\int_0^1 x^\alpha |u'(x)|^2\,dx\right)^{1/2} \quad \forall u \in H^1_\alpha(0,1).$$

Actually, $|\cdot|_{H^1_{0,\alpha}(0,1)}$ is an equivalent norm on the closed subspace $H^1_{0,\alpha}(0,1)$ to the norm of $H^1_\alpha(0,1)$. This fact is a simple consequence of the following version of Poincaré's inequality.

Proposition 2.1. *There is a positive constant $C_* = C(\alpha)$ such that*

$$\|u\|_{L^2(\Omega)} \leq C_* |u|_{H^1_{0,\alpha}(0,1)} \quad \forall u \in H^1_{0,\alpha}(0,1). \tag{4}$$

Proof. Let $u \in H^1_{0,\alpha}(0,1)$. For any $x \in]0,1]$ we have that

$$|u(x)| = \left|\int_x^1 u'(s)\,ds\right| \leq |u|_{H^1_{0,\alpha}(0,1)} \left\{\int_0^1 \frac{1}{x^\alpha}\,dx\right\}^{1/2}.$$

Therefore

$$\int_0^1 |u(x)|^2\,dx \le \frac{1}{1-\alpha}|u|^2_{H^1_{0,\alpha}(0,1)}.$$

Next, we define

$$H^2_\alpha(0,1) = \{u \in H^1_\alpha(0,1) :\ x^\alpha u' \in H^1(0,1)\},$$

where $H^1(0,1)$ denotes the classical Sobolev space.

Now we reformulate (P) into an augmented system. For that, we need the following proposition.

Remark 2.2. *Notice that if $u \in H^2_\alpha(0,1), \alpha \in [1,2)$, we have $(x^\alpha u_x)(0) \equiv 0$. Indeed, if $x^\alpha u_x(x) \to L$ when $x \to 0$, then $x^\alpha |u_x(x)|^2 \sim L/x^\alpha$ and therefore $L = 0$ otherwise $u \notin H^1_\alpha(0,1)$.*

Proposition 2.3 (see [20]). *Let ν be the function:*

$$\nu(\xi) = |\xi|^{(2\tau-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \tau < 1. \tag{5}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t\varphi(\xi,t) + (\xi^2+\eta)\varphi(\xi,t) - U(t)\nu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \ge 0, t > 0, \tag{6}$$

$$\varphi(\xi,0) = 0, \tag{7}$$

$$O(t) = (\pi)^{-1}\sin(\tau\pi)\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi,t)\,d\xi \tag{8}$$

is given by

$$O = I^{1-\tau,\eta}U. \tag{9}$$

where

$$[I^{\tau,\eta}f](t) = \frac{1}{\Gamma(\tau)}\int_0^t (t-s)^{\tau-1}e^{-\eta(t-s)}f(s)\,ds.$$

Lemma 2.4 (see [1]). *If $\lambda \in D_\eta = \mathbb{C}\backslash\,]-\infty,-\eta]$ then*

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{\nu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \frac{\pi}{\sin\tau\pi}(\lambda+\eta)^{\tau-1}.$$

Using now Proposition 2.3 and relation (9), system (P) may be recast into the following augmented system

$$\begin{cases} u_{tt}(x,t) - (x^\alpha u_x(x,t))_x = 0, & \\ \varphi_t(\xi,t) + (\xi^2+\eta)\varphi(\xi,t) - u_t(0,t)\nu(\xi) = 0, & -\infty < \xi < +\infty,\ t > 0, \\ -mu_{tt}(0,t) + (x^\alpha u_x)(0,t) = \zeta\displaystyle\int_{-\infty}^{+\infty}\nu(\xi)\varphi(\xi,t)\,d\xi, & \\ u(1,t) = 0, & \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \end{cases} \tag{P'}$$

where $\zeta = \varrho(\pi)^{-1}\sin(\tau\pi)$. Thus, we shall consider problem (P') instead of (P).

3. Well-posedness

In this section, we will use the semigroup approach to study the well-posedness of system (P'). To define the semigroup associated with (P'), we consider the right-end boundary condition

$$u_t(0,t) = \theta(t), t > 0,$$

where θ solve the equation

$$-m\theta_t(t) + (x^\alpha u_x)(0,t) - \zeta \int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi,t)\,d\xi = 0. \tag{10}$$

with the initial condition

$$\theta(0) = u_1(0) = \theta_0. \tag{11}$$

Considering $U := (u, u_t, \varphi, \theta)^T$ and $U_0 = (u_0, u_1, 0, \theta_0)^T$, system (P') can be written in the following abstract framework

$$\partial_t U = \mathcal{P}U, \qquad U(0) = U_0, \tag{12}$$

where the operator $\mathcal{P}$ is given by

$$\mathcal{P}\begin{pmatrix} u \\ v \\ \varphi \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ (x^\alpha u_x)_x \\ -(\xi^2+\eta)\varphi + v(0)\nu(\xi) \\ \dfrac{1}{m}(x^\alpha u_x)(0) - \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\,d\xi \end{pmatrix}. \tag{13}$$

This operator will be defined in an appropriate subspace of the Hilbert space

$$\mathcal{H} = H^1_{0,\alpha}(0,1) \times L^2(0,1) \times L^2(-\infty,+\infty) \times \mathbb{C},$$

endowed with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ \varphi \\ \theta \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\varphi} \\ \tilde{\theta} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 x^\alpha u_x \overline{\tilde{u}}_x dx + \int_0^1 v\overline{\tilde{v}} dx + \zeta \int_{-\infty}^{+\infty} \varphi\overline{\tilde{\varphi}}\,d\xi + m\theta\overline{\tilde{\theta}}.$$

We choose the domain for the operator $\mathcal{P}$ as

$$D(\mathcal{P}) = \left\{ \begin{array}{l} (u,v,\varphi,\theta) \text{ in } \mathcal{H} : u \in H^2_\alpha(0,1) \cap H^1_{0,\alpha}(0,1), v \in H^1_{0,\alpha}(0,1), \theta \in \mathbb{C}, \\ -(\xi^2+\eta)\varphi + v(0)\nu(\xi) \in L^2(-\infty,+\infty), v(0) = \theta, \\ |\xi|\varphi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{14}$$

Our main result is giving by the following theorem.

Theorem 3.1. *The operator $\mathcal{P}$ defined by (13) and (14), generates a C_0-semigroup of con- tractions $e^{t\mathcal{P}}$ in the Hilbert space $\mathcal{H}$.*

Proof. To prove this result we shall use the Lumer-Phillips theorem. Since for every $U = (u, v, \varphi, \theta) \in D(\mathcal{P})$ we have

$$\Re\langle \mathcal{P}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)|\varphi(\xi)|^2\,d\xi. \tag{15}$$

then the operator $\mathcal{P}$ is dissipative.

Let $\lambda > 0$. we prove that the operator $(\lambda I - \mathcal{P})$ is a surjection. Let $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, the vector $U = (u, v, \varphi, \theta) \in D(\mathcal{P})$ is a solution of the system $\lambda I - \mathcal{P}U = F$ if its components satisfy

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - (x^\alpha u_x)_x = f_2, \\ \lambda\varphi + (\xi^2 + \eta)\varphi - v(0)\nu(\xi) = f_3. \\ \lambda\theta - \frac{1}{m}(x^\alpha u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi = f_4. \end{cases} \tag{16}$$

Suppose u is found with the appropriate regularity. Then, $(16)_1$ and $(16)_3$ yield

$$v = \lambda u - f_1 \in H^1_{0,\alpha}(0,1), \tag{17}$$

$$\varphi(\xi) = \frac{f_3(\xi)}{\xi^2 + \eta + \lambda} + \frac{\lambda u(0)\nu(\xi)}{\xi^2 + \eta + \lambda} - \frac{f_1(0)\nu(\xi)}{\xi^2 + \eta + \lambda}. \tag{18}$$

By using (16) and (17) it can easily be shown that u satisfies

$$\lambda^2 u - (x^\alpha u_x)_x = f_2 + \lambda f_1. \tag{19}$$

Solving equation (19) is equivalent to finding $u \in H^2_\alpha(0,1) \cap H^1_{0,\alpha}(0,1)$ such that

$$\int_0^1 (\lambda^2 u\overline{w} - (x^\alpha u_x)_x\overline{w})\, dx = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx, \tag{20}$$

for all $w \in H^1_{0,\alpha}(0,1)$. By using (20), the boundary condition $(16)_4$, the fact that $\theta = v(0)$ and (18), the function u satisfying the following equation

$$\begin{cases} \displaystyle\int_0^1 (\lambda^2 u\overline{w} + x^\alpha u_x\overline{w}_x)\, dx + \lambda(m\lambda + \tilde{\zeta})u(0)\overline{w}(0) \\ = \displaystyle\int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\nu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\, d\xi\overline{w}(0) + (m\lambda + \tilde{\zeta})f_1(0)\overline{w}(0) - mf_4\overline{w}(0). \end{cases} \tag{21}$$

where $\tilde{\zeta} = \zeta\displaystyle\int_{-\infty}^{+\infty} \frac{\nu^2(\xi)}{\xi^2 + \eta + \lambda}\, d\xi$. Problem (21) is of the form

$$\mathcal{B}(u, w) = \mathcal{L}(w), \qquad \forall w \in H^1_{0,\alpha}(0,1), \tag{22}$$

where $\mathcal{B} : [H^1_{0,\alpha}(0,1) \times H^1_{0,\alpha}(0,1)] \to \mathbb{C}$ is the the sesquilinear form defined by

$$\mathcal{B}(u, w) = \int_0^1 (\lambda^2 u\overline{w} + x^\alpha u_x\overline{w}_x)\, dx + \lambda(m\lambda + \tilde{\zeta})u(0)\, \overline{w}(0)$$

and $\mathcal{L} : H^1_{0,\alpha}(0,1) \to \mathbb{C}$ is the antilinear form given by

$$\mathcal{L}(w) = \int_0^1 (f_2 + \lambda f_1)\overline{w}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\nu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\, d\xi\overline{w}(0) + (m\lambda + \tilde{\zeta})f_1(0)\overline{w}(0) - mf_4\overline{w}(0).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Therefore, using the Lax-Milgram Theorem, we conclude that (22) has a unique solution $u \in H^1_{0,\alpha}(0,1)$. By classical regularity arguments, we conclude that the solution u of (22) belongs into $H^2_\alpha(0,1)$. Therefore, the operator $\lambda I - \mathcal{P}$ is surjective for any $\lambda > 0$.

□

As a consequence of Theorem 3.1, the system (P') is well-posed in the energy space $\mathcal{H}$ and we have the following proposition.

Proposition 3.2. *For $(u_0, u_1, 0, \theta_0) \in \mathcal{H}$, the problem (P') admits a unique weak solution*

$$(u, u_t, \varphi, \theta) \in C^0(\mathbb{R}_+, \mathcal{H}).$$

and for $(u_0, u_1, 0, \theta_0) \in D(\mathcal{P})$, the problem (P') admits a unique strong solution

$$(u, u_t, \varphi, \theta) \in C^0(\mathbb{R}_+, D(\mathcal{P})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Moreover, from the density $D(\mathcal{P})$ in $\mathcal{H}$ the energy of $(u(t), \varphi(t))$ at time $t \geq 0$ by

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + x^\alpha |u_x|^2)dx + \frac{m}{2}|u_t(0,t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\varphi(\xi,t)|^2\, d\xi \tag{23}$$

decays as follow

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\varphi(\xi,t)|^2\, d\xi \leq 0. \tag{24}$$

Proof of Proposition 3.2. Noting that the regularity of the solution of the problem (P') is consequence of the semigroup properties. We have just to prove (24).

Multiplying the first equation in (P') by $\overline{u}_t$, integrating over $(0,1)$ and using integration by parts, we get

$$\int_0^1 u_{tt}(x,t)\overline{u}_t dx - \int_0^1 (x^\alpha u_x(x,t))_x \overline{u}_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 |u_t(x,t)|^2 dx\right) + \frac{1}{2}\frac{d}{dt}\int_0^1 x^\alpha |u_x(x,t)|^2\, dx - \Re\Big[(x^\alpha u_x)(x,t)\overline{u}_t\Big]_0^1 = 0.$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(|u_t(x,t)|^2 + x^\alpha |u_x(x,t)|^2\right) dx + \frac{m}{2}|u_t(0,t)|^2 + \zeta \Re \overline{u}_t(0,t)\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi,t)\, d\xi = 0. \tag{25}$$

Multiplying the second equation in (P') by $\zeta\overline{\varphi}$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\zeta \int_{-\infty}^{+\infty} \varphi_t(\xi,t)\overline{\varphi}(\xi,t)d\xi + \zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)|\varphi(\xi,t)|^2 d\xi - \zeta u_t(0,t)\int_{-\infty}^{+\infty} \nu(\xi)\overline{\varphi}(\xi,t)d\xi = 0.$$

Hence

$$\frac{\zeta}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} |\varphi(\xi,t)|^2 d\xi + \zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)|\varphi(\xi,t)|^2 d\xi - \zeta \Re u_t(0,t)\int_{-\infty}^{+\infty} \nu(\xi)\overline{\varphi}(\xi,t)d\xi = 0. \tag{26}$$

From (23), (25) and (26) we get

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\varphi(\xi,t)|^2\, d\xi \leq 0.$$

This completes the proof of the lemma.

Remark 3.3. • *We can easily extend the global existence result for a general function $d(x)$ instead of x^α with $0 < \mu_d < 1$ (see (2)).*

• *In the case $\tau = 1$, we take $\varrho u_t(0,t)$ instead of $\varrho\partial_t^{\tau,\eta}u(0,t)$. We do not need to introduce a diffusive representation technique to bring the problem back into the semigroup theory. Indeed the operator $\mathcal{P}$ takes the form*

$$\tilde{\mathcal{P}}\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ (x^\alpha u_x)_x \\ \frac{1}{m}(x^\alpha u_x)(0) - \frac{\varrho}{m}\theta \end{pmatrix}. \tag{27}$$

with domain

$$D(\tilde{\mathcal{P}}) = \left\{ \begin{matrix} (u,v,\theta) \text{ in } \mathcal{H} : u \in H^2_\alpha(0,1) \cap H^1_{0,\alpha}(0,1), v \in H^1_{0,\alpha}(0,1), \theta \in \mathbb{C}, \\ v(0) = \theta, \end{matrix} \right\}. \tag{28}$$

where

$$\mathcal{H} = H^1_{0,\alpha}(0,1) \times L^2(0,1) \times \mathbb{C},$$

with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\theta} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 x^\alpha u_x \overline{\tilde{u}}_x dx + \int_0^1 v\overline{\tilde{v}} dx + m\theta\overline{\tilde{\theta}}.$$

The well-posedness result follows exactly as in the case $0 < \tau < 1$. Moreover, the energy function is defined as

$$\tilde{E}(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + x^\alpha |u_x|^2)dx + \frac{m}{2}|u(0,t)|^2 \tag{29}$$

and decays as follows

$$\tilde{E}'(t) = -\varrho|u_t(0,t)|^2 \leq 0.$$

□

4. Strong stability and lack of exponential stability

4.1. Strong Stability

We need the following Theorem to prove strong stability of solutions.

Theorem 4.1 ([3]). *Let $\mathcal{P}$ be the generator of a uniformly bounded C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{X}$. If:*

(i) $\mathcal{P}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{P})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{X}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{X}$.

Our main result is the following theorem:

Theorem 4.2. *The C_0-semigroup $e^{t\mathcal{P}}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0 \in \mathcal{H}$, the solution of (12) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{P}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 4.2, we need the following two lemmas.

Lemma 4.3. *$\mathcal{P}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof.
We make a distinction between $i\lambda = 0$ and $i\lambda \neq 0$.
Step 1. Solving for $\mathcal{P}U = 0$ leads to the system

$$\begin{cases} v = 0, \\ (x^\alpha u_x)_x = 0, \\ (\xi^2+\eta)\varphi - v(0)\nu(\xi) = 0. \\ -\frac{1}{m}(x^\alpha u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty}\nu(\xi)\varphi(\xi)\,d\xi = 0. \end{cases} \tag{30}$$

Then $v = 0, \varphi = 0, (x^\alpha u_x)(0) = 0$ and

$$(x^\alpha u_x)(x) = c.$$

As $(x^\alpha u_x)(0) = 0$, we have $(x^\alpha u_x)(x) = 0$. Hence

$$u_x(x) = 0 \text{ for } x \in (0,1).$$

As $u(1) = 0$, then $u = 0$. we have $U = 0$. Hence, $i\lambda = 0$ is not an eigenvalue of $\mathcal{P}$.

Step 2. Let $\lambda \in \mathbb{R} - \{0\}$. We prove that $i\lambda$ is not an eigenvalue of $\mathcal{P}$ by proving that the unique solution $U \in D(\mathcal{P})$ of the equation

$$\mathcal{P}U = i\lambda U \tag{31}$$

is $U = 0$. Let $U = (u, v, \varphi, \theta)^T$. The equation (31) means that

$$\begin{cases} i\lambda u - v = 0, \\ i\lambda v - (x^\alpha u_x)_x = 0, \\ i\lambda\varphi + (\xi^2+\eta)\varphi - v(0)\nu(\xi) = 0. \\ i\lambda\theta - \frac{1}{m}(x^\alpha u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty}\nu(\xi)\varphi(\xi)\,d\xi = 0. \end{cases} \tag{32}$$

Using (15) and (31), we find

$$\varphi \equiv 0, \tag{33}$$

then, using the third equation in (32), we deduce that

$$v(0) = 0. \tag{34}$$

Therefore, from the first and last equation in (32), we find

$$u(0) = 0 \quad \text{and} \quad (x^\alpha u_x)(0) = 0. \tag{35}$$

Thus, by eliminating v, the system (32) implies that

$$\begin{cases} \lambda^2 u + (x^\alpha u_x)_x = 0 \text{ on } (0,1), \\ u(0) = u(1) = 0, \\ (x^\alpha u_x)(0) = 0. \end{cases} \tag{36}$$

The solution of the equation (36) is given by

$$u(x) = C_1\Phi_+(x) + C_2\Phi_-(x),$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right), \quad \Phi_-(x) = x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right). \tag{37}$$

From boundary conditions $(36)_2$ and $(36)_3$, we deduce that

$$u \equiv 0.$$

Therefore $U = 0$. Consequently, $\mathcal{P}$ does not have purely imaginary eigenvalues.

Lemma 4.4.
If $\lambda \neq 0$, the operator $i\lambda I - \mathcal{P}$ is surjective.
If $\lambda = 0$ and $\eta \neq 0$, the operator $i\lambda I - \mathcal{P}$ is surjective.

Proof.
Case 1: $\lambda \neq 0$. Let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ be given, and let $U = (u, v, \varphi, \theta)^T \in D(\mathcal{P})$ be such that

$$(i\lambda I - \mathcal{P})U = F. \tag{38}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\alpha u_x)_x = f_2, \\ i\lambda\varphi + (\xi^2 + \eta)\varphi - v(0)\nu(\xi) = f_3. \\ i\lambda\theta - \frac{1}{m}(x^\alpha u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi = f_4 \end{cases} \tag{39}$$

with boundary conditions. Then we deduce from these equations a weak formulation (similar computation as in Theorem 3.1):

$$\mathcal{B}(u, w) = l(w), \quad \forall w \in H^1_{0,\alpha}(0, 1), \tag{40}$$

where

$$\mathcal{B}(u, w) = \mathcal{B}_1(u, w) + \mathcal{B}_2(u, w)$$

with

$$\begin{cases} \mathcal{B}_1(u, w) = \displaystyle\int_0^1 x^\alpha u_x \overline{w}_x\, dx + i\lambda\varrho(i\lambda + \eta)^{\tau-1} u(0)\overline{w}(0), \\ \mathcal{B}_2(u, w) = -\displaystyle\int_0^1 \lambda^2 u\overline{w}\, dx - m\lambda^2 u(0)\overline{w}(0), \end{cases} \tag{$*$}$$

and

$$l(w) = \int_0^1 (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta\int_{-\infty}^{+\infty} \frac{\nu(\xi)}{\xi^2 + \eta + i\lambda} f_3(\xi)\, d\xi\overline{w}(0) + (mi\lambda + \varrho(i\lambda + \eta)^{\tau-1})f_1(0)\overline{w}(0) - mf_4\overline{w}(0).$$

Let $(H^1_{0,\alpha}(0, 1))'$ be the dual space of $H^1_{0,\alpha}(0, 1)$. Let us define the following operators

$$\begin{array}{cc} B : H^1_{0,\alpha}(0, 1) \to (H^1_{0,\alpha}(0, 1))' & B_i : H^1_{0,\alpha}(0, 1) \to (H^1_{0,\alpha}(0, 1))' \quad i \in \{1, 2\} \\ u \mapsto Bu & u \mapsto B_i u \end{array} \tag{$**$}$$

such that

$$\begin{array}{l} (Bu)w = \mathcal{B}(u, w),\ \forall w \in H^1_{0,\alpha}(0, 1), \\ (B_i u)w = \mathcal{B}_i(u, w),\ \forall w \in H^1_{0,\alpha}(0, 1), i \in \{1, 2\}. \end{array} \tag{$***$}$$

We need to prove that the operator B is an isomorphism. For this aim, we divide the proof into three steps:
Step 1. In this step, we want to prove that the operator B_1 is an isomorphism. For this aim, it is easy to see that $\mathcal{B}_1$ is sesquilinear, continuous form on $H^1_{0,\alpha}(0, 1)$. Furthermore

$$\begin{array}{rcl} \Re\mathcal{B}_1(u, u) & = & \|x^{\alpha/2} u_x\|_2^2 + \varrho\lambda\Re\left(i(i\lambda + \eta)^{\tau-1}\right)|u(0)|^2 \\ & \geq & \|x^{\alpha/2} u_x\|_2^2, \end{array}$$

where we have used the fact that

$$\varrho\lambda\Re\left(i(i\lambda+\eta)^{\tau-1}\right)=\zeta\lambda^2\int_{-\infty}^{+\infty}\frac{\nu(\xi)^2}{\lambda^2+(\eta+\xi^2)^2}\,d\xi>0.$$

Thus $\mathcal{B}_1$ is coercive. Then, from (**) and Lax-Milgram theorem, the operator B_1 is an isomorphism.
Step 2. In this step, we want to prove that the operator B_2 is compact. For this aim, from (*) and (* * *), we have

$$|\mathcal{B}_2(u,w)|\leq c\|u\|_{L^2(0,1)}\|w\|_{L^2(0,1)}+c'|u|_{H^1_{0,\alpha}(0,1)}|w|_{H^1_{0,\alpha}(0,1)},$$

and consequently, using the compact embedding from $H^1_{0,\alpha}(0,1)$ to $L^2(0,1)$ we deduce that B_2 is a compact operator. Therefore, from the above steps, we obtain that the operator $B=B_1+B_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we still need to prove that the operator B is injective to obtain that the operator B is an isomorphism.
Step 3. Let $u\in ker(B)$, then

$$\mathcal{B}(u,w)=0\quad\forall w\in H^1_{0,\alpha}(0,1). \tag{41}$$

In particular for $w=u$, it follows that

$$\lambda^2\|u\|^2_{L^2(0,1)}+m\lambda^2|u(0)|^2-i\varrho\lambda(i\lambda+\eta)^{\tau-1}|u(0)|^2=\|x^{\alpha/2}u_x\|^2_{L^2(0,1)}.$$

Hence, we have

$$u(0)=0. \tag{42}$$

From (41), we obtain

$$(x^{\alpha/2}u_x)(0)=0 \tag{43}$$

and then

$$\begin{cases}-\lambda^2u-(x^\alpha u_x)_x=0,\\ u(0)=(x^{\alpha/2}u_x)(0)=0,\\ u(1)=0.\end{cases} \tag{44}$$

Then, according to Lemma 4.3, we deduce that $u=0$ and consequently $Ker(B)=\{0\}$. Finally, from Step 3 and Fredholm alternative, we deduce that the operator B is isomorphism. It is easy to see that the operator l is a antilinear and continuous form on $H^1_{0,\alpha}(0,1)$. Consequently, (40) admits a unique solution $u\in H^1_{0,\alpha}(0,1)$. By using the classical elliptic regularity, we deduce that $U\in D(\mathcal{P})$ is a unique solution of (38). Hence $i\lambda-\mathcal{P}$ is surjective for all $\lambda\in\mathbb{R}^*$.

Case 2: $\lambda=0$ **and** $\eta\neq 0$**.** Using Lax-Milgram Lemma, we obtain the result.

Taking account of Lemmas 4.3, 4.4 and from Theorem 4.1 The C_0-semigroup $e^{t\mathcal{P}}$ is strongly stable in $\mathcal{H}$. □

4.2. Lack of exponential stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (12). In order to state and prove our stability results, we need some lemmas.

Theorem 4.5 ([23]). *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space X with generator $\mathcal{P}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{P})\supseteq\{i\beta:\beta\in\mathbb{R}\}\equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}}\|(i\beta I-\mathcal{P})^{-1}\|_{\mathcal{L}(X)}<\infty.$$

Our main result is the following.

Theorem 4.6. *The semigroup generated by the operator $\mathcal{P}$ is not exponentially stable.*

Proof. We will examine two cases.
•**Case 1** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{P}$. Indeed, noting that $F = (\sin(x-1), 0, 0, 0)^T \in \mathcal{H}$, and suppose that there exists $U = (u, v, \varphi, \theta)^T \in D(\mathcal{P})$ such that $-\mathcal{P}U = F$. We get $\varphi(\xi) = |\xi|^{\frac{2\tau-5}{2}} \sin 1$. But, then $\varphi \notin L^2(-\infty, +\infty)$, since $\tau \in]0, 1[$. So $(u, v, \varphi, \theta)^T \notin D(\mathcal{P})$ and the operator $\mathcal{P}$ is not invertible.
• **Case 2** $\eta \neq 0$:

We aim to show that an infinite number of eigenvalues of $\mathcal{P}$ approach the imaginary axis which prevents the system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{P}$. Let λ be an eigenvalue of $\mathcal{P}$ with associated eigenvector $U = (u, v, \varphi, \theta)^T$. Then $\mathcal{P}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda u - v = 0, \\ \lambda v - (x^\alpha u_x)_x = 0, \\ \lambda\varphi + (\xi^2 + \eta)\varphi - v(0)\nu(\xi) = 0, \\ \lambda\theta - \frac{1}{m}(x^\alpha u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi = 0. \end{cases} \tag{45}$$

It is well-known that Bessel functions play an important role in this type of problem. From $(45)_1 - (45)_2$ for such λ, we find

$$\lambda^2 u - (x^\alpha u_x)_x = 0. \tag{46}$$

Using the boundary conditions and $(45)_3$, we deduce that

$$\begin{cases} \lambda^2 u - (x^\alpha u_x)_x = 0, \\ (x^\alpha u_x)(0) - (m\lambda^2 + \varrho\lambda(\lambda+\eta)^{\tau-1})u(0) = 0, \\ u(1) = 0. \end{cases} \tag{47}$$

Assume that u is a solution of (47) associated to eigenvalue $-\lambda^2$, then one easily checks that the function

$$u(x) = x^{\frac{1-\alpha}{2}} \Psi\left(\frac{2}{2-\alpha} i\lambda x^{\frac{2-\alpha}{2}}\right)$$

is a solution of the following problem:

$$y^2\Psi''(y) + y\Psi'(y) + (y^2 - (\frac{\alpha-1}{2-\alpha})^2)\Psi(y) = 0. \tag{48}$$

We have

$$u(x) = c_+\Phi_+ + c_-\Phi_-, \tag{49}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) := x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(\frac{2}{2-\alpha} i\lambda x^{\frac{2-\alpha}{2}}\right)$$

and

$$\Phi_-(x) := x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\left(\frac{2}{2-\alpha} i\lambda x^{\frac{2-\alpha}{2}}\right),$$

where

$$J_\nu(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{y}{2}\right)^{2m+\nu} = \sum_{m=0}^{\infty} c^+_{\nu,m} y^{2m+\nu}, \tag{50}$$

$$J_{-\nu}(y)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m-\nu+1)}\left(\frac{y}{2}\right)^{2m-\nu}=\sum_{m=0}^{\infty}c^-_{\nu,m}y^{2m-\nu} \tag{51}$$

$$\nu_\alpha=\frac{1-\alpha}{2-\alpha}$$

and J_{ν_α} and $J_{-\nu_\alpha}$ are Bessel functions of the first kind of order ν_α and $-\nu_\alpha$. As $\nu_\alpha\notin\mathbb{N}$, so J_{ν_α} and $J_{-\nu_\alpha}$ are linearly independent and therefore the pair $(J_{\nu_\alpha},J_{-\nu_\alpha})$ (classical result) forms a fundamental system of solutions (48).

Then, using the series expansion of J_{ν_τ} and $J_{-\nu_\tau}$, one obtains

$$\Phi_+(x)=\sum_{m=0}^{\infty}\tilde{c}^+_{\nu_\alpha,m}x^{1-\alpha+(2-\alpha)m},\quad \Phi_-(x)=\sum_{m=0}^{\infty}\tilde{c}^-_{\nu_\alpha,m}x^{(2-\alpha)m},$$

with

$$\tilde{c}^+_{\nu_\alpha,m}=c^+_{\nu_\alpha,m}\left(\frac{2}{2-\alpha}i\lambda\right)^{2m+\nu_\alpha},\quad \tilde{c}^-_{\nu_\alpha,m}=c^-_{\nu_\alpha,m}\left(\frac{2}{2-\alpha}i\lambda\right)^{2m-\nu_\alpha}.$$

Next one easily verifies that $\Phi_+,\Phi_-\in H^1_\alpha(0,1)$: indeed,

$$\begin{array}{ll}\Phi_+(x)\sim_0\tilde{c}^+_{\nu_\alpha,0}x^{1-\alpha}, & x^{\alpha/2}\Phi_+'(x)\sim_0(1-\alpha)\tilde{c}^+_{\nu_\alpha,0}x^{-\alpha/2},\\ \Phi_-(x)\sim_0\tilde{c}^-_{\nu_\alpha,0}, & x^{\alpha/2}\Phi_-'(x)\sim_0(2-\alpha)\tilde{c}^-_{\nu_\alpha,0}x^{1-\alpha/2},\end{array}$$

where we have used the following relation

$$xJ'_\mu(x)=\mu J_\mu(x)-xJ_{\mu+1}(x). \tag{52}$$

Hence, given c_+ and c_-, $u(x)=c_+\Phi_+(x)+c_-\Phi_-(x)\in H^1_\alpha(0,1)$ with the following boundary conditions

$$\begin{cases}(x^\alpha u_x)(0)-(m\lambda^2+\varrho\lambda(\lambda+\eta)^{\tau-1})u(0)=0,\\ u(1)=0.\end{cases}$$

Then

$$M(\lambda)C(\lambda)=\begin{pmatrix}(1-\alpha)\tilde{c}^+_{\nu_\alpha,0} & -(m\lambda^2+\varrho\lambda(\lambda+\eta)^{\tau-1})\tilde{c}^-_{\nu_\alpha,0}\\ J_{\nu_\alpha}\left(\frac{2}{2-\alpha}i\lambda\right) & J_{-\nu_\alpha}\left(\frac{2}{2-\alpha}i\lambda\right)\end{pmatrix}\begin{pmatrix}c_+\\ c_-\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{53}$$

Hence, a non-trivial solution u exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda)=\det M(\lambda)$. Thus the characteristic equation is $f(\lambda)=0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

Since $\mathcal{P}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{P}$ in the strip $-\tau_0\le\Re(\lambda)\le 0$, for some $\tau_0>0$ large enough and for such λ, we remark that Φ_+,Φ_- remains bounded.

Lemma 4.7. *There exists $N\in\mathbb{N}$ sufficiently large and a sequence $(\lambda_k)_{k\in\mathbf{Z}^*,|k|\ge N}$ of simple roots of $\det M$ (that are also simple eigenvalues of $\mathcal{P}$) and satisfying the following asymptotic behavior:*

$$\begin{aligned}\lambda_k\ =\ &-\frac{2-\alpha}{2}i\left(k+\frac{\nu_\alpha}{2}+\frac{3}{4}\right)\pi-i\frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin\nu_\alpha\pi}{(k\pi)^{2-2\nu_\alpha}}\\ &+i\frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{(2-2\nu_\alpha)(\frac{\nu_\alpha}{2}+\frac{3}{4})}{\pi^{2-2\nu_\alpha}k^{3-2\nu_\alpha}}\sin\nu_\alpha\pi\\ &-\left(\frac{1-\alpha}{m}\right)^2\left(\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\right)^2\frac{8}{(2-\alpha)^3}\frac{\sin\nu_\alpha\cos\nu_\alpha}{(\pi k)^{4-4\nu_\alpha}}i\\ &-i\left(\frac{2}{2-\alpha}\right)^{3-\tau}\frac{\varrho(1-\alpha)}{m^2}\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin\nu_\alpha\pi\sin(1-\tau)\frac{\pi}{2}}{\pi^{4-\tau-2\nu_\alpha}}\frac{1}{k^{4-\tau-2\nu_\alpha}}\\ &-\left(\frac{2}{2-\alpha}\right)^{3-\tau}\frac{\varrho(1-\alpha)}{m^2}\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin\nu_\alpha\pi\cos(1-\tau)\frac{\pi}{2}}{\pi^{4-\tau-2\nu_\alpha}}\frac{1}{k^{4-\tau-2\nu_\alpha}}+o\left(\frac{1}{k^\omega}\right)\end{aligned} \tag{54}$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

where $\omega = \max\{4-\tau-2\nu_\alpha, 4-4\nu_\alpha\}$*. Moreover for all* $|k| \geq N$*, the eigenvalues* λ_k *are simple.*

Proof. We look at the roots of $f(\lambda)$. From (53), we have

$$f(\lambda) = (1-\alpha)\tilde{c}^{+}_{\nu_\alpha,0} J_{-\nu_\alpha}\left(\frac{2}{2-\alpha} i\lambda\right) + (m\lambda^2 + \varrho\lambda(\lambda+\eta)^{\tau-1}))\tilde{c}^{-}_{\nu_\alpha,0} J_{\nu_\alpha}\left(\frac{2}{2-\alpha} i\lambda\right) = 0.$$

We will use the following classical asymptotic development (see **[17]** p. 122, (5.11.6)): for all $\delta > 0$, the following development holds when $|argz| \leq \pi - \delta$:

$$J_\mu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \mu\frac{\pi}{2} - \frac{\pi}{4}\right)\left(1 + O(\frac{1}{|z|^2})\right) - \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \mu\frac{\pi}{2} - \frac{\pi}{4}\right) O\left(\frac{1}{|z|^2}\right). \tag{55}$$

We divide the proof into five steps:

Step 1. First, using the asymptotic expansion, we get

$$\frac{1}{(\lambda+\eta)^{1-\tau}} = \frac{1}{\lambda^{1-\tau}}(1 + O(\lambda^{-1})) \tag{56}$$

Next, using (55) and (56), we get

$$f(\lambda) = m\left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \lambda^{2-\nu_\alpha} c^{-}_{\nu_\alpha,0}\left(\frac{2}{2-\alpha} i\right)^{-\nu_\alpha} \frac{e^{-i(\tilde{z}-\nu_\alpha\frac{\pi}{2}-\frac{\pi}{4})}}{2} \tilde{f}(\lambda), \tag{57}$$

where

$$\tilde{z} = \frac{2}{2-\alpha} i\lambda$$

and

$$\begin{aligned} \tilde{f}(\lambda) &= (e^{2i(\tilde{z}-\nu_\alpha\frac{\pi}{2}-\frac{\pi}{4})} + 1) + \frac{1-\alpha}{m}\frac{c^{+}_{\nu_\alpha,0}}{c^{-}_{\nu_\alpha,0}}\left(\frac{2}{2-\alpha} i\right)^{2\nu_\alpha} \frac{e^{2i(\tilde{z}-\frac{\pi}{4})} + e^{-i\nu_\alpha\pi}}{\lambda^{2-2\nu_\alpha}} \\ &\qquad + \frac{\varrho}{m}\frac{e^{2i(\tilde{z}-\nu_\alpha\frac{\pi}{2}-\frac{\pi}{4})} + 1}{\lambda^{2-\tau}} + O\left(\frac{1}{\lambda^2}\right) \\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{2-2\nu_\alpha}} + \frac{f_2(\lambda)}{\lambda^{2-\tau}} + O\left(\frac{1}{\lambda^2}\right), \end{aligned} \tag{58}$$

where

$$f_0(\lambda) = e^{2i(\tilde{z}-\nu_\alpha\frac{\pi}{2}-\frac{\pi}{4})} + 1, \tag{59}$$

$$f_1(\lambda) = \frac{1-\alpha}{m}\frac{c^{+}_{\nu_\alpha,0}}{c^{-}_{\nu_\alpha,0}}\left(\frac{2}{2-\alpha} i\right)^{2\nu_\alpha} (e^{2i(\tilde{z}-\frac{\pi}{4})} + e^{-i\nu_\alpha\pi}), \tag{60}$$

$$f_2(\lambda) = \frac{\varrho}{m}(e^{2i(\tilde{z}-\nu_\alpha\frac{\pi}{2}-\frac{\pi}{4})} + 1). \tag{61}$$

Note that f_0, f_1 and f_2 remain bounded in the strip $-\tau_0 \leq \Re(\lambda) \leq 0$.

Step 2. We look at the roots of f_0. From (59), f_0 has one family of roots that we denote λ^0_k.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2i(\tilde{z}-\nu_\alpha\frac{\pi}{2}-\frac{\pi}{4})} + 1 = 0$$

Hence

$$2i\left(\frac{2}{2-\alpha} i\lambda - \nu_\alpha\frac{\pi}{2} - \frac{\pi}{4}\right) = i(2k+1)\pi, \quad k \in \mathbf{Z},$$

i.e.,

$$\lambda_k^0 = -\frac{2-\alpha}{2} i \left(k + \frac{\nu_\alpha}{2} + \frac{3}{4}\right)\pi, \quad k \in \mathbf{Z}.$$

Now with the help of Rouché's Theorem and the asymptotic Equation (58), we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (58) the unknown λ by $u = 2iz$ then (58) becomes

$$\tilde{f}(u) = (e^u + 1) + O\left(\frac{1}{u^{\varpi}}\right) = f_0(u) + O\left(\frac{1}{u^{\varpi}}\right),$$

where $\varpi = \max\{2-2\nu_\alpha, 2-\tau\}$. The roots of f_0 are $u_k = -\frac{2-\alpha}{2} i \left(k + \frac{\nu_\alpha}{2} + \frac{3}{4}\right)\pi, k \in \mathbf{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $-\frac{2-\alpha}{2} i \left(k + \frac{\nu_\alpha}{2} + \frac{3}{4}\right)\pi$ of f_0. Finally for $|k| \geq N$, λ_k is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = -\frac{2-\alpha}{2} i \left(k + \frac{\nu_\alpha}{2} + \frac{3}{4}\right)\pi + \varepsilon_k. \tag{62}$$

Using (62), we get

$$\begin{aligned} e^{2i\left(\left(\frac{2}{2-\alpha} i\lambda_k\right) - \nu_\alpha \frac{\pi}{2} - \frac{\pi}{4}\right)} &= -e^{-\frac{4}{2-\alpha}\varepsilon_k} \\ &= -1 + \tfrac{4}{2-\alpha}\varepsilon_k + O(\varepsilon_k^2). \end{aligned} \tag{63}$$

Substituting (63) into (58), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = \frac{4}{2-\alpha}\varepsilon_k + \frac{1-\alpha}{m}\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\left(\frac{2}{2-\alpha}\right)^2 \frac{2i \sin \nu_\alpha \pi}{(k\pi)^{2-2\nu_\alpha}} + o(\varepsilon_k) + o\left(\frac{1}{k^{2-2\nu_\alpha}}\right) = 0 \tag{64}$$

and hence

$$\varepsilon_k = -\frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}} i \frac{\sin \nu_\alpha \pi}{(k\pi)^{2-2\nu_\alpha}}.$$

Step 4. From Step 3, we can write

$$\lambda_k = -\frac{2-\alpha}{2} i \left(k + \frac{\nu_\alpha}{2} + \frac{3}{4}\right)\pi - i\frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin \nu_\alpha \pi}{(k\pi)^{2-2\nu_\alpha}} + \varepsilon_k. \tag{65}$$

Using (62), we get

$$\begin{aligned} e^{2i\left(\left(\frac{2}{2-\alpha} i\lambda_k\right) - \nu_\alpha \frac{\pi}{2} - \frac{\pi}{4}\right)} &= -e^{-\frac{4}{2-\alpha}\varepsilon_k + \frac{4c}{2-\alpha}\frac{1}{k^{2-2\nu_\alpha}}} \\ &= -1 + \tfrac{4}{2-\alpha}\varepsilon_k - \tfrac{4c}{2-\alpha}\tfrac{1}{k^{2-2\nu_\alpha}} + O(\varepsilon_k^2), \end{aligned} \tag{66}$$

where

$$c = \frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin \nu_\alpha \pi}{\pi^{2-2\nu_\alpha}} i.$$

Substituting (66) into (58), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\begin{aligned} \tilde{f}(\lambda_k) = \tfrac{4}{2-\alpha}\varepsilon_k - i\frac{1-\alpha}{m}\left(\frac{8}{(2-\alpha)^2}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{(2-2\nu_\alpha)(\frac{\nu_\alpha}{2} + \frac{3}{4})}{\pi^{2-2\nu_\alpha} k^{3-2\nu_\alpha}} \sin \nu_\alpha \pi \\ + o(\varepsilon_k) + o\left(\tfrac{1}{k^{3-2\nu_\alpha}}\right) = 0 \end{aligned} \tag{67}$$

and hence

$$\varepsilon_k = i\frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{(2-2\nu_\alpha)(\frac{\nu_\alpha}{2}+\frac{3}{4})}{\pi^{2-2\nu_\alpha}k^{3-2\nu_\alpha}}\sin\nu_\alpha\pi.$$

Step 5. From Step 4, we can write

$$\begin{aligned}\lambda_k &= -\frac{2-\alpha}{2}i\left(k+\frac{\nu_\alpha}{2}+\frac{3}{4}\right)\pi - i\frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin\nu_\alpha\pi}{(k\pi)^{2-2\nu_\alpha}}\\ &+i\frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{(2-2\nu_\alpha)(\frac{\nu_\alpha}{2}+\frac{3}{4})}{\pi^{2-2\nu_\alpha}k^{3-2\nu_\alpha}}\sin\nu_\alpha\pi+\varepsilon_k.\end{aligned} \tag{68}$$

Using (62), we get

$$\begin{aligned}e^{2i((\frac{2}{2-\alpha}i\lambda_k)-\nu_\alpha\frac{\pi}{2}-\frac{\pi}{4})} &= -e^{-\frac{4}{2-\alpha}\varepsilon_k+\frac{4c}{2-\alpha}\frac{1}{k^{2-2\nu_\alpha}}-\frac{4\tilde{c}}{2-\alpha}\frac{1}{k^{3-2\nu_\alpha}}}\\ &= -1+\tfrac{4}{2-\alpha}\varepsilon_k-\tfrac{4c}{2-\alpha}\tfrac{1}{k^{2-2\nu_\alpha}}+\tfrac{4\tilde{c}}{2-\alpha}\tfrac{1}{k^{3-2\nu_\alpha}}-\tfrac{1}{2}(\tfrac{4c}{2-\alpha})^2\tfrac{1}{k^{4-4\nu_\alpha}}+O(\varepsilon_k^2),\end{aligned} \tag{69}$$

where

$$\begin{aligned}c &= \frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin\nu_\alpha\pi}{\pi^{2-2\nu_\alpha}}i,\\ \tilde{c} &= i\frac{1-\alpha}{m}\left(\frac{2}{2-\alpha}\right)\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{(2-2\nu_\alpha)(\frac{\nu_\alpha}{2}+\frac{3}{4})}{\pi^{2-2\nu_\alpha}k^{3-2\nu_\alpha}}\sin\nu_\alpha\pi.\end{aligned}$$

Substituting (69) into (58), using that $\tilde{f}(\lambda_k)=0$, we get:

$$\begin{aligned}\tilde{f}(\lambda_k) &= \tfrac{4}{2-\alpha}\varepsilon_k-\tfrac{4c}{2-\alpha}\tfrac{1}{k^{2-2\nu_\alpha}}+\tfrac{4\tilde{c}}{2-\alpha}\tfrac{1}{k^{3-2\nu_\alpha}}\\ &-\tfrac{1}{2}(\tfrac{4c}{2-\alpha})^2\tfrac{1}{k^{4-4\nu_\alpha}}-2i\frac{\tilde{\tilde{c}}}{\delta^{2-2\nu_\alpha}}\frac{\sin\nu_\alpha\pi}{k^{2-2\nu_\alpha}}\\ &+2i(2-2\nu_\alpha)(\tfrac{\nu_\alpha}{2}+\tfrac{3}{4})\frac{\tilde{\tilde{c}}}{\delta^{2-2\nu_\alpha}}\frac{\sin\nu_\alpha\pi}{k^{3-2\nu_\alpha}}-\frac{\tilde{\tilde{c}}c}{\delta^{2-2\nu_\alpha}}\frac{4}{2-\alpha}\frac{e^{i\nu_\alpha\pi}}{k^{4-4\nu_\alpha}}-\frac{\varrho}{m}\frac{4}{2-\alpha}\frac{c}{\delta^{2-\tau}}\frac{1}{k^{4-\tau-2\nu_\alpha}}\\ &+o(\varepsilon_k)+o\left(\tfrac{1}{k^\omega}\right)\\ &= \tfrac{4}{2-\alpha}\varepsilon_k+\left(\frac{1-\alpha}{m}\right)^2\left(\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\right)^2\frac{32}{(2-\alpha)^4}\frac{\sin\nu_\alpha\cos\nu_\alpha}{(\pi k)^{4-4\nu_\alpha}}i-\frac{\varrho}{m}\frac{4}{2-\alpha}\frac{c}{\delta^{2-\tau}}\frac{1}{k^{4-\tau-2\nu_\alpha}}\\ &+o(\varepsilon_k)+o\left(\tfrac{1}{k^\omega}\right)=0,\end{aligned} \tag{70}$$

where $\omega=\max\{4-\tau-2\nu_\alpha, 4-4\nu_\alpha\}$ and

$$\delta=-\frac{2-\alpha}{2}i\pi,\quad \tilde{\tilde{c}}=\frac{1-\alpha}{m}\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\left(\frac{2}{2-\alpha}i\right)^{2\nu_\alpha}$$

and hence

$$\begin{aligned}\varepsilon_k &= -\left(\frac{1-\alpha}{m}\right)^2\left(\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\right)^2\frac{8}{(2-\alpha)^3}\frac{\sin\nu_\alpha\cos\nu_\alpha}{(\pi k)^{4-4\nu_\alpha}}i\\ &-i\left(\frac{2}{2-\alpha}\right)^{3-\tau}\frac{\varrho(1-\alpha)}{m^2}\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin\nu_\alpha\pi\sin(1-\tau)\frac{\pi}{2}}{\pi^{4-\tau-2\nu_\alpha}}\frac{1}{k^{4-\tau-2\nu_\alpha}}\\ &-\left(\frac{2}{2-\alpha}\right)^{3-\tau}\frac{\varrho(1-\alpha)}{m^2}\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin\nu_\alpha\pi\cos(1-\tau)\frac{\pi}{2}}{\pi^{4-\tau-2\nu_\alpha}}\frac{1}{k^{4-\tau-2\nu_\alpha}}+o\left(\frac{1}{k^\omega}\right).\end{aligned}$$

As (54) shows that the eigenvalues λ_k of $\mathcal{P}$ approach the imaginary axis as k goes to infinity, clearly system (12) is not uniformly stable. From (54), we have

$$|k|^{4-\tau-2\nu_\alpha}\Re\lambda_k \approx -\left(\frac{2}{2-\alpha}\right)^{3-\tau}\frac{\varrho(1-\alpha)}{m^2}\frac{c^+_{\nu_\alpha,0}}{c^-_{\nu_\alpha,0}}\frac{\sin\nu_\alpha\pi\cos(1-\tau)\frac{\pi}{2}}{\pi^{4-\tau-2\nu_\alpha}}.$$

The operator $\mathcal{P}$ has a non exponential decaying branche of eigenvalues. Thus the proof is complete.

5. Polynomial Stability (for $\eta \neq 0$)

To state and prove our stability results, we need some results from semigroup theory.

Theorem 5.1 ([7]). *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{X}$ with generator $\mathcal{P}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{P}) \text{ and } \overline{\lim_{|\beta|\to\infty}} \frac{1}{\beta^l} \|(i\beta I - \mathcal{P})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \infty$$

for some l, then there exist c such that

$$\|e^{\mathcal{P}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Our main result is the following.

Theorem 5.2. *The semigroup $S_{\mathcal{P}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{P}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{(4-\tau-2\nu_\alpha)}}}\|U_0\|^2_{D(\mathcal{P})}.$$

Proof. We will need to study the resolvent equation $(i\lambda - \mathcal{P})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (x^\alpha u_x)_x = f_2, \\ i\lambda\varphi + (\xi^2 + \eta)\varphi - v(0)\nu(\xi) = f_3. \\ i\lambda\theta - \frac{1}{m}(x^\alpha u_x)(0) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \nu(\xi)\varphi(\xi)\, d\xi = f_4. \end{cases} \tag{71}$$

where $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$. From $(71)_1$ and $(71)_2$, we have

$$\lambda^2 u + (x^\alpha u_x)_x = -(f_2 + i\lambda f_1) \tag{72}$$

with

$$\begin{cases} \lambda^2 u - (x^\alpha u_x)_x = 0, \\ -(x^\alpha u_x)(0) + (-m\lambda^2 + i\varrho\lambda(i\lambda + \eta)^{\tau-1})u(0) \\ \qquad\qquad = mf_4 - \zeta\displaystyle\int_{-\infty}^{+\infty} \frac{\nu(\xi)f_3(\xi)}{i\lambda + \eta + \xi^2}\, d\xi + (mi\lambda + \varrho(i\lambda + \eta)^{\tau-1})f_1(0), \\ u(1) = 0. \end{cases} \tag{73}$$

Assume that Φ is a solution of (72), then one easily checks that the function Ψ defined by

$$\Phi(x) = x^{\frac{1-\alpha}{2}}\Psi\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right) \tag{74}$$

is solution of the following inhomogeneous Bessel equation:

$$\begin{aligned} &y^2\Psi''(y) + y\Psi'(y) + \left(y^2 - \left(\frac{\alpha-1}{2-\alpha}\right)^2\right)\Psi(y) = \\ &-(\tfrac{2}{2-\alpha})^2(\tfrac{2-\alpha}{2}\tfrac{1}{\lambda}y)^{\frac{3-\alpha}{2-\alpha}}\left(f_2\left((\tfrac{2-\alpha}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\alpha}}\right) + i\lambda f_1\left((\tfrac{2-\alpha}{2}\tfrac{1}{\lambda}y)^{\frac{2}{2-\alpha}}\right)\right). \end{aligned} \tag{75}$$

The general solution of (75) is easily seen to be

$$\Psi(y) = AJ_{\nu_\alpha}(y) + BJ_{-\nu_\alpha}(y) - \frac{\pi}{2\sin\nu_\alpha\pi}\int_0^y \frac{f(s)}{s}\left(J_{\nu_\alpha}(s)J_{-\nu_\alpha}(y) - J_{\nu_\alpha}(y)J_{-\nu_\alpha}(s)\right)ds,$$

where A and B are constants free to be determined later and

$$f(s) = -(\frac{2}{2-\alpha})^2(\frac{2-\alpha}{2}\frac{1}{\lambda}s)^{\frac{3-\alpha}{2-\alpha}}\left(f_2\left((\frac{2-\alpha}{2}\frac{1}{\lambda}s)^{\frac{2}{2-\alpha}}\right)+i\lambda f_1\left((\frac{2-\alpha}{2}\frac{1}{\lambda}s)^{\frac{2}{2-\alpha}}\right)\right).$$

Thus,

$$\begin{aligned}
&u(x) = Ax^{\frac{1-\alpha}{2}}J_{\nu_\alpha}\left(\tfrac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right)+Bx^{\frac{1-\alpha}{2}}J_{-\nu_\alpha}\left(\tfrac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right)\\
&+\frac{\pi}{2\sin\nu_\alpha\pi}\left(\frac{2}{2-\alpha}\right)x^{\frac{1-\alpha}{2}}\int_0^x s^{\frac{1-\alpha}{2}}(f_2(s)+i\lambda f_1(s))\left(J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda s^{\frac{2-\alpha}{2}}\right)J_{-\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right)\right.\\
&\left.-J_{\nu_\alpha}\left(\tfrac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right)J_{-\nu_\alpha}\left(\tfrac{2}{2-\alpha}\lambda s^{\frac{2-\alpha}{2}}\right)\right)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(x) &= A\Phi_+(x)+B\Phi_-(x)\\
&\quad+\frac{\pi}{2\sin\nu_\alpha\pi}\left(\frac{2}{2-\alpha}\right)\int_0^x(f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(x)-\Phi_+(x)\Phi_-(s))\,ds,
\end{aligned}\tag{76}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = x^{\frac{1-\alpha}{2}}J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right),\quad \Phi_-(x) = x^{\frac{1-\alpha}{2}}J_{-\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right).\tag{77}$$

We thus have

$$\begin{aligned}
u_x(x) &= A\Phi_+'(x)+B\Phi_-'(x)\\
&\quad+\frac{\pi}{2\sin\nu_\alpha\pi}\left(\frac{2}{2-\alpha}\right)\int_0^x(f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-'(x)-\Phi_+'(x)\Phi_-(s))\,ds.
\end{aligned}\tag{78}$$

It remains to determine the constants A and B. Using $(73)_2$, (78) and (76), we conclude that

$$\begin{aligned}
&(1-\alpha)\tilde{c}^+_{\nu_\alpha,0}A-(-m\lambda^2+\varrho i\lambda(i\lambda+\eta)^{\tau-1})\tilde{c}^-_{\nu_\alpha,0}B\\
&\quad= -mf_4+\zeta\int_{-\infty}^{+\infty}\frac{\nu(\xi)f_3(\xi)}{i\lambda+\eta+\xi^2}\,d\xi-(mi\lambda+\varrho(i\lambda+\eta)^{\tau-1})f_1(0),
\end{aligned}\tag{79}$$

$$A\Phi_+(1)+B\Phi_-(1) = -\frac{\pi}{2\sin\nu_\alpha\pi}(\frac{2}{2-\alpha})\int_0^1(f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds,\tag{80}$$

where

$$\tilde{c}^+_{\nu_\alpha,m} = c^+_{\nu_\alpha,m}\left(\frac{2}{2-\alpha}\lambda\right)^{2m+\nu_\alpha},\quad \tilde{c}^-_{\nu_\alpha,m} = c^-_{\nu_\alpha,m}\left(\frac{2}{2-\alpha}\lambda\right)^{2m-\nu_\alpha}$$

and

$$\Phi_+(1) = J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda\right),\qquad \Phi_-(1) = J_{-\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda\right).$$

We write equations (79) and (80) in matrix form as

$$\begin{pmatrix} r_{11} & r_{12}\\ r_{21} & r_{22}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}C\\ \tilde{C}\end{pmatrix},\tag{81}$$

where

$$\begin{aligned}
r_{11} &= (1-\alpha)\tilde{c}^+_{\nu_\alpha,0},\\
r_{12} &= (m\lambda^2-\varrho i\lambda(i\lambda+\eta)^{\tau-1})\tilde{c}^-_{\nu_\alpha,0},\\
r_{21} &= J_{\nu_\alpha}\left(\tfrac{2}{2-\alpha}\lambda\right),\\
r_{22} &= J_{-\nu_\alpha}\left(\tfrac{2}{2-\alpha}\lambda\right),
\end{aligned}$$

$$\begin{aligned} C &= -mf_4 + \zeta\int_{-\infty}^{+\infty}\frac{\nu(\xi)f_3(\xi)}{i\lambda+\eta+\xi^2}\,d\xi - (mi\lambda + \varrho(i\lambda+\eta)^{\tau-1})f_1(0),\\ \tilde{C} &= -\frac{\pi}{2\sin\nu_\alpha\pi}(\frac{2}{2-\alpha})\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds. \end{aligned}$$

Let the determinant of the linear system given in (81) be denoted by D. Then Note that

$$\begin{aligned} D &= (1-\alpha)\tilde{c}^+_{\nu_\alpha,0}J_{-\nu_\alpha}\left(\tfrac{2}{2-\alpha}\lambda\right) - (m\lambda^2 - \varrho i\lambda(i\lambda+\eta)^{\tau-1})\tilde{c}^-_{\nu_\alpha,0}J_{\nu_\alpha}\left(\tfrac{2}{2-\alpha}\lambda\right)\\ &= (1-\alpha)c^+_{\nu_\alpha,0}\left(\tfrac{2}{2-\alpha}\right)^{\nu_\alpha}\lambda^{\nu_\alpha}\left[\left(\frac{2-\alpha}{\pi\lambda}\right)^{1/2}\cos\left(\tfrac{2}{2-\alpha}\lambda+\nu_\alpha\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)+O(\frac{1}{\lambda^{5/2}})\right]\\ &\quad -(m\lambda^2-\varrho i\lambda(i\lambda+\eta)^{\tau-1})c^-_{\nu_\alpha,0}\left(\tfrac{2}{2-\alpha}\right)^{-\nu_\alpha}\lambda^{-\nu_\alpha}\left[\left(\frac{2-\alpha}{\pi\lambda}\right)^{1/2}\cos\left(\tfrac{2}{2-\alpha}\lambda-\nu_\alpha\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)+O(\frac{1}{\lambda^{5/2}})\right]\\ &= -mc^-_{\nu_\alpha,0}\left(\tfrac{2}{2-\alpha}\right)^{-\nu_\alpha}\left(\frac{2-\alpha}{\pi}\right)^{1/2}\lambda^{2-\nu_\alpha-\frac12}\cos\left(\tfrac{2}{2-\alpha}\lambda-\nu_\alpha\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)\\ &\quad +(1-\alpha)c^+_{\nu_\alpha,0}\left(\tfrac{2}{2-\alpha}\right)^{\nu_\alpha}\left(\frac{2-\alpha}{\pi}\right)^{1/2}\lambda^{\nu_\alpha-\frac12}\cos\left(\tfrac{2}{2-\alpha}\lambda+\nu_\alpha\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)\\ &\quad +\varrho i^\tau c^-_{\nu_\alpha,0}\left(\tfrac{2}{2-\alpha}\right)^{-\nu_\alpha}\left(\frac{2-\alpha}{\pi}\right)^{1/2}\lambda^{\tau-\nu_\alpha-\frac12}\cos\left(\tfrac{2}{2-\alpha}\lambda-\nu_\alpha\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)+O(\frac{1}{\lambda^{3/2+\nu_\alpha-\tau}}). \end{aligned}$$

As $D \neq 0$ for all $\lambda \neq 0$, then A and B are uniquely determined by (81).

Now, it is easy to prove that

$$|D| \geq c|\lambda|^{-5/2+\nu_\alpha+\tau} \text{ for large } \lambda. \tag{82}$$

In the following lemma we will use some technical inequalities which will be useful for showing the optimal polynomial decay of the solution.

Lemma 5.3 ([24]). **(I)** *for all $\lambda \in \mathbb{R}-\{0\}$ large, we have*

$$\|\Phi_+\|_{L^2(0,1)}, \|\Phi_-\|_{L^2(0,1)} \leq \frac{c}{\sqrt{|\lambda|}}. \tag{83}$$

(II)

$$\left\|x^{-\frac12}J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right)\right\|_{L^2(0,1)}, \left\|x^{-\frac12}J_{-\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right)\right\|_{L^2(0,1)} \leq c\sqrt{|\lambda|}. \tag{84}$$

(III) *There exists a constant $C > 0$ such that, for all $f_1 \in H^1_{0,\alpha}(0,1)$, $f_2 \in L^2(0,1)$ and $\lambda \in \mathbb{R}-\{0\}$,*

$$\left|\int_0^1 (f_2(s)+i\lambda f_1(s))(\Phi_+(s)\Phi_-(1)-\Phi_+(1)\Phi_-(s))\,ds\right| \leq \frac{1}{|\lambda|}\left(\|f_1\|_{H^1_{0,\alpha}(0,1)}+\|f_2\|_{L^2(0,1)}\right). \tag{85}$$

Now, inverting the matrix in (81) we obtain

$$\begin{cases} A &= \dfrac{1}{D}(Cr_{22}-\tilde{C}r_{12})\\ B &= \dfrac{1}{D}(-Cr_{21}+\tilde{C}r_{11}) \end{cases}$$

Considering only the dominant terms of λ, the following is obtained:

$$\begin{aligned} |D||A| &\leq c_1|\lambda|^{\frac12}+c_2|\lambda|^{1-\nu_\alpha} \leq c_3|\lambda|^{1-\nu_\alpha},\\ |D||B| &\leq c_1|\lambda|^{\frac12}+c_2|\lambda|^{\nu_\alpha-1} \leq c|\lambda|^{\frac12}. \end{aligned}$$

Hence, using (82), we deduce that

$$|A| \le c|\lambda|^{\frac{7}{2}-\tau-2\nu_\alpha} \tag{86}$$

$$|B| \le c|\lambda|^{3-\tau-\nu_\alpha}. \tag{87}$$

Also, we have

$$\begin{aligned} &\left\| \int_0^x f_2(s)\Phi_\pm(x)\Phi_\mp(s)\, ds \right\|_{L^2(0,1)} \le \|f_2\|_{L^2(0,1)} \|\Phi_\pm\|_{L^2(0,1)} \|\Phi_\mp\|_{L^2(0,1)} \le \frac{c}{|\lambda|}, \\ &\left\| i\lambda \int_0^x f_1(s)\Phi_\pm(x)\Phi_\mp(s)\, ds \right\|_{L^2(0,1)} \le \|f_1\|_{L^2(0,1)} \|\Phi_\pm\|_{L^2(0,1)} \|\Phi_\mp\|_{L^2(0,1)} \le c. \end{aligned} \tag{88}$$

Then, from (76), (86), (87) and (88), we get

$$\|u\|_{L^2(0,1)} \le c|\lambda|^{3-\tau-2\nu_\alpha} \left(\|f_1\|_{H^1_{0,\alpha}(0,1)} + \|f_2\|_{L^2(0,1)} + \|f_3\|_{L^2(-\infty,+\infty)} \right),$$

consequently, from $(71)_2$ and (76), we get

$$\|v\|_{L^2(0,1)} \le c|\lambda|^{4-\tau-2\nu_\alpha} \left(\|f_1\|_{H^1_{0,\alpha}(0,1)} + \|f_2\|_{L^2(0,1)} + \|f_3\|_{L^2(-\infty,+\infty)} \right).$$

Using (77) and (52), we obtain

$$\begin{cases} x^{\alpha/2}\Phi'_+(x) &= (\frac{1-\alpha}{2} + \frac{2-\alpha}{2}\nu_\alpha)x^{-1/2}J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right) - \lambda x^{\frac{1-\alpha}{2}} J_{1+\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right), \\ x^{\alpha/2}\Phi'_-(x) &= (\frac{1-\alpha}{2} - \frac{2-\alpha}{2}\nu_\alpha)x^{-1/2}J_{-\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right) - \lambda x^{\frac{1-\alpha}{2}} J_{1-\nu_\alpha}\left(\frac{2}{2-\alpha}\lambda x^{\frac{2-\alpha}{2}}\right). \end{cases}$$

Then from (78), (83) and (84), we can get

$$\|x^{\alpha/2}u_x\|_{L^2(0,1)} \le c|\lambda|^{4-\tau-2\nu_\alpha} \left(\|f_1\|_{H^1_{0,\alpha}(0,1)} + \|f_2\|_{L^2(0,1)} + \|f_3\|_{L^2(-\infty,+\infty)} \right).$$

Moreover from (15), we have

$$\|\varphi\|^2_{L^2(-\infty,\infty)} \le \frac{1}{\eta} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\varphi(\xi)|^2\, d\xi \le c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|(i\lambda I - \mathcal{P})^{-1}\|_{\mathcal{H}} \le c|\lambda|^{4-\tau-2\nu_\alpha} \text{ as } |\lambda| \to \infty.$$

The conclusion then follows by applying Theorem 5.1.

Besides, we prove that the decay rate is optimal. Indeed, the decay rate is consistent with the asymptotic expansion of eigenvalues.

□

Remark 5.4. *We can extend the results of this paper to more general measure density (see* **[10]***) instead of (5). Indeed, let us suppose that ν is an even nonnegative measurable function such that*

$$\int_{-\infty}^{\infty} \frac{\nu(\xi)^2}{1+\xi^2}\, d\xi < \infty. \tag{89}$$

We easily obtain the following Theorem.

Theorem 5.5. *Let*

$$\Lambda(\lambda) = \frac{|\lambda|^{3-2\nu_\alpha}}{(\Re S(i\lambda))},$$

where $S(i\lambda) = \int_{-\infty}^{+\infty} \frac{\nu(\xi)^2}{i\lambda+\eta+\xi^2}\, d\xi$. Then the semigroup $S_{\mathcal{P}}(t)_{t\ge0}$ associated to (P') satisfies the following decay estimate

$$\|e^{\mathcal{P}t}U_0\| \le C\frac{1}{\Lambda^{-1}(t)}\|U_0\|_{D(\mathcal{P})}, \quad t \to \infty,$$

where Λ^{-1} is any asymptotic inverse of Λ.

Open problem

It seems to be interesting to study a qualitative propreties of (P) *with* $d(x)$ *instead of* x^α *(see (2)) with* $0 < \mu_d < 1$.

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